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ON THE SOMIGLIANA'S FORMULA

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1. Let a solid of a known mass M be rotating uniformly around a fixed axis with the velocity ω. Let a level surface S of gravity surrounding totally the mass M be given. Then, according to the Stokes' theorem, (see N. P. Grušinskiĭ [2], Ch. 10) the potential function W of gravity is determined on the surface S, and in the space out side of the surface S. The potential of gravity is equal to the sum of the potentials of the gravitational and centrifugal forces

$$W = V + U$$

where the potential of the centrifugal force $U = \frac{1}{2}\omega^2(x^2 + y^2)$ does not depend on the mass and form of the solid. The potential function V is the solution of the exterior Dirichlet's problem for the surface S. The searched function V has to satisfy the known conditions, among them two let be quoted:

$$\Delta_2 V = 0 \tag{1,1}$$

and

$$V_0 = W_0 - \tfrac{1}{2}\omega^2(x^2 + y^2) \tag{1,2}$$

where W_0 is the potential of gravity on the level surface S.

Solution of the problem for a special case is quoted by M. S. Molodenskiĭ, V. F. Eremeev, M. I. Jurkina ([7], Ch. 2), who immersed the oblate ellipsoid of rotation defined by the equation

$$(x^2 + y^2)/a^2 + z^2/b^2 = 1\,, \quad a > b > 0\,, \tag{1,3}$$

in the rectangular coordinates x, y, z into a triply orthogonal system

$$\begin{aligned} x &= c \sin u \cos v \cosh w\,, \\ y &= c \sin u \sin v \cosh w\,, \\ z &= c \cos u \sinh w\,, \end{aligned} \tag{1,4}$$

where $c = (a^2 - b^2)^{1/2}$ and $u \in \langle 0, \pi \rangle$, $v \in \langle 0, 2\pi \rangle$, $w \in (-\infty, \infty)$. Let the ellipsoid of rotation (1,3) be characterized by the value $w = w_0$, hence

$$(1,5) \qquad a = c \cosh w_0 \,, \quad b = c \sinh w_0 \,.$$

By the solution of the exterior Dirichlet's problem, M. S. Molodenskiĭ, V. F. Eremeev, M. I. Jurkina reduce the proceeding by which C. Somigliana [9] modified the former solutions of P. Pizzetti [8] (cf. also C. F. Baeschlin [1], Ch. IX, W. A. Heiskanen, H. Moritz [3]) and got a relation among three values of the gravity at different latitudes. C. Somigliana had deduced the formula:

$$(1,6) \qquad \gamma = (\gamma_p\, b \sin^2 B + \gamma_e a \cos^2 B) : (b^2 \sin^2 B + a^2 \cos^2 B)^{1/2}$$

expressing the law of the change of the gravity γ on a surface of an oblate ellipsoid of rotation (1 3). Here γ_p and γ_e are the values of γ on the poles and on the equator.

In the further text, there will be shown that Molodenskiĭ's proceeding, limited only to the oblate ellipsoid of rotation, can be very substantially generalized. It will be done in a detailed way for the case of an oblate ellipsoid of rotation (see Sec. 3). For other cases – including the formal analogies for hyperboloids of rotation – it will be quoted in a concise way. But, first of all, the introduction of a system (1,4) shall be motivated. That has not been done by M. S. Molodenskiĭ (see Sec. 2 and 3).

2. Let the ellipsoid of rotation (x, y, z signify always rectangular coordinates)

$$x = a \sin u \cos v \,, \quad y = a \sin u \sin v \,, \quad z = b \cos u \,;$$
$$a, b = \text{const.} \,, \quad u \in \langle 0, \pi \rangle \,, \quad v \in \langle 0, 2\pi \rangle \,,$$

be immersed into a system

$$(2,1) \qquad \begin{aligned} x &= \sin u \cos v\, f(w) \,, \\ y &= \sin u \sin v\, g(w) \,, \\ z &= \cos u\, h(w) \,, \end{aligned}$$

where $f(w)$, $g(w)$, $h(w)$ are functions of the first class on a certain interval of the variable w which will be chosen later. For a fixed $w = w_0$ it holds that

$$(2,2) \qquad f(w_0) = g(w_0) = a \,, \quad h(w_0) = b \,.$$

Let be found out when the system (2,1) is triply orthogonal. Let $\boldsymbol{r}$ mean the radius vector of the point (x, y, z).

Then

$$\frac{\partial \mathbf{r}}{\partial u} = \{\cos u \cos v\, f(w),\ \cos u \sin v\, g(w),\ -\sin u\, h(w)\}\,, \tag{2,3}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \{-\sin u \sin v\, f(w),\ \sin u \cos v\, g(w),\ 0\}\,,$$

$$\frac{\partial \mathbf{r}}{\partial w} = \{\sin u \cos v\, f'(w),\ \sin u \sin v\, g'(w),\ \cos u\, h'(w)\}\,,$$

hence

$$\frac{\partial \mathbf{r}}{\partial u}\frac{\partial \mathbf{r}}{\partial v} = \cos u \sin u \cos v \sin v[g^2(w) - f^2(w)]\,, \tag{2,4}$$

$$\frac{\partial \mathbf{r}}{\partial v}\frac{\partial \mathbf{r}}{\partial w} = \sin^2 u \cos v \sin v[g(w)\, g'(w) - f(w) f'(w)]\,,$$

$$\frac{\partial \mathbf{r}}{\partial w}\frac{\partial \mathbf{r}}{\partial u} = \cos u \sin u[\cos^2 v\, f(w) f'(w) + \sin^2 v\, g(w)\, g'(w) - h(w)\, h'(w)]\,.$$

The conditions of the orthogonality are

$$g^2(w) - f^2(w) = 0\,, \tag{2,5}$$

$$g(w)\, g'(w) - f(w) f'(w) = 0\,, \tag{2,6}$$

$$\cos^2 v\, f(w) f'(w) + \sin^2 v\, g(w)\, g'(w) - h(w)\, h'(w) = 0\,. \tag{2,7}$$

The condition (2,6) follows from the condition (2,5). On the basis of (2,6) there is possible to transcribe (2,7) to a form which after an integration gives

$$f^2(w) - h^2(w) = \varkappa\,, \quad \varkappa = \text{const}\,. \tag{2,8}$$

The constant $\varkappa$ can take arbitrary values.

It can be supposed

$$f(w) > 0\,, \quad g(w) > 0\,, \quad h(w) > 0\,. \tag{2,9}$$

According to (2,1), this can be reached by a suitable orientation of the axes of coordinates. When $\varkappa > 0$, then $f(w) = g(w) > h(w)$ and in the system (2,1), the surfaces $w =$ const. are oblate ellipsoids of rotation; when $\varkappa = 0$, then $f(w) = g(w) = h(w)$ and $w =$ const. are spherical surfaces, and finally when $\varkappa < 0$, then $f(w) = g(w) < h(w)$ and $w =$ const. are prolate spheroids. In the first and the third cases ($\varkappa > 0, \varkappa < 0$), the ellipsoids have obviously z for the axis of rotation and any meridian plane is intersecting them in a system of confocal ellipses.

The surfaces u = const. in the family (2,1) became for $\varkappa > 0$ hyperboloids of one sheet, for $\varkappa = 0$ a conic surface and for $\varkappa < 0$ hyperboloids of two sheets. The axis z is even now the axis of rotation and any meridian plane is intersecting them in a system of confocal hyperbolas.

In the system (2,1), the coordinate z does not depend on v, and the surfaces $v =$ = const. form a family of half-planes with the axis z as the boundary.

In any case, the demand of the orthogonality of the system (2,1) leads to the conclusion that (2,1) is a special triply orthogonal system of quadrics.

3. Let the case $\varkappa > 0$ be discussed. Let the function of the first class $\Phi(w)$ be chosen and put

$$f(w) = \cosh \Phi(w) \sqrt{\varkappa}\,, \qquad h(w) = \sinh \Phi(w) \sqrt{\varkappa}\,. \tag{3,1}$$

It is obvious that the introduction of the hyperbolic function $\sinh \Phi(w)$ and $\cosh \Phi(w)$ is motivated by the identical fulfilment of the conditions of orthogonality (2,8). The functional determinant of the transformation (2,1) is, according to (2,3), equal to

$$I = \Phi'(w)\, \varkappa^{3/2} \sin u \cosh \Phi(w) \left[\sinh^2 \Phi(w) + \cos^2 u\right] \tag{3,2}$$

and if it is supposed that

$$\Phi'(w) > 0 \tag{3,3}$$

it assumes the values of zero only for $u = k\pi$ (where k is a integer) or for $(\sinh^2 \Phi(w) + \cos^2 u) = 0$. The second case can occur only when $\sinh^2 \Phi(w) = 0$ and $\cos^2 u = 0$. According to the presumption (2,9), $h(w) \neq 0$ and hence according to (3,1) also $\sinh^2 \Phi(w) \neq 0$. According to (2,1), to the $\cos u = 0$ there corresponds the plane $z = 0$. If the axis z and the plane $z = 0$ is eliminated from our considerations, a one-to-one transformation is obtained.

The symmetrical components of the basic tensor of the space in the considered system (2,1) with (3,1) are

$$\begin{aligned} a_{11} &= \varkappa\left[\sinh^2 \Phi(w) + \cos^2 u\right], \\ a_{22} &= \varkappa \sin^2 u \cosh^2 \Phi(w)\,, \\ a_{33} &= \varkappa(\Phi'(w))^2 \left[\sinh^2 \Phi(w) + \cos^2 u\right]. \end{aligned} \tag{3,4}$$

Considering that $a_{ij} = 0$, it follows

$$A^2 = |a_{ik}| = \varkappa^3 (\Phi'(w))^2 \left[\sinh^2 \Phi(w) + \cos^2 u\right] . \sin^2 u \cosh^2 \Phi(w) \neq 0\,. \tag{3,5}$$

The Laplace equation (1,1) can be by means of (3,4) and (3,5) (see V. Hlavatý, [3], p. 204) and after a simple arrangement written as

$$(3,6) \qquad \Delta_2 V = \frac{\partial}{\partial u}\left(\varkappa^{1/2}\,\Phi'(w)\sin u\cosh\Phi(w)\,\frac{\partial V}{\partial u}\right) + \\ + \frac{\partial}{\partial v}\left(\varkappa^{1/2}\,\Phi'(w)\,\frac{\cos^2 u + \sinh^2\Phi(w)}{\sin u\cosh\Phi(w)}\,\frac{\partial V}{\partial v}\right) + \\ + \frac{\partial}{\partial w}\left(\varkappa^{1/2}(\Phi'(w))^{-1}\sin u\cosh\Phi(w)\,\frac{\partial V}{\partial w}\right) = 0\,.$$

The solution of Laplace equation (3,6), i.e. the potential V of the gravitational force, is searched in the form

$$(3,7) \qquad V = \sum_{n=1}^{\infty} A_n\,\varphi_n(u)\,\psi_n(w)$$

where $A_n = \text{const.}$, φ_n and ψ_n are functions only of the argument u and w respectively and the coefficients of the terms of the series (3,7) are solutions of (3,6). The Laplace equation (3,6), in the relation to the form of the searched solution (3,7) shall be divided after separation of variables into two equations, where $k = \text{const.}$:

$$(3,8) \qquad \frac{d^2\psi_n(w)}{dw^2}(\Phi'(w))^{-1} + \frac{d\psi_n(w)}{dw}\left(\operatorname{tgh}\Phi(w) + \frac{d(\Phi'(w))^{-1}}{dw}\right) - k\Phi'(w)\,\psi_n(w) = 0\,,$$

$$(3,9) \qquad \frac{d^2\varphi_n(u)}{du^2} + \operatorname{cotg} u\,\frac{d\varphi_n(u)}{du} + k\,\varphi_n(u) = 0\,.$$

The substitution

$$(3,10) \qquad x = i\sinh\Phi(w)$$

shall transform (3,8) to the Legendre's equation

$$(3,11) \qquad (x^2 - 1)\frac{d^2\psi_n(x)}{dx^2} + 2x\,\frac{d\psi_n(x)}{dx} - n(n+1)\,\psi_n(x) = 0\,.$$

The equation (3,9) shall change by the transformation $x = \cos u$ also to the Legendre's equation. Its bounded solutions corresponding to the eigenvalues $k = n(n+1)$, $n \neq 0$ are Legendre's polynomials of the first order $P_n(\cos u)$ in $\cos u$; $u \in \langle 0, \pi\rangle$ (see E. Kamke, [5], p. 455). The searched solution (3,7) can be hence written in the form

$$(3,12) \qquad V = \sum_{n=1}^{\infty} A_n P_n(\cos u)\,\psi_n(w)\,.$$

The potential of the centrifugal force is

$$\tfrac{1}{2}\omega^2(x^2 + y^2)$$

where ω is the angular velocity. According to (2,1) and (3,1), this potential is equal to

$$(3,13) \qquad \tfrac{1}{2}\omega^2 \sin^2 u \cosh^2 \Phi(w)\, \varkappa\,.$$

Let the boundary condition (1,2) be rewritten to the form:

$$(3,14) \qquad V_0 = W_0 - \tfrac{1}{2}\omega^2\varkappa \sin^2 u \cosh^2 \Phi(w_0)\,.$$

It does not depend on the coordinate v and the solution of the problem for the ellipsoid of rotation shall be equal to the sum of partial solutions $A_n\, \varphi_n(u)\, \psi_n(w)$ where $\psi_n(w)$ is the solution of the equation (3,8) and $\varphi_n(u)$ the solution of the equation (3,9).

In order to satisfy the condition (3,14), the potential of the gravitational force (3,12), for $w = w_0$, has to change into (3,14):

$$(3,15) \qquad V_0 = \sum_{n=1}^{\infty} A_n P_n(\cos u)\, \psi_n(w_0) = W_0 - \tfrac{1}{2}\varkappa\omega^2 \cosh^2 \Phi(w_0) \sin^2 u\,.$$

The right-hand side in (3,15) is of the form: const. + const. . $\sin^2 u$ and it follows from the known properties of Legendre's polynomials (see J. Lense, [6], part. I, par. 4) that it can be written in the unique way as a linear combination of Legendre's polynomials of the degree 0 and 2. Hence in (3,12) there is

$$(3,16) \qquad A_1 = A_3 = A_4 = \ldots = 0\,.$$

The potential W of gravity is according to (3,12), (3,14) and (3,16):

$$(3,17) \qquad W = A_0 P_0(\cos u)\, \psi_0(w) + A_2 P_2(\cos u)\, \psi_2(w) + \\ + \tfrac{1}{2}\omega^2\varkappa \cosh^2 \Phi(w) \sin^2 u\,.$$

The decomposition of gravity on a boundary surface S (in this case on the surface of an ellipsoid of rotation) is determined by the derivative of the potential (3,17) in the direction of the normal to the surface S, where $w = w_0$, i.e.:

$$(3,18) \qquad \gamma = \left(-\frac{\partial W}{\partial v}\right)_0 = \left(\frac{1}{\sqrt{a_{33}}} \cdot \frac{\partial W}{\partial w}\right)_0\,.$$

According to (3,4) and (3,17) with regard to the presumption (3,3) and to the elimination of the axis z and the plane $z = 0$ from the consideration (i.e. $(\sinh^2 \Phi(w) + \cos^2 u) \neq 0$), it follows that

$$(3,19) \quad \gamma = -\frac{A_0 P_0\, \psi_0'(w_0) + A_2 P_2\, \psi_2'(w_0) + \omega^2\varkappa\, \Phi'(w_0) \cosh \Phi(w_0) \sinh \Phi(w_0) \sin^2 u}{\Phi'(w_0)\, [\varkappa(\sinh^2 \Phi(w_0) + \cos^2 u)]^{1/2}}$$

i.e.

$$(3,20) \qquad \gamma = (B \cos^2 u + C \sin^2 u) : [\varkappa(\sinh^2 \Phi(w_0) + \cos^2 u)]^{1/2}$$

where B and C are certain constants.

In the denominator (3,20), it is possible to multiply $\sinh^2 \Phi(w_0)$ by the expression $(\sin^2 u + \cos^2 u)$ and considering (3,1) to arrange (3,20) to the form

$$(3,21) \qquad \gamma = (B \cos^2 u + C \sin^2 u) : [f^2(w_0) \cos^2 u + h^2(w_0) \sin^2 u]^{1/2}\,.$$

For $u = 0$, the equation (3,21) has the form

$$\gamma_0 = B : f(w_0)\,,$$

for $u = \pi/2$ it has the form

$$\gamma_{\pi/2} = C : h(w_0)\,.$$

The substitution of the constants B and C into (3,21) yields

$$(3,22) \quad \gamma = [\gamma_0 f(w_0) \cos^2 u + \gamma_{\pi/2} h(w_0) \sin^2 u] : [f^2(w_0) \cos^2 u + h^2(w_0) \sin^2 u]^{1/2}$$

and this is Somigliana's formula expressing the exact law of change of gravity on a surface S. The formula (3,22) when using (2,2),

$$(3,23) \qquad \gamma = (\gamma_0 a \cos^2 u + \gamma_{\pi/2} b \sin^2 u) : [a^2 \cos^2 u + b^2 \sin^2 u]^{1/2}\,,$$

is the equation quoted by M. S. Molodenskiĭ, V. F. Eremeev, M. I. Jurkina ([7], p. 48, eq. (II.20)).

The general proceeding of the derivation of the equation (3,23) used above, leads to the same result as a very special proceeding of Molodenskiĭ.

4. In the case that $\varkappa < 0$, let $\varkappa^* = -\varkappa > 0$ and

$$(4,1) \qquad \begin{aligned} f(w) &= \sinh \Phi(w) \sqrt{\varkappa^*}\,, \\ h(w) &= \cosh \Phi(w) \sqrt{\varkappa^*} \end{aligned}$$

where $\Phi(w)$ is a certain function of the first class. The condition of orthogonality (2,8) is then identically fulfilled.

By an analogous calculation as for the case $\varkappa > 0$ in Sec. 3, the Laplace equation for the potential V is obtained:

$$(4,2) \qquad \begin{aligned} \Delta_2 V = {} & \frac{\partial}{\partial u}\left((\varkappa^*)^{1/2}\, \Phi'(w) \sin u \sinh \Phi(w) \frac{\partial V}{\partial u}\right) + \\ & + \frac{\partial}{\partial v}\left((\varkappa^*)^{1/2}\, \Phi'(w) \frac{\sin^2 u + \sinh^2 \Phi(w)}{\sin u \sinh \Phi(w)} \frac{\partial V}{\partial v}\right) + \\ & + \frac{\partial}{\partial w}\left((\varkappa^*)^{1/2}\, (\Phi'(w))^{-1} \sin u \sinh \Phi(w) \frac{\partial V}{\partial w}\right) = 0\,. \end{aligned}$$

If the further proceeding in order to find solution in the form (3,7) is analogous to that in Sec. 3, then from (4,2) two diferential equations are obtained for $\psi_n(w)$ and $\varphi_n(u)$, and if now in the first equation instead of substitution (3,10) the substitution $x = \cosh \Phi(w)$ is employed, and in the second equation the substitution $x = \cos u$ is used, the equation (3,11) follows again from both differential equations. By an analogous consideration as in Sec. 3, the formula for decomposition of gravity on a surface S with $w = w_0$ representing now a prolate spheroid is obtained as follows:

(4,3)
$$\gamma = \left[\gamma_0 f(w_0) \cos^2 u + \gamma_{\pi/2} h(w_0) \sin^2 u\right] : \\ : \left[f^2(w_0) \cos^2 u + h^2(w_0) \sin^2 u\right]^{1/2} .$$

5. For the case $\varkappa = 0$, it holds according to (2,5), (2,8) and (2,9) that

$$f(w) = g(w) = h(w) .$$

The surface S is now spherical and the discussion according to the model of Sec. 3 shall be substantially simplified. The formula for the decomposition of gravity on the spherical surface S has the form:

(5,1)
$$\gamma = \gamma_0 \cos^2 u + \gamma_{\pi/2} \sin^2 u$$

6. It follows, from the comparison of the resulting expressions (3,22), (4,3) a (5,1) for the individual cases of the choice of the constant $\varkappa$, that Somigliana's formula for the decomposition of the gravity is formally identical for all ellipsoids of rotation having the axis z as the axis of rotation. Hence it is possible, considering (2,2), to summarize equations (3,22), (4,3) and (5,1) into a unique form:

(6,1)
$$\gamma = (\gamma_0 a \cos^2 u + \gamma_{\pi/2} b \sin^2 u) : (a^2 \cos^2 u + b^2 \sin^2 u)^{1/2} .$$

The support function of the ellipse (having the semi-axes a, b) with regard to its centre (i.e. the distance of the centre to the tangent) is

(6,2)
$$h(u) = (a^2 \cos^2 u + b^2 \sin^2 u)^{1/2} .$$

Let

(6,3)
$$\Gamma(u) = \gamma\, h(u) , \quad \Gamma(0) = \gamma_0 a , \quad \Gamma(\pi/2) = \gamma_{\pi/2} b ,$$

hence the formula (6,1) can be transcribed to the form

(6,4)
$$\Gamma(u) = \Gamma(0) \cos^2 u + \Gamma(\pi/2) \sin^2 u ,$$

which is formally totally identical with the classical Euler formula for the normal curvature of the surface or with the Tissot's formula for deformation.

7. The equation of a hyperboloid of rotation of one sheet whose axis is the axis z, with the centre at the origin and with the semi-axes a, b is

$$(x^2 + y^2) : a^2 - z^2 : b^2 = 1 . \tag{7,1}$$

Let the hyperboloid (7,1) be immersed into a triply orthogonal system

$$\begin{aligned} x &= \cosh u \cos v\, f(w) , \\ y &= \cosh u \sin v\, g(w) , \\ z &= \sinh u\, h(w) , \end{aligned} \tag{7,2}$$

$u \in \langle 0, \pi \rangle$, $v \in \langle 0, 2\pi \rangle$; $f(w)$, $h(w)$ and $g(w)$ are functions of the first class on a certain interval of the variable w which will be chosen later; for definite $w = w_0$ it holds (2,2).

The conditions of the orthogonality of the system (7,2) are

$$g^2(w) - f^2(w) = 0 , \tag{7,3}$$

$$g(w)\, g'(w) - f(w)\, f'(w) = 0 , \tag{7,4}$$

$$f(w)\, f'(w) + h(w)\, h'(w) = 0 . \tag{7,5}$$

From the integration of the condition (7,5) it follows

$$f^2(w) + h^2(w) = \varkappa , \quad \varkappa = \text{const} . \tag{7,6}$$

Since the consideration is limited to the real space, (7,6) is possible only when $\varkappa > 0$. During the further discussion, analogous to Sec. 3, let be chosen a function of the first class $\Phi(w)$ and let

$$f(w) = \sin \Phi(w) \sqrt{\varkappa} , \quad h(w) = \cos \Phi(w) \sqrt{\varkappa} . \tag{7,7}$$

The Laplace's equation for the solution of the Dirichlet's problem, analogous to the equation (3,6) is

$$\begin{aligned} \Delta_2 V = {} & \frac{\partial}{\partial u} \left(\varkappa^{1/2}\, \Phi'(w) \sin \Phi(w) \cosh u \frac{\partial V}{\partial u} \right) + \\ & + \frac{\partial}{\partial v} \left(\varkappa^{1/2}\, \Phi'(w) \frac{\sinh^2 u + \cos^2 \Phi(w)}{\cosh u \,.\, \sin \Phi(w)} \frac{\partial V}{\partial v} \right) + \\ & + \frac{\partial}{\partial w} \left(\varkappa^{1/2} (\Phi'(w))^{-1} \sin \Phi(w) \cosh u \frac{\partial V}{\partial w} \right) = 0 . \end{aligned} \tag{7,8}$$

The solution of the equation (7,8) in the form of (3,7) is searched, i.e.

$$V = A_n\, \varphi_n(u)\, \psi_n(w) .$$

By the same proceeding as in Sec. 3, using the substitution $x = i \sinh u$ and $x = \cos \Phi(w)$ instead of transformation (3,8), the equation will be transformed into two Legendre's equations of the form (3,11). The bounded solution of the equation with the variable u is formed by Legendre's polyncmials $P_n(i \sinh u)$, which enables us to find out, by formally the same proceeding as in Sec. 3, a formula analogous to (3,21), however, in this case, with regard to the surface S with $w = w_0$ being a hyperboloid of rotation of one sheet

$$(7,9) \qquad \gamma = (B \cos h^2 u + C \sin h^2 u) : [f^2(w_0) \sinh^2 u + h^2(w_0) \cosh^2 u]^{1/2} .$$

8. Let a hyperboloid of rotation of two sheets

$$(8,1) \qquad (x^2 + y^2) : a^2 - z^2 : b^2 = -1$$

be immersed into a system

$$(8,2) \qquad \begin{aligned} x &= \sinh u \cos v\, f(w) , \\ y &= \sinh u \sin v\, g(w) , \\ z &= \cosh u\, h(w) . \end{aligned}$$

The conditions of the orthogonality are identical with (7,3), (7,4) and (7,6). The choice of the function $f(w)$ and $h(w)$ is analogous to the choice (7,7). By an analogous proceeding as in Sec. 7, the substitution $x = \cosh u$ and $x = \cos \Phi(w)$ is employed, and it follows

$$(8.3) \qquad \gamma = (B \cosh^2 u + C \sinh^2 u) : [h^2(w_0) \sinh^2 u + f^2(w_0) \cosh^2 u]^{1/2} ,$$

in this case with regard to the hyperboloid of rotation of two sheets S with $w = w_0$. In the expressions (8,3) and (7,9), B and C are certain constants; the constant B for the hyperboloid of rotation of one sheet is $B = h(w_0)\, \gamma|_{u=0}$; for the hyperboloid of rotation of two sheets it is $B = f(w_0)\, \gamma|_{u=0}$.

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Souhrn

O SOMIGLIANOVĚ VZORCI

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Existuje-li těleso známé hmoty M rovnoměrně se otáčející kolem pevné osy a je-li dána hladinová plocha S síly tíže úplně obepínající hmotu M, potom podle Stokesova teorému bude potenciálová funkce W síly tíže určena na ploše S a v prostoru vně plochy S. Potenciál síly tíže je roven součtu potenciálů síly přitažlivé a síly odstředivé $W = V + U$. Potenciálová funkce V je řešením vnější Dirichletovy úlohy pro plochu S. Řešením úlohy pro speciální případ se zabýval M. S. Moloděnskij, V. F. Jeremejev, M. I. Jurkina.

V pojednání je ukázáno, že řešení úlohy lze velmi podstatně zobecnit. Podrobněji je to provedeno s rotačním zploštělým elipsoidem, kde je rovněž motivováno zavedení hyperbolických funkcí, a ostatní případy — včetně formálních analogií u rotačních hyperboloidů — jsou jen stručně naznačeny.

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