Jan Zítko

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EXTRAPOLATION OF S.O.R. ITERATIONS

JAN ZÍTKO

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1. INTRODUCTION

Let us consider the system of n linear algebraic equations

$$\mathbf{x} = M\mathbf{x} + \mathbf{b}\,, \tag{1}$$

where M is a normalizable square matrix (i.e. similar to a diagonal one) with $\varrho(M) < 1$. Let $\mathbf{V}_n(C)$ be the n-dimensional vector space over the field of complex numbers C of column vectors with n complex components. Let us define, for arbitrary nonzero vector $\mathbf{x}_0 \in \mathbf{V}_n(C)$, a sequence of vectors $\{\mathbf{x}_k\}_{k=0}^{\infty}$ by the recurrence relation

$$\mathbf{x}_k = M\mathbf{x}_{k-1} + \mathbf{b} \tag{2}$$

for $k = 1, 2, 3, \ldots$.

Let $\mathbf{u} \in \mathbf{V}_n(C)$ and let $\{\mathbf{y}_k\}_{k=0}^{\infty}$ be a sequence of vectors which belong to $\mathbf{V}_n(C)$. Suppose that there exist $q \in C$, a constant $K > 0$ and a sequence of vectors $\{\mathbf{z}_k\}_{k=0}^{\infty}$ such that the following conditions are satisfied:

1) $0 < |q| < 1$,

2) $\|\mathbf{z}_k\| > 1/K$ for an infinite number of integers k,

3) $\|\mathbf{z}_k\| < K$ and $\mathbf{y}_k - \mathbf{u} = q^k \mathbf{z}_k$ for all k.

Then we shall say that the rate of convergence of $\mathbf{y}_k$ to the vector $\mathbf{u}$ equals q and write

$$\mathbf{y}_k \xrightarrow{(q)} \mathbf{u}\,.$$

Remark 1. Symbol $\|\cdot\|$ denotes a vector norm.

If $\mathbf{y}_k \xrightarrow{(q)} \mathbf{u}$, $\mathbf{y}'_k \xrightarrow{(q_1)} \mathbf{u}$ and $|q| < |q_1|$ then we shall say that the sequence $\{\mathbf{y}_k\}_{k=0}^{\infty}$ converges to $\mathbf{u}$ faster than $\{\mathbf{y}'_k\}_{k=0}^{\infty}$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be all mutually different and nonzero eigenvalues of the matrix M and

$$(3) \qquad |\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_r| .$$

Generally, $\mathbf{x}_k \xrightarrow{(\lambda_1)} \mathbf{x}$, and if $|\lambda_1| > |\lambda_2|$ then the sequence

$$\left\{\frac{1}{1 - \lambda_1} (\mathbf{x}_k - \lambda_1 \mathbf{x}_{k-1})\right\}_{k=1}^{\infty}$$

converges to the solution $\mathbf{x}$ faster than $\{\mathbf{x}_k\}_{k=0}^{\infty}$. This problem was studied in the papers [8] and [9]. A generalization of this result is given in the assertion of Theorem 1 in Section 2. In Section 3, it is applied to the S.O.R. method (Successive Overrelaxation Iterative Method). We seek the solution of the system

$$(4) \qquad A\mathbf{x} = \mathbf{b} ,$$

where A is a 2-cyclic, consistently ordered and positive definite matrix. If

$$\mu_1 > \mu_2 > \ldots > \mu_r$$

are all mutually different and positive eigenvalues of the Jacobi matrix B derived from A then the rate of convergence of the optimal S.O.R. equals $\omega_1 - 1$, where $\omega_1 = 2/(1 + \sqrt{(1 - \mu_1^2)})$. However, using the construction from Section 2 we obtain a method (if eigenvalues $\mu_1, \ldots, \mu_s$ are known) which converges by the rate $\omega_s - 1$, where $\omega_s = 2/(1 + \sqrt{(1 - \mu_s^2)})$. This method will be called (in this paper) the extrapolation of the S.O.R. iterations for s eigenvalues. Moreover, inequalities

$$\omega_1 - 1 > \omega_2 - 1 > \ldots > \omega_s - 1$$

hold. The S.O.R. method is a special case of the extrapolation for $s = 1$. If $\mu_1, \ldots, \mu_s$ for any $s \geqq 1$ are not given, there appear some difficulties in calculating this eigenvalues. Estimation of μ_1 was done for example in the papers [4] and [12]. We present one theorem for estimating $\mu_2, \mu_3, \ldots$. Other relevant theorems can be found for example in the book [2]. Practically it is possible to calculate on the digital computer, on account of roundoff errors, the eigenvalues μ_1, μ_2, μ_3 (if double precission arithmetic is not used). This problem is discussed in Section 3, too.

In practice an eigenvalue problem $\lambda A\mathbf{x} = L\mathbf{x}$ with 2-cyclic sparse matrix A of large order is often met with. We use usually Kellogg's iteration process or its modification (see [12]) in order to find a maximal eigenvalue. This leads to the solution of the systems

$$A\mathbf{v} = \mathbf{f}$$

for many vectors $\mathbf{f}$. If eigenvalues of the Jacobi matrix derived from A are not given, then the calculation of two or three eigenvalues μ_1, μ_2, μ_3 of B represents an additional work. Nevertheless, this calculation is excuted only once while the solution of $A\mathbf{v} = \mathbf{f}$ is repeated, and therefore it is worth while.

A numerical example is given in Section 4.

2. PROOF OF GENERAL THEOREM

Let us consider the system (1), let $\lambda_1, \ldots, \lambda_r$, $(r > 1)$, be all mutually different and nonzero eigenvalues of the matrix M. Let inequalities (3) hold. Denote by $\mathbf{x}_0$ the nonzero initial complex vector and by $\{\mathbf{x}_k\}_{k=0}^{\infty}$ the sequence defined by the relation (2). If

$$p(z) = z^s + t_1 z^{s-1} + \ldots + t_s$$

is a polynomial with complex coefficients, then we put

$$f_k^{(m)}(t_1, \ldots, t_s) = \frac{1}{p(1)} (\mathbf{x}_k + t_1 \mathbf{x}_{k-m} + \ldots + t_s \mathbf{x}_{k-sm}) .$$

Let us denote by P_i $(i = 1, 2, \ldots, m)$ and P the projection of $\mathbf{V}_n(C)$ on the subspace of eigenvectors corresponding to the eigenvalue λ_i and 0 respectively.

We can express the initial vector $\mathbf{x}_0$ as the sum

$$\mathbf{x}_0 = \mathbf{w}_1 + \mathbf{w}_2 + \ldots + \mathbf{w}_r + \mathbf{w}' , \tag{5}$$

where $\mathbf{w}_i = P_i \mathbf{x}_0$ for $i = 1, 2, \ldots, r$ and $\mathbf{w}' = P\mathbf{x}_0$. This equality implies

$$\mathbf{x}_k = \lambda_1^k \mathbf{w}_1 + \lambda_2^k \mathbf{w}_2 + \ldots + \lambda_r^k \mathbf{w}_r + \sum_{t=0}^{k-1} M^t \mathbf{b} . \tag{5'}$$

Theorem 1. *Let the matrix M of the system* (1), *i.e.*,

$$\mathbf{x} = M\mathbf{x} + \mathbf{b} ,$$

be normalizable with $\varrho(M) < 1$. Let $\lambda_1, \ldots, \lambda_r$ be all mutually different and nonzero eigenvalues of the matrix M, $r > 1$ and

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_r| .$$

Let s, m be integers, $s \in \langle 1, r)$, $m > 0$. Suppose that the initial vector $\mathbf{x}_0 \in \mathbf{V}_n(C)$ satisfies the relations

$$\mathbf{x}_0 \neq \Theta \text{ (null vector)} ,$$

$$P_{s+1}\mathbf{x}_0 - P_{s+1}\mathbf{b}/(1 - \lambda_{s+1}) \neq \Theta .$$

Construct the sequence of vectors $\{\mathbf{y}_k\}_{k=0}^{\infty}$, $\mathbf{y}_k \in \mathbf{V}_n(C)$ *in the following way:*

(6)
$$\mathbf{y}_k = \mathbf{x}_k \quad \text{for} \quad k = 0, 1, \ldots, sm - 1\,,$$
$$\mathbf{y}_k = f_k^{(m)}(\sigma_1^{(s,m)}, \ldots, \sigma_s^{(s,m)}) \quad \textit{for} \quad k > sm\,,$$

where $\mathbf{x}_k$ is defined by the relation (2) and

(7)
$$\sigma_i^{(s,m)} = (-1)^i \sum_{\substack{j_1, j_2, \ldots, j_i = 1 \\ j_1 < j_2 < \ldots < j_i}}^{s} \lambda_{j_1}^m \lambda_{j_2}^m \ldots, \lambda_{j_i}^m$$

for $i = 1, 2, \ldots, s$.

Then

(8)
$$\mathbf{y}_k \xrightarrow{(\lambda_{s+1})} \mathbf{x}$$

and for $k > sm$,

(8′)
$$\mathbf{y}_k = M\mathbf{y}_{k-1} + \mathbf{b}$$

holds. Moreover, for $k \geqq sm$ *the relation*

(8″)
$$\mathbf{y}_k - \mathbf{x} = \frac{1}{p(1)}\, p(M^m)\,(\mathbf{x}_{k-sm} - \mathbf{x})$$

holds, where $p(z) = z_s + \sigma_1^{(s,m)} z^{s-1} + \ldots + \sigma_s^{(s,m)}$.

Proof. Let us write

(9)
$$\mathbf{b} = \sum_{i=1}^{r} \mathbf{v}_i + \hat{\mathbf{v}}\,,$$

where $\mathbf{v}_i = P_i\mathbf{b}$ and $\hat{\mathbf{v}} = P\mathbf{b}$. It follows from (9) that

(10)
$$\sum_{t=0}^{k-1} M^t \mathbf{b} = \sum_{i=1}^{r} \frac{1 - \lambda_i^k}{1 - \lambda_i} \mathbf{v}_i + \hat{\mathbf{v}}\,.$$

The following $(s + 1)$ equations are obtained easily from the relations (5) and (10).

$$\mathbf{x}_k = \sum_{i=1}^{r} \left(\lambda_i^k \mathbf{w}_i + \frac{1 - \lambda_i^k}{1 - \lambda_i} \mathbf{v}_i\right) + \hat{\mathbf{v}}$$

$$\mathbf{x}_{k-m} = \sum_{i=1}^{r} \left(\lambda_i^{k-m} \mathbf{w}_i + \frac{1 - \lambda_i^{k-m}}{1 - \lambda_i} \mathbf{v}_i\right) + \hat{\mathbf{v}}$$

. . . .

$$\mathbf{x}_{k-sm} = \sum_{i=1}^{r} \left(\lambda_i^{k-sm} \mathbf{w}_i + \frac{1 - \lambda_i^{k-sm}}{1 - \lambda_i} \mathbf{v}_i\right) + \hat{\mathbf{v}}\,.$$

Multiplying these equations successively by the numbers 1, $\sigma_1^{(s,m)}, \ldots, \sigma_s^{(s,m)}$ and summing them all, we obtain the equation

$$p(1)\,\mathbf{y}_k = \sum_{i=s+1}^{r} \lambda_i^k \frac{p(\lambda_i^m)}{\lambda_i^{sm}} \mathbf{w}_i + \sum_{i=1}^{r} \frac{p(1)}{1-\lambda_i} \mathbf{v}_i + \sum_{i=s+1}^{r} \frac{-\lambda_i^k\, p(\lambda_i^m)}{\lambda_i^{sm}(1-\lambda_i)} \mathbf{v}_i + p(1)\,\hat{\mathbf{v}}\,.$$

(Recall that

$$p(z) = z^s + \sigma_1^{(s,m)} z^{s-1} + \ldots + \sigma_s^{(s,m)})\,.$$

It is $p(\lambda_{s+1}^m) \neq 0$ since λ_{s+1}^m is not a root of the equation $p(z) = 0$ as well as $p(1) \neq 0$ since the roots $\lambda_1^m, \ldots, \lambda_s^m$ of $p(z) = 0$ lie inside the unite sphere.

From (1) it follows that

$$\mathbf{x} = (I - M)^{-1}\mathbf{b} = \sum_{i=1}^{r} \frac{1}{1-\lambda_i} \mathbf{v}_i + \hat{\mathbf{v}} \tag{12}$$

and therefore

$$\mathbf{y}_k - \mathbf{x} = \sum_{i=s+1}^{r} \lambda_i^k \left[\frac{p(\lambda_i^m)}{p(1)\,\lambda_i^{sm}} (\mathbf{w}_i - \mathbf{v}_i/(1-\lambda_i)) \right]. \tag{13}$$

Let us denote for $i = s+1, \ldots, r$

$$\mathbf{z}_i = \frac{p(\lambda_i^m)}{p(1)\,\lambda_i^{sm}} (\mathbf{w}_i - \mathbf{v}_i/(1-\lambda_i))\,.$$

The proof of (8) will be done by showing that there exists $K > 0$ such that $\| \sum_{i=s+1}^{r} (\lambda_i/\lambda_{s+1})^k \mathbf{z}_i \| < K$ for all k and that for an infinite number of integers k the inequality $\| \sum_{i=s+1}^{r} (\lambda_i/\lambda_{s+1})^k \mathbf{z}_i \| > 1/K$ holds. The first inequality is evident since $|\lambda_i/\lambda_{s+1}| \leqq 1$ and holds for every $K \geqq \sum_{i=s+1}^{r} \|\mathbf{z}_i\|$. Both inequalities are consequently equivalent to the following auxiliary assertion:

There exists $\delta > 0$, $1/\delta \geqq \sum_{i=s+1}^{r} \|\mathbf{z}_i\|$ such that

$$\left\| \sum_{i=s+1}^{r} \left(\frac{\lambda_i}{\lambda_{s+1}} \right)^k \mathbf{z}_i \right\| > \delta$$

holds for an infinite number of k.

The proof of the auxiliary assertion. Let

$$|\lambda_{s+1}| = \ldots = |\lambda_{s+t}| > |\lambda_{s+t+1}| \geqq \ldots \geqq |\lambda_r|$$

and denote by $\mathbf{W}$ the subspace of $\mathbf{V}_n(C)$ generated by all eigenvectors corresponding to the eigenvalues $\lambda_{s+1}, \ldots, \lambda_{s+t}$. Let $\dim(\mathbf{W}) = t_1$. Let $M_{\mathbf{W}}$ denote the restriction of M to $\mathbf{W}$ and

$$B_{\mathbf{W}} = \frac{1}{\lambda_{s+1}} M_{\mathbf{W}} .$$

It is clear that $B_{\mathbf{W}}$ is a linear operator on the space $\mathbf{W}$ and all its eigenvalues lie on the unite sphere in the complex plane.

Let us denote

$\mathbf{w}'_1, \ldots, \mathbf{w}'_{t_1}$: the basis of $\mathbf{W}$, where $\mathbf{w}'_i$, $(i = 1, \ldots, t_1)$ are eigenvectors of M,

$\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{t_1}$: the orthonormal basis of $\mathbf{W}$.

Let

$$\mathbf{w}'_i = \sum_{j=1}^{t_1} \beta_{ij} \hat{\mathbf{v}}_j$$

and denote by R the matrix $R = (\beta_{ji})_{i,j=1}^{t_1}$. For every $g \in \mathbf{W}$,

$$g = \sum_{i=1}^{t_1} \alpha_i \mathbf{w}'_i = \sum_{i=1}^{t_1} \gamma_i \hat{\mathbf{v}}_i ,$$

the equality

$$(\gamma_1, \ldots, \gamma_{t_1})^T = R(\alpha_1, \ldots, \alpha_{t_1})^T$$

holds, the superscript T being used here for transpose. Let l be an integer. If $(\delta_1, \ldots, \delta_{t_1})^T$ are coordinates of the vector $B_{\mathbf{W}}^{-l} g$ in the orthonormal basis of W then[1])

$$\begin{aligned}(\delta_1, \ldots, \delta_{t_1})^T &= R(\alpha_1 \varepsilon_1, \ldots, \alpha_{t_1} \varepsilon_{t_1})^T = \\ &= R\{\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{t_1})\} (\alpha_1, \ldots, \alpha_{t_1})^T = \\ &= R\{\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{t_1})\} R^{-1}\{R(\alpha_1, \ldots, \alpha_{t_1})^T\} = F(\gamma_1, \ldots, \gamma_{t_1})^T ,\end{aligned}$$

where we put

$$F = R\{\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{t_1})\} R^{-1}$$

and $\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{t_1})$ denotes the diagonal matrix $t_1 \times t_1$ with the diagonal elements $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{t_1}$ successively.

Let us define for every $h \in \mathbf{W}$, $h = \sum_{i=1}^{t_1} \tau_i \hat{\mathbf{v}}_i$ the norm in $\mathbf{W}$:

$$\|h\|_{\mathbf{W}} = \Big(\sum_{i=1}^{t_1} |\tau_i|^2\Big)^{1/2}$$

[1]) ε_i lie on the unite circle in the complex plane.

i.e., $\|h\|_{\mathbf{W}}$ is equal to the spectral norm of the vector $(\tau_1, \tau_2, \ldots, \tau_{t_1})^T$. For a linear operator H on $\mathbf{W}$ define

$$\|H\|_{\mathbf{W}} = \sup_{\tilde{g} \neq \Theta} \frac{\|H\tilde{g}\|_{\mathbf{W}}}{\|\tilde{g}\|_{\mathbf{W}}}.$$

It is

$$\frac{\|B_W^{-l} g\|_{\mathbf{W}}}{\|g\|_{\mathbf{W}}} = \frac{\|F(\gamma_1, \ldots, \gamma_{t_1})^T\|_S}{\|(\gamma_1, \ldots, \gamma_{t_1})^T\|_S} \leqq \|F\|_S \leqq \varkappa(R) = \|R\|_S \|R^{-1}\|_S$$

where $\|\cdot\|_S$ denotes spectral norm of vector and matrix respectively. From the definition it follows

$$\|B_{\mathbf{W}}^{-l}\|_{\mathbf{W}} \leqq \varkappa(R).$$

However, if we put $g_1 = \alpha_1 \mathbf{w}_1'$, $\alpha_1 \neq 0$ then

$$\frac{\|B_{\mathbf{W}}^{-l} g_1\|_{\mathbf{W}}}{\|g_1\|_{\mathbf{W}}} = 1.$$

This implies that $\|B_{\mathbf{W}}^{-l}\|_{\mathbf{W}} \geqq 1$.

According to the assumption of the theorem, it is $\mathbf{z}_{s+1} \neq \Theta$, and therefore $\sum_{i=s+1}^{s+t} \mathbf{z}_i \neq \Theta$.

Let us put

$$\left\| \sum_{i=s+1}^{s+t} \mathbf{z}_i \right\|_S = 2\tilde{\delta}\varkappa(R) > 0,$$

and let $\tilde{k}$ be such an integer that for $k > \tilde{k}$ the relation

$$\left\| \sum_{i=s+t+1}^{r} \left(\frac{\lambda_i}{\lambda_{s+1}} \right)^k \mathbf{z}_i \right\|_S < \tilde{\delta}$$

holds. It is

$$2\tilde{\delta}\varkappa(R) = \left\| \sum_{i=s+1}^{s+t} \mathbf{z}_i \right\|_S = \left\| \sum_{i=s+1}^{s+t} \mathbf{z}_i \right\|_{\mathbf{W}} = \left\| B_{\mathbf{W}}^{-k} B_{\mathbf{W}}^{k} \sum_{i=s+1}^{s+t} \mathbf{z}_i \right\|_{\mathbf{W}} \leqq$$

$$\leqq \|B_{\mathbf{W}}^{-k}\|_{\mathbf{W}} \cdot \left\| B_{\mathbf{W}}^{k} \sum_{i=s+1}^{s+t} \mathbf{z}_i \right\|_{\mathbf{W}} \leqq \left\| \sum_{i=s+1}^{s+t} \left(\frac{\lambda_i}{\lambda_{s+1}} \right)^k \mathbf{z}_i \right\|_S \varkappa(R).$$

Consequently,

$$\left\| \sum_{i=s+1}^{r} \left(\frac{\lambda_i}{\lambda_{s+1}} \right)^k \mathbf{z}_i \right\|_S \geqq \left\| \sum_{i=s+1}^{s+t} \left(\frac{\lambda_i}{\lambda_{s+1}} \right)^k \mathbf{z}_i \right\|_S - \tilde{\delta} \geqq \tilde{\delta} > 0.$$

Recall that $\|\cdot\|$ and $\|\cdot\|_S$ satisfy the relation $\|\cdot\|_S < \vartheta \|\cdot\|$ for some $\vartheta > 0$. If we put

$$\delta = \min\left(\tilde{\delta}/\vartheta, \left(\sum_{i=s+1}^{r} \|\mathbf{z}_i\| \right)^{-1}\right)$$

then the auxiliary assertion holds for this δ, therefore

$$\mathbf{y}_k \xrightarrow{(\lambda_{s+1})} \mathbf{x} .$$

If $k > sm$ then (13) implies

$$\mathbf{y}_{k-1} = x + \sum_{i=s+1}^{r} \lambda_i^{k-1} \mathbf{z}_i$$

and hence

$$M\mathbf{y}_{k-1} = M\mathbf{x} + \sum_{i=s+1}^{r} \lambda_i^k \mathbf{z}_i = \mathbf{x} - \mathbf{b} + \sum_{i=s+1}^{r} \lambda_i^k \mathbf{z}_i = \mathbf{y}_k - \mathbf{b} .$$

To complete the proof we show (8″). It is easy to see that

$$\mathbf{x}_{k-(s-l)m} = (M^m)^l \, \mathbf{x}_{k-sm} + \sum_{j=0}^{lm-1} M^j \mathbf{b}$$

for $l = 0, 1, \ldots, s$ and

$$\mathbf{x} = (M^m)^l \, \mathbf{x} + \sum_{j=0}^{lm-1} M^j \mathbf{b} .$$

Evidently

$$(\mathbf{x}_{k-(s-l)m} - \mathbf{x}) = (M^m)^l \, (\mathbf{x}_{k-sm} - \mathbf{x})$$

for $l = 0, 1, \ldots, s$. If we multiply the l-th equation by $\sigma_{s-l}^{(s,m)}$ and sum all these equations we obtain

$$p(1)\,(\mathbf{y}_k - \mathbf{x}) = p(M^m)\,(\mathbf{x}_{k-sm} - \mathbf{x}) .$$

Dividing by $p(1)$ we obtain (8″), which completes the proof.

Remark 1. We have assumed that $s < r$. The case $s = r$ is evident since $\mathbf{y}_k = \mathbf{x}$ for $k \geqq sm$.

Now we assume that the eigenvalues $\lambda_1, \ldots, \lambda_t$ of the matrix M are given. We shall show how to find the other eigenvalues of M.

Theorem 2. *Let M be a normalizable convergent matrix $n \times n$ and $\lambda_1, \lambda_2, \ldots, \lambda_r$ all mutually different and nonzero eigenvalues of the matrix M. We assume that $r > 1$. Let $m > 0$, $t \in \langle 1, r)$ be integers and let the inequalities*

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_t| \geqq |\lambda_{t+1}| > |\lambda_{t+2}| \geqq \ldots \geqq |\lambda_r|$$

hold. Suppose that the eigenvalues $\lambda_1, \ldots, \lambda_t$ are given. Let $\mathbf{x}_0 \in \mathbf{V}_n(C)$ and $\mathbf{P}_{t+1}\mathbf{x}_0 \neq \Theta$.

Construct sequences of vectors $\{\mathbf{x}_k\}_{k=0}^{\infty}$ and $\{\mathbf{y}_k\}_{k=0}^{\infty}$ in the following way:

(14) $$\mathbf{x}_k = M\mathbf{x}_{k-1} ,$$

$$\mathbf{y}_k = \mathbf{x}_k \ \textit{for} \ k = 1, 2, \ldots, tm - 1 ,$$

(15) $$\mathbf{y}_k = f_k^{(m)}(\sigma_1^{(t,m)}, \ldots, \sigma_t^{(t,m)}) \quad \text{for} \quad k \geqq tm .$$

If we put

(16) $$v_k^{(t+1)} = \frac{\mathbf{y}_{k-1}^H \mathbf{y}_k}{\mathbf{y}_{k-1}^H \mathbf{y}_{k-1}}$$

then

(17) $$\lim v_k^{(t+1)} = \lambda_{t+1} .$$

Remark 2. The superscript H denotes transpose and conjugate.

Proof. If we assume that $\mathbf{x}_0$ is given by the relation (5) then

(18) $$\mathbf{y}_k = \sum_{i=t+1}^{r} \lambda_i^k \frac{p(\lambda_i^m)}{\lambda_i^{tm}\, p(1)} \mathbf{w}_i$$

or

(19) $$\mathbf{y}_k = \lambda_{t+1}^k \mathbf{w}''_{t+1} + \sum_{i=t+2}^{r} \lambda_i^k \mathbf{w}''_i$$

where we denote

$$\mathbf{w}''_i = \frac{p(\lambda_i^m)}{\lambda_i^{tm}\, p(1)} \mathbf{w}_i .$$

It follows from (19) that

(20) $$\mathbf{y}_{k-1}^H \mathbf{y}_k = \lambda_{t+1} |\lambda_{t+1}|^{2k-2} (\mathbf{w}''_{t+1})^H \mathbf{w}''_{t+1} (1 + \sum_{i,j=t+1}^{r}{}' \alpha_{ij} \bar{q}_i^{k-1} q_j^k) ,$$

where

$$\alpha_{ij} = \frac{(\mathbf{w}''_i)^H \mathbf{w}''_j}{(\mathbf{w}''_{t+1})^H \mathbf{w}''_{t+1}} \quad \text{and} \quad q_i = \frac{\lambda_i}{\lambda_{t+1}} .$$

It is easy to see that $|q_i| < 1$ for $i = t + 2, \ldots, r$. From (20) we obtain

$$v_k^{(t+1)} = \lambda_{t+1} \frac{1 + \sum_{i,j=t+1}^{r}{}' \alpha_{ij} \bar{q}_i^{k-1} q_j^k}{1 + \sum_{i,j=t+1}^{r}{}' \alpha_{ij} \bar{q}_i^{k-1} q_j^{k-1}} ,$$

where $\sum'$ denotes that we do not sum for $i = j = t + 1$. The assertion of the theorem is evident from this formulae.

Theorem 2'. *Let the assumptions of Theorem 2 be valid. Moreover, let the eigenvalues of the matrix M be real.*

If we denote

$$\tau_k^{(t+1)} = \left(\mathbf{y}_k^H \mathbf{y}_k / \mathbf{y}_{k-1}^H \mathbf{y}_{k-1}\right)^{1/2}$$

then

$$\lim_{k\to\infty} \tau_k^{(t+1)} = \left|\lambda_{t+1}\right| .$$

Discussion of Theorem 1. The assertions of Theorems 2 or 2′ are the same. For the sake of brevity, take $s = 3$ and $m = 1$. There are two ways how to interpret the theorem.

A) We calculate successively $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$,

$$\mathbf{y}_3 = \{\mathbf{x}_3 + \sigma_1\mathbf{x}_2 + \sigma_2\mathbf{x}_1 + \sigma_3\mathbf{x}_0\}/\{(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)\}$$

where

$$\sigma_1 = -(\lambda_1 + \lambda_2 + \lambda_3), \quad \sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$
$$\sigma_3 = -\lambda_1\lambda_2\lambda_3,$$

and then

$$\mathbf{y}_k = M\mathbf{y}_{k-1} + \mathbf{b} \quad \text{for} \quad k = 4, 5, \ldots.$$

B) For sufficiently great k it is

$$\mathbf{x} \doteq \mathbf{y}_k = \{\mathbf{x}_k + \sigma_1\mathbf{x}_{k-1} + \sigma_2\mathbf{x}_{k-2} + \sigma_3\mathbf{x}_{k-3}\}/\{(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)\} . (*)$$

Now we take a special case. Let $\|\mathbf{x}\|_E = 1$, i.e., for some j the j-th component $(\mathbf{x})_j = 1$. Moreover, let $\lambda_1, \lambda_2, \lambda_3$ be close to unity. Then $\{(1 - \lambda_1)(1 - \lambda_2) . (1 - \lambda_3)\}^{-1}$ is a great number. It is $(\mathbf{y}_k)_j \doteq 1$ since $\mathbf{x} \doteq \mathbf{y}_k$. According to (*) and the relation $(\mathbf{x})_j = 1$, the exponent of the number

$$(\mathbf{x}_k + \sigma_1\mathbf{x}_{k-1} + \sigma_2\mathbf{x}_{k-2} + \sigma_3\mathbf{x}_{k-3})_j$$

equals $\log_{10}[(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)]$.

From this it follows that in numerical calculation, in which every real number is correctly rounded to d decimal places, the expression

$$d - \left|\log_{10}[(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)]\right|$$

represents a number of significant digits while

$$\left|\log_{10}[(1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)]\right|$$

represents the loss of decimal places. In many practical cases this loss is not small in the procedure B) nor in A). We suggest in such cases to take $m > 1$ since

$$\{\prod(1 - \lambda_i^m)\}^{-1} < \{\prod(1 - \lambda_i)\}^{-1}$$

or to use the procedure A) and to calculate an initial approximation in double precission arithmetics.

Remark 3. For $s = 1$ we obtain the well-known formulae

$$\mathbf{y}_k = \frac{1}{1 - \lambda_1^m} \{\mathbf{x}_k - \lambda_1^m \mathbf{x}_{k-1}\},$$

(see [8], [9], [10]).

3. APPLICATION ON SUCCESSIVE OVERRELAXATION ITERATIVE METHOD

Suppose that we seek the solution of the matrix equation

$$A\mathbf{x} = \mathbf{b}, \tag{21}$$

where A is a given $n \times n$ positive definite matrix, $n \geqq 2$. Let us write

$$A = D(I - L - U),$$

where D is the diagonal of A, L and U are strictly lower and upper triangular matrices respectively. We rewrite the system (21) in the form

$$\mathbf{x} = B\mathbf{x} + \mathbf{c}, \tag{22}$$

where $B = L + U$ and $\mathbf{c} = D^{-1}\mathbf{b}$.

Let the matrix B satisfy the following conditions:

(**a**) B is weakly cyclic of index 2;

(**b**) B is consistently ordered;

(**c**) $\varrho(B) < 1$.

The matrix B is normalizable with real eigenvalues since

$$D^{1/2}BD^{-1/2} = I - D^{-1/2}AD^{-1/2}$$

and the matrix on the right hand side is Hermitian.

Let

$$\mu_1 \geqq \mu_2 \geqq \ldots \geqq \mu_t \tag{23}$$

be all positive eigenvalues of the matrix B. There exists a matrix U such that

$$UBU^{-1} = \operatorname{diag}(\mu_1, \ldots, \mu_t, -\mu_1, \ldots, -\mu_t, 0, \ldots, 0).$$

Recall that we have used the symbol diag (...) for the diagonal matrix with diagonal elements $\mu_1, \ldots, \mu_t, -\mu_1, \ldots, -\mu_t$ and $(n - 2t)$-times zero.

From (22) we obtain an equivalent equation

$$\mathbf{x} = \mathscr{L}_\omega \mathbf{x} + \mathbf{d}\,, \tag{24}$$

where

$$\mathscr{L}_\omega = (I - \omega L)^{-1} (\omega U + (1 - \omega) I) \quad \text{and} \quad \mathbf{d} = \omega(I - \omega L)^{-1} \mathbf{c}\,. \tag{25}$$

It is

$$\varrho(\mathscr{L}_\omega) < 1 \quad \text{for} \quad \omega \in (0, 2)\,. \tag{26}$$

This follows immediately from Ostrowski's theorem.

Now we recall two well-known assertions.

Lemma 1. If $\mathscr{L}_\omega$ *is the matrix from* (25) *then* $\det(\mathscr{L}_\omega) = 0$ *if and only if* $\omega = 1$.

Lemma 2. *Let the matrix* A *of the system* (21) *be positive definite, let the Jacobi matrix* B *derived from* A *satisfy the conditions* (**a**), (**b**), (**c**). *Assume that* $\omega \neq 0, 1$.

If μ *is an eigenvalue of* B *and* λ *satisfies the relation*

$$\mu = \frac{\lambda + \omega - 1}{\omega \lambda^{1/2}}\,, \tag{27}$$

then λ *is an eigenvalue of* $\mathscr{L}_\omega$.

Conversely if λ *is an eigenvalue of* $\mathscr{L}_\omega$ *and* μ *satisfies* (27) *then* μ *is an eigenvalue of* B.

The proof is given for example in [4], [6], [11].

Theorem 3. *Let* A *be an* $n \times n$ *positive definite matrix,* $n \geqq 2$. *Suppose that the* $n \times n$ *Jacobi matrix* B *derived from* A *satisfies conditions* (**a**), (**b**) *and* (**c**). *Let us denote by*

$$\mu_1 > \mu_2 > \ldots > \mu_r \tag{28}$$

all positive, mutually different eigenvalues of B. *Let* $r > 1$, $\omega \neq 0, 1$.

Then

1) *The numbers* $\mu_1, \ldots, \mu_r, -\mu_1, \ldots, -\mu_r$ *and* 0, *in virtue of* $\det(B) = 0$, *are all mutually different eigenvalues of* B.

2) *For every admissible* ω *the matrix* $\mathscr{L}_\omega$ *is normalizable. The numbers*

$$\lambda_{2i-1}(\omega) = \left(\frac{\omega\mu_i + \sqrt{(\omega^2\mu_i^2 - 4(\omega - 1))}}{2}\right)^2, \tag{29}$$

$$\lambda_{2i}(\omega) = \left(\frac{(\omega\mu_i - \sqrt{(\omega^2\mu_i^2 - 4(\omega - 1))}}{2}\right)^2, \tag{29}$$

for $i = 1, 2, \ldots, r$, *and*

$$\lambda_{2i+1}(\omega) = \ldots = \lambda_n(\omega) = 1 - \omega\,, \tag{30}$$

are eigenvalues of $\mathscr{L}_\omega$ *while no other complex number is an eigenvalue of* $\mathscr{L}_\omega$.

3) *Let i be an integer,* $i \in \langle 1, r \rangle$. *In the interval* $(1, 2)$, *there is just one root of the equation*

$$\omega^2 \mu_i^2 - 4(\omega - 1) = 0\,. \tag{31}$$

If we denote it by ω_i, *then*

$$\omega_i = \frac{2}{1 + \sqrt{(1 - \mu_i^2)}} \tag{32}$$

and

$$2 > \omega_1 > \omega_2 > \ldots > \omega_r > 1\,.$$

4) *The eingevalues* $\lambda_1(\omega_i), \lambda_2(\omega_i), \ldots, \lambda_{2i}(\omega_i)$ *of the matrix* $\mathscr{L}_{\omega_i}$ *are real,* $\lambda_{2i+1}(\omega_i), \ldots, \lambda_{2r}(\omega_i)$ *are not real and inequalities*

$$\lambda_1(\omega_i) > \lambda_3(\omega_i) > \ldots > \lambda_{2i-1}(\omega_i) = \lambda_{2i}(\omega_i) > \lambda_{2i-2}(\omega_i) > \ldots > \lambda_2(\omega_i) \tag{34}$$

hold. If $i = 1$ *then* (34) *reduces to* $\lambda_1(\omega_1) = \lambda_2(\omega_1)$. *Moreover,*

$$\lambda_{2i-1}(\omega_i) = \lambda_{2i}(\omega_i) = \omega_i - 1\,, \tag{35}$$

$$\lambda_{2l-1}(\omega_i) = \lambda_{2l}(\omega_i) \tag{36}$$

for $l = i + 1, \ldots, r$, *and*

$$|\lambda_{2l-1}(\omega_i)| = \omega_i - 1 \tag{37}$$

for $l = i, i + 1, \ldots, r$.

Proof. It follows from the definition of the Jacobi matrix B that

$$D^{1/2} B D^{-1/2} = I - D^{-1/2} A D^{-1/2}\,.$$

This shows that B is similar to a Hermitian matrix. This together with (**a**) implies the assertion 1).

2) The proof that $\mathscr{L}_\omega$ is normalizable is due to G. J. Tee (see [5]). The other assertion in 2) follows immediately from (**a**), (**b**) and Lemma 2 (see [11], [7]).

The assertions 3) and 4) follows immediately from the relations (27), (28) and (29).

The main result of this paper is the following theorem which we obtain immediately from Theorems 1 and 3.

Theorem 4. *Let A be an $n \times n$ positive definite matrix, $n > 2$,*

$$A = D(I - L - U),$$

where D is the diagonal of A, L and U are strictly lower and upper triangular matrices respectively. Suppose that

$B = L + U$ *is weakly cyclic of index 2,*

B *is consistently ordered,*

$\varrho(B) < 1$.

Let

$$\mu_1 > \mu_2 > \ldots > \mu_r$$

be all mutually different positive eigenvalues of the matrix B, $r > 1$. Let m, i be integers, $m > 0$, $i \in \langle 1, r \rangle$. Let $\mathbf{x} \in \mathbf{V}_n(C)$ be a solution of the equation

$$A\mathbf{x} = \mathbf{b},$$

where $\mathbf{b} \in \mathbf{V}_n(C)$. Let us denote

$$\omega_i = \frac{2}{1 + \sqrt{(1 - \mu_i^2)}}, \tag{38}$$

$$\Lambda_j = \left(\frac{\omega_i \mu_j + \sqrt{(\omega_i^2 \mu_j^2 - 4(\omega_i - 1))}}{2}\right)^2 \tag{39}$$

for $j = 1, 2, \ldots, r$,

$$S_j^{(i-1)} = (-1)^j \sum_{\substack{l_1, l_2, \ldots, l_j = 1 \\ l_1 < l_2 < \ldots < l_j}}^{i-1} \Lambda_{l_1}^m \Lambda_{l_2}^m \ldots \Lambda_{l_j}^m \tag{40}$$

for $j = 1, 2, \ldots, i - 1$, $S_l^{(0)} = 0$,

$$S = 1 + S_1^{(i-1)} + \ldots + S_{i-1}^{(i-1)}. \tag{41}$$

Let P_j be a projection of the space $\mathbf{V}_n(C)$ on the subspace of eigenvectors corresponding to the eigenvalue Λ_j of $\mathscr{L}_{\omega_i}$. Further, let us denote

$$\mathbf{d} = \omega_i(I - \omega_i L)^{-1} D^{-1}\mathbf{b}, \tag{42}$$

$$\mathbf{x}_k = \mathscr{L}_{\omega_i}\mathbf{x}_{-1} + \mathbf{d}, \tag{43}$$

where $\mathbf{x}_0 \in \mathbf{V}_n(C)$, $\mathbf{x}_0 \neq \Theta$ and

$$P_i\mathbf{x}_0 - P_i\mathbf{d}/(1 - \Lambda_i) \neq \Theta.$$

If we put

$$\mathbf{y}_k = \mathbf{x}_k \quad \textit{for} \quad k = 1, 2, \ldots, i - 1 \tag{44}$$

and

(45) $$\mathbf{y}_k = \frac{1}{S} (\mathbf{x}_k + S_1^{(i-1)} \mathbf{x}_{k-m} + \ldots + S_{i-1}^{(i-1)} \mathbf{x}_{k-(i-1)m})$$

for $k \geqq i$

then

$$\mathbf{y}_k \xrightarrow{(\omega_i - 1)} \mathbf{x} .$$

Proof. Theorem 3 implies that the eigenvalues of the matrix B are real. If we take eigenvalues $\Lambda_1, \ldots, \Lambda_{i-1}$ of the matrix $\mathscr{L}_{\omega_i}$ then according to (34),

$$|\Lambda_1| > |\Lambda_2| > \ldots > |\Lambda_{i-1}| > |\Lambda_i| = \omega_i - 1$$

and the absolute value of any other eigenvalue of the matrix $\mathscr{L}_{\omega_i}$ is less or equal to $\omega_i - 1$.

From Theorem 1 it follows

$$\mathbf{y}_k \xrightarrow{(\Lambda_i)} \mathbf{x} .$$

However,

$$\Lambda_i = \left(\frac{\omega_i \mu_i}{2}\right)^2 = \left(\frac{\mu_i}{1 + \sqrt{(1 - \mu_i^2)}}\right)^2 = \left(\frac{\mu_i (1 - \sqrt{(1 - \mu_i^2)})}{\mu_i^2}\right)^2 =$$

$$= \left(\frac{1 - \sqrt{(1 - \mu_i^2)}}{\mu_i}\right)^2 = \omega_i - 1$$

which completes the proof.

4. NUMERICAL RESULTS

Now we consider a model problem. Let a rectangle $\mathbf{G} = \mathsf{ABCD}$ in the plane be given. Suppose the coordinates of the points A, B, C, D are $\mathsf{A} = (x_0, y_0)$, $\mathsf{B} = (x_{N+1}, y_0)$, $\mathsf{C} = (x_0, y_{M+1})$, $\mathsf{D} = (x_{N+1}, y_{M+1})$. Moreover, assume that a uniform mesh exists with the mesh size h such that

$$x_{N+1} = x_0 + (N + 1) h , \quad y_{M+1} = y_0 + (M + 1) h .$$

Consider now the equation

$$-\Delta u = 0 \quad \text{on} \quad \mathbf{G}^0$$

$$u(x, y) = 0 \quad \text{for} \quad (x, y) \in \Gamma$$

where Γ is the boundary of the rectangle $\mathbf{G}$. By the five point difference approximation we obtain the system of linear algebraic equations

(46) $$A\mathbf{x} = 0 .$$

Let us assume that $\operatorname{diag}(A) = I$ (unite matrix) and express

$$A = I - L - U .$$

Let us rewrite the system (46) in the form

$$\mathbf{x} = \mathscr{L}_\omega \mathbf{x} ,$$

where

$$\mathscr{L}_\omega = (I - \omega L)^{-1} (\omega U + (1 - \omega) I) .$$

For the initial approximation $\mathbf{x}_0$ we choose

$$\mathbf{x}_0 = (1, 1, \ldots, 1)^T .$$

Now we compare the following three methods:

(I) Optimal S.O.R.
(II) Extrapolation of S.O.R. for $s = 2$.
(III) Extrapolation for $s = 3$.

We take $M = 5$, $N = 7$. It is well-known that the eigenvalues of the Jacobi matrix are

$$\mu_{kl} = \frac{1}{2}\left(\cos \frac{k\pi}{M + 1} + \cos \frac{l\pi}{N + 1}\right)$$

for

$$1 \leqq k \leqq M , \quad 1 \leqq l \leqq N .$$

The vector $\mathbf{y}_k$ is equal to the error vector while the right hand side of (46) is equal to zero.

Table

k \ $\|\varepsilon_k\|_S$	Method (I)	Method (II)	Method (III)
3	1.46332999	0.83364992	1.14735982
4	0.93064849	0.50714034	0.52746601
7	0.16818544	0.08956832	0.07079159
10	0.01158962	0.00354345	0.00348324
13	0.00094126	0.00005976	0.00001853
16	0.00007315	0.00000089	0.00000012
18	0.00001098	0.00000007	0
19	0.00000490	0.00000002	0
20	0.00000209	0	0
25	0.00000002	0	0
27	0	0	0

If we calculate the eigenvalues of the Jacobi matrix using Theorem 2′, we obtain

$$\mu_1 = 0.89495247\ [0.89495247]$$
$$\mu_2 = 0.78527137\ [0.78656609]$$
$$\mu_3 = 0.71527443\ [0.71193977]$$

Remark 3. We have calculated eigenvalues of the matrix $\mathscr{L}_1$ since the relation $\lambda = \mu^2$ holds (see Lemma 2) for eigenvalues of the matrices $\mathscr{L}_1$ and B.

The numbers in brackets are the exact values.

Remark 4. All calculations were executed on the computer MINSK 22.

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Souhrn

EXTRAPOLACE ITERACÍ PŘI METODĚ S.O.R.

JAN ZÍTKO

Mějme dánu soustavu n lineárních algebraických rovnic

$$\mathbf{x} = M\mathbf{x} + \mathbf{b},$$

kde o matici M předpokládáme, že je normalisovatelná a $\varrho(M) < 1$. Definujme si posloupnost

$$\mathbf{x}_k = M\mathbf{x}_{k-1} + \mathbf{b} \quad \text{pro} \quad k = 1, 2, \ldots$$

Přitom $\mathbf{x}_0$ je libovolný nenulový vektor z n-rozměrného komplexního vektorového prostoru který budeme značit $\mathbf{V}_n(C)$.

Nechť $\{y_k\}_{k=0}^{\infty}$ je posloupnost vektorů z $V_n(C)$, $u \in V_n(C)$. Existuje-li číslo q, $0 < |q| < 1$, kladná konstanta K a posloupnost vektorů $\{z_k\}_{k=0}^{\infty}$ taková, že pro všechna k je

$$\|\mathbf{z}_k\| < K, \quad \mathbf{y}_k - \mathbf{u} = q^k \mathbf{z}_k$$

a pro nekonečně mnoho k je $\|\mathbf{z}_k\| > 1/K$ pak řekneme, že posloupnost $\mathbf{y}_k$ konverguje k vektoru $\mathbf{u}$ rychlostí q a budeme psát $\mathbf{y}_k \xrightarrow{(q)} \mathbf{u}$.

Nechť $\lambda_1, \ldots, \lambda_r$ jsou všechna nenulová a navzájem různá vlastní čísla matice M a nechť

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_r| .$$

Obecně zřejmě platí $\mathbf{x}_k \xrightarrow{(\lambda_i)} \mathbf{x}$. Předpokládáme-li, že $|\lambda_1| > |\lambda_2|$, pak posloupnost

$$\frac{1}{1 - \lambda_1} (\mathbf{x}_k - \lambda_1 \mathbf{x}_{k-1}) \xrightarrow{(\lambda_2)} \mathbf{x} ,$$

což značí, že konverguje k vektoru $\mathbf{x}$ rychleji než poloupnost $\{\mathbf{x}_k\}$. Toto bylo vyšetřováno na příklad v pracích [8] a [9].

V předložené práci tento výsledek nejprve zobecňujeme a sestrojujeme posloupnost aproximací $\mathbf{y}_1, \mathbf{y}_2, \ldots$ takovou, že $\mathbf{y}_k \xrightarrow{(\lambda_s)} \mathbf{x}$. Obecná tvrzení jsou obsažena ve větách 1, 2, 2'.

Uvedenou konstrukci aplikujeme na metodu horní relaxace. Toto tvoří hlavní část předložené práce. Metodu, kterou obdržíme, nazýváme extrapolací metody horní relaxace. Mějme dánu soustavu n lineárních algebraických rovnic

$$A\mathbf{x} = \mathbf{b} ,$$

kde A je positivně definitní matice. Předpokládáme dále, že příslušná Jacobiho matice je slabě cyklická s indexem 2, shodně uspořadaná a konvergentní. Nechť

$$\mu_1 > \mu_2 > \ldots > \mu_r$$

jsou všechna kladná a navzájem různá vlastní čísla Jacobiho matice, nechť přirozené číslo $s \leqq r$. Označíme

$$\omega_1 = \frac{2}{1 + \sqrt{(1 - \mu_1^2)}}, \quad \omega_s = \frac{2}{1 + \sqrt{(1 - \mu_s^2)}} .$$

Optimální S.O.R. konverguje k řešení $\mathbf{x}$ rychlostí $\omega_1 - 1$. Známe-li vlastní čísla $\mu_1, \ldots, \mu_s$ pak příslušná extrapolovaná metoda sestrojená v této práci, dává posloupnost aproximací, které konvergují k řešení rovnice $A\mathbf{x} = \mathbf{b}$ rychlostí $\omega_s - 1$. Přitom je $\omega_1 - 1 > \omega_s - 1$.

V článku je dále diskuse k praktickému použití předložené metody. V závěru je uveden jeden numerický příklad a seznam použité literatury.

Author's address: RNDr *Jan Zítko*, Katedra numerické matematiky na Matematicko-fyzikální fakultě KU, Malostranské nám. 25, 118 00 Praha 1.