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A PROCEDURE FOR DETERMINING NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A SOLUTION TO THE MULTI-INDEX PROBLEM

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INTRODUCTION

The multi-index problem was described by Haley [2] [3], and can be defined as:

Maximize

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} c_{ijk} x_{ijk} .$$

Subject to

$$\sum_{k=1}^{n} x_{ijk} = A_{ij} , \quad (i = 1, 2, \ldots, l ; \quad j = 1, 2, \ldots, m) \tag{1.1}$$

$$\sum_{j=1}^{m} x_{ijk} = B_{ik} , \quad (i = 1, 2, \ldots, l ; \quad k = 1, 2, \ldots, n) \tag{1.2}$$

$$\sum_{i=1}^{l} x_{ijk} = C_{jk} , \quad (j = 1, 2, \ldots, m ; \quad k = 1, 2, \ldots, n) . \tag{1.3}$$

Where:

$$x_{ijk} \geqq 0 , \quad (i = 1, 2, \ldots, l ; \quad j = 1, 2, \ldots, m ; \quad k = 1, 2, \ldots, n)$$

$$\sum_{i=1}^{l} A_{ij} = \sum_{k=1}^{n} C_{jk} , \quad (j = 1, 2, \ldots, m) \tag{2.1}$$

$$\sum_{j=1}^{m} C_{jk} = \sum_{i=1}^{l} B_{ik} , \quad (k = 1, 2, \ldots, n) \tag{2.2}$$

$$\sum_{k=1}^{n} B_{ik} = \sum_{j=1}^{m} A_{ij} , \quad (i = 1, 2, \ldots, l) . \tag{2.3}$$

Although it is possible to solve the multi-index problem using the simplex method, the fact that the multi-index problem is an extension of the transportation problem has stimulated search for a more efficient algorithm [2], [3], [11]. This is particularly important since multi-index problems are often quite large, and may be extraordinarily degenerate.

One of the principal difficulties is that of finding a feasible solution, and it is therefore of interest to determine necessary and sufficient conditions for the existence of a solution, particularly if these conditions are of a form which aids in determining a feasible solution.

Several sets of necessary conditions have been determined [4], [5], [6], [7], [9], but necessary and sufficient conditions have been determined only for problems where at least one of l, m, n is less than or equal to 2.

Take for instance the case where $n = 2$.

$$\begin{aligned} \text{Let } L &= \{1, 2, \ldots, l\} \\ M &= \{1, 2, \ldots, m\} \\ N &= \{1, 2\} \\ I &\subseteq L \\ J &\subseteq M \\ K &\subseteq N \\ \bar{K} &= N - K . \end{aligned}$$

Then necessary and sufficient conditions for the existence of a solution are [5]

$$\sum_{i\in I} \sum_{j\in J} A_{ij} \leqq \sum_{i\in I} \sum_{k\in K} B_{ik} + \sum_{j\in J} \sum_{k\in K} C_{jk} \quad \text{for all} \quad I, J, K . \tag{3}$$

DETERMINING NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A SOLUTION

It has been shown [8] that the constraints 1 are equivalent to:

$$x_{111} + \sum_{i=2}^{l} \sum_{j=2}^{m} \sum_{k=2}^{n} x_{ijk} = \sum_{i=1}^{l} \sum_{k=1}^{n} B_{ik} - \sum_{i=2}^{l} A_{i1} - \sum_{k=2}^{n} B_{1k} - \sum_{j=2}^{m} C_{j1} \tag{4.1}$$

$$x_{i11} - \sum_{j=2}^{m} \sum_{k=2}^{n} x_{ijk} = B_{i1} - \sum_{j=2}^{m} A_{ij} , \quad (i = 2, 3, \ldots, l) \tag{4.2}$$

$$x_{1j1} - \sum_{i=2}^{l} \sum_{k=2}^{n} x_{ijk} = A_{1j} - \sum_{k=2}^{n} C_{jk} , \quad (j = 2, 3, \ldots, m) \tag{4.3}$$

$$x_{11k} - \sum_{i=2}^{l} \sum_{j=2}^{m} x_{ijk} = C_{1k} - \sum_{i=2}^{l} B_{ik} , \quad (k = 2, 3, \ldots, n) \tag{4.4}$$

$$x_{ij1} + \sum_{k=2}^{n} x_{ijk} = A_{ij}\,, \quad (i = 2, 3, \ldots, l; \quad j = 2, 3, \ldots, m) \tag{4.5}$$

$$x_{i1k} + \sum_{j=2}^{m} x_{ijk} = B_{ik}\,, \quad (i = 2, 3, \ldots, l; \quad k = 2, 3, \ldots, n) \tag{4.6}$$

$$x_{1jk} + \sum_{i=2}^{l} x_{ijk} = C_{jk}\,, \quad (j = 2, 3, \ldots, m; \ k = 2, 3, \ldots, n)\,. \tag{4.7}$$

The constraints 4 form a linear programming problem with the addition of the objective function:

Maximize:

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{ijk}\,. \tag{5}$$

Note that summing equation 1.1 over i and j gives

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{ijk} = \sum_{i=1}^{l} \sum_{j=1}^{m} A_{ij}$$

so that the objective function 5 is constant for all feasible solutions, and so does not restrict the problem in any way.

The dual to the linear programming problem 4 and 5 is:

$$\alpha \geqq 1 \tag{6.1}$$

$$\beta_i \geqq 1\,, \quad (i = 2, 3, \ldots, l) \tag{6.2}$$

$$\gamma_j \geqq 1\,, \quad (j = 2, 3, \ldots, m) \tag{6.3}$$

$$\delta_k \geqq 1\,, \quad (k = 2, 3, \ldots, n) \tag{6.4}$$

$$\xi_{ij} \geqq 1\,, \quad (i = 2, 3, \ldots, l; \ j = 2, 3, \ldots, m) \tag{6.5}$$

$$\eta_{ik} \geqq 1\,, \quad (i = 2, 3, \ldots, l; \quad k = 2, 3, \ldots, n) \tag{6.6}$$

$$\theta_{jk} \geqq 1\,, \quad (j = 2, 3, \ldots, m; \quad k = 2, 3, \ldots, n) \tag{6.7}$$

$$\alpha - \beta_i - \gamma_j - \delta_k + \xi_{ij} + \eta_{ik} + \theta_{jk} \geqq 1\,, \qquad (i = 2, 3, \ldots, l; \ j = 2, 3, \ldots, m; \ k = 2, 3, \ldots, n)\,. \tag{6.8}$$

Minimize:

$$\Big[\sum_{i=1}^{l} \sum_{k=1}^{n} B_{ik} - \sum_{i=2}^{l} A_{i1} - \sum_{k=2}^{n} B_{1k} - \sum_{j=2}^{m} C_{j1}\Big] \alpha + \tag{7}$$

$$+ \sum_{i=2}^{l} \Big[B_{i1} - \sum_{j=2}^{m} A_{ij}\Big] \beta_i + \sum_{j=2}^{m} \Big[A_{1j} - \sum_{k=2}^{n} C_{jk}\Big] \gamma_j + \sum_{k=2}^{n} \Big[C_{1k} - \sum_{i=2}^{l} B_{ik}\Big] \delta_k +$$

$$+ \sum_{i=2}^{l} \sum_{j=2}^{m} A_{ij}\xi_{ij} + \sum_{i=2}^{l} \sum_{k=2}^{n} B_{ik}\eta_{ik} + \sum_{j=2}^{m} \sum_{k=2}^{n} C_{jk}\theta_{jk}\,.$$

Transforming the dual inequalities into equations gives:

(8.1) $\alpha \; - y_{111} = 1$

(8.2) $\beta_i \; - y_{i11} = 1\,, \quad (i = 2, 3, \ldots, l)$

(8.3) $\gamma_j \; - y_{1j1} = 1\,, \quad (j = 2, 3, \ldots, m)$

(8.4) $\delta_k \; - y_{11k} = 1\,, \quad (k = 2, 3, \ldots, n)$

(8.5) $\xi_{ij} - y_{ij1} = 1\,, \quad (i = 2, 3, \ldots, l; \quad j = 2, 3, \ldots, m)$

(8.6) $\eta_{ik} - y_{i1k} = 1\,, \quad (1 = 2, 3, \ldots, l; \quad k = 2, 3, \ldots, n)$

(8.7) $\theta_{jk} - y_{1jk} = 1\,, \quad (j = 2, 3, \ldots, m; \quad k = 2, 3, \ldots, n)$

(8.8) $$\alpha - \beta_i - \gamma_j - \delta_k + \xi_{ij} + \eta_{ik} + \theta_{jk} - y_{ijk} = 1\,,$$
$$(i = 2, 3, \ldots, l;\ j = 2, 3, \ldots, m;\ k = 2, 3, \ldots, n)$$

(8.9) $y_{ijk} \geqq 0\,, \quad (i = 1, 2, \ldots, l;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$

with the objective function 7.

Since the primal constraints are equations, the original dual variables need not be sign restricted, but the form of constraints 6.1 to 6.7 ensures that this is so. Moreover, since all of the original dual variables must be $\geqq 1$, they must always all be basic in any feasible simplex tableau.

The dual problem has a feasible solution with all of the original dual variables equal to 1. Therefore, from the duality and existence theorems of linear programming [10], the primal either:

(i) has no solution (in which case the dual objective function is unbounded); or

(ii) has a solution (in which case the dual objective function is finite).

Therefore, the necessary and sufficient condition for the existence of a solution to 4 is that the objective function to the dual is bounded.

The dual problem is bounded if and only if, for all possible basis sets, there is no non-basic variable having all non-positive coefficients in the constraints, which also has (with the objective function in the form of 7) a negative coefficient in the objective function.

In any simplex tableau representing the dual problem, let y_{pqr} be non-basic and let the coefficients of y_{pqr} be:

g_{111} in the row with α basic,

g_{i11} in the row with β_i basic $(i = 2, 3, \ldots, l)$,

g_{1j1} in the row with γ_j basic $(j = 2, 3, \ldots, m)$,

g_{11k} in the row with δ_k basic $(k = 2, 3, \ldots, n)$,

g_{ij1} in the row with ξ_{ij} basic $(i = 2, 3, \ldots, l; \quad j = 2, 3, \ldots, m)$,

g_{i1k} in the row with η_{ik} basic $(i = 2, 3, \ldots, l; \quad k = 2, 3, \ldots, n)$,

g_{1jk} in the row with θ_{jk} basic $(j = 2, 3, \ldots, m; \; k = 2, 3, \ldots, n)$,

g_{ijk} in the remaining rows $(i = 2, 3, \ldots, l; \; j = 2, 3, \ldots, m; \; k = 2, 3, \ldots, n)$.

Then the objective function coefficient of y_{pqr} is

$$C'_{pqr} = - \{[\sum_{i=1}^{l} \sum_{k=1}^{n} B_{ik} - \sum_{i=2}^{l} A_{i1} - \sum_{k=2}^{n} B_{1k} - \sum_{j=2}^{m} C_{j1}] \, g_{111} + \\ + \sum_{i=2}^{l} [B_{i1} - \sum_{j=2}^{m} A_{ij}] \, g_{i11} + \sum_{j=2}^{m} [A_{1j} - \sum_{k=2}^{n} C_{jk}] \, g_{1j1} + \\ + \sum_{k=2}^{n} [C_{1k} - \sum_{i=2}^{l} B_{ik}] \, g_{11k} + \sum_{i=2}^{l} \sum_{j=2}^{m} A_{ij} g_{ij1} + \\ + \sum_{i=2}^{l} \sum_{k=2}^{n} B_{ik} g_{i1k} + \sum_{j=2}^{m} \sum_{k=2}^{n} C_{jk} g_{1jk}\} . \tag{9}$$

The dual is unbounded and no solution to the primal exists if:

$$g_{ijk} \leqq 0 \quad (i = 1, 2, \ldots, l; \; j = 1, 2, \ldots, m; \; k = 1, 2, \ldots, n) \quad \text{and} \quad C'_{pqr} < 0$$

This leads to the following procedure for determining necessary and sufficient conditions for the existence of a solution to a multi-index problem of certain dimensions (say $l = l'$, $m = m'$, $n = n'$):

For each possible basis of the dual problem 8, inspect each column of the simplex tableau. For each column which has

$$g_{ijk} \leqq 0 \; (i = 1, 2, \ldots, l; \; j = 1, 2, \ldots, m; \; k = 1, 2, \ldots, n)$$

record:

$$g_{1jk} \; (j = 1, 2, \ldots, m; \; k = 1, 2, \ldots, n)$$
$$g_{i1k} \; (i = 1, 2, \ldots, l, \quad k = 1, 2, \ldots, n)$$
$$g_{ij1} \; (i = 1, 2, \ldots, l; \quad j = 1, 2, \ldots, m),$$

unless an identical set of coefficients has been recorded previously.

The sets of coefficients can then be used to determine whether or not any problem for which $l = l'$, $m = m'$, $n = n'$ has a solution. For each set of coefficients, the objective function coefficient is calculated from expression 9. If the results of any such calculation is negative, then the problem has no solution. Otherwise the problem has a solution.

The problem of finding each possible basis has been dealt with by Balinski [1].

SOME RESULTS

The procedure for determining necessary and sufficient conditions for the existence of a solution to the multi-index problem was used with several small problems. For problems smaller than $l = 3$, $m = 3$, $n = 3$ the procedure succeeded in determining necessary and sufficient conditions, but, processing of the largest problem attempted ($l = 3$, $m = 3$, $n = 3$) was abandoned before completion because there was no indication that the end of the run was near after 50 hours of processing on an IBM 360/50 computer.

The problems solved completely all had at least one dimension (l, m or n) of 2, and therefore the necessary and sufficient conditions for the existence of a solution are known to be the conditions 3. The conditions generated by the procedure were found to be equivalent to 3 which to some extent validated the computer program.

CONCLUDING REMARKS

A procedure has been given which, (given enough computer time) will determine necessary and sufficient conditions for the existence of a solution to the multi-index problem of any given dimensions.

The conditions are of the form that functions of the right hand sides of 1 (see expression 9) are non-negative. It therefore follows from 1 that the conditions provide lower bounds (zero) for functions of the variables x_{ijk}: a fact which could probably be used to advantage in any algorithm for finding a feasible solution.

Furthermore it is evident that since the number of basis sets is finite and also the number of columns in the dual tableau is finite, the number of conditions is also finite.

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Souhrn

METODA NALEZENÍ NUTNÝCH A POSTAČUJÍCÍCH PODMÍNEK EXISTENCE ŘEŠENÍ VÍCEINDEXOVÉHO DOPRAVNÍHO PROBLÉMU

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Je udán postup, který dává nutné a postačující podmínky existence řešení víceindexového dopravního problému libovolných dimensí. Je dokázáno, že počet podmínek je konečný, a že tyto podmínky dávají dolní meze pro funkce neznámých problému.

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