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STABILITY OF ITERATIVE SCHEMES FOR NONSELFADJOINT EQUATIONS

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1. INTRODUCTION

In this paper we study the stability of a class of iterative schemes which may be used to obtain numerical solutions of a partial differential equation. The differential equation is normally replaced by a finite difference approximation at a set of mesh points. When the boundary conditions have been applied, one obtains a system of algebraic equations which is solved to give the required numerical solution. Such a system may be written in the form

$$Au = f\,, \tag{1.1}$$

where A is the coefficient matrix of the algebraic system and u is the discrete solution vector.

In order to solve (1.1), one frequently employs an iterative procedure of the form

$$Bu^{(m+1)} = Bu^{(m)} - \tau(Au^{(m)} - f)\,, \tag{1.2}$$

where B is matrix and τ an iteration parameter, both of which are chosen in order to make the iterative scheme (1.2) stable and convergent. The problem of stability has attracted the attention of several authors. The well known Von Neumann condition of stability [6], [7] can be applied to virtually every problem; however, it is a necessary condition and does not always guarantee stability. Kreiss [4] obtained a set of equivalent conditions which were shown to be sufficient for stability. These conditions are rather of theoretical interest and do not yield a practical stability criterion. One of these conditions proves the existence of a similarity transformation which puts A into a form easily tested for stability. Another one puts a limit on the growth of the resolvent. Similar shortcomings are associated with other known results (see [7], [12]).

In 1968, Samarskii [10] obtained several equivalent necessary and sufficient conditions of stability for the case when the matrix A is symmetric. These conditions could be used to ascertain the stability before starting the computations. In the case of a nonsymmetric matrix A, Samarskii obtained some sufficient conditions of stability but they require the knowledge of A^{-1} which is rarely known. We try to overcome this difficulty and obtain several sufficient conditions of stability that can be applied *a priori*. We also obtain some estimates of the rate of convergence of the iterative schemes and show their superiority to the existing results.

It may be noted at this point that an iterative scheme of the form (1.2) can also be used to obtain the time dependent solution of the abstract Cauchy problem

$$(1.3) \qquad \frac{\partial u}{\partial t}(x, t) = Lu(x, t) + f(x, t)\,;$$

$$u(x, 0) = u_0(x)\,, \quad 0 \leqq t \leqq t_0$$

where L is a matrix differential operator in the space variable $x = (x_1, x_2, \dots x_d) \in R^d$.

In order to consider the stability of (1.2) in a general setting, we introduce a family of real Hilbert spaces $\{H_h\}$ depending upon a parameter h which is a vector in a normed space (e.g. $h \in R^d$); $|h|$ is the norm of the vector h. We introduce the network

$$(1.4) \qquad \omega_\tau = \{t = m\tau / m = 0, 1, \dots, m_0;\ t_0 = m_0\tau\}\,.$$

Let $y(t) \equiv y_{h\tau}(t)$, $f(t) \equiv f_{h\tau}(t)$, etc. be abstract functions of the argument $t \in \omega_\tau$ with values in H_h; $A(t) \equiv A_{h\tau}(t)$, $B(t) \equiv B_{h\tau}(t)$, etc., be linear operators mapping H_h into H_h for each $t \in \omega_\tau$. An iterative scheme of the type (1.2) can be written in the following operator form:

$$(1.5) \qquad B_{h\tau}(t)\,\frac{y_{h\tau}(t + \tau) - y_{h\tau}(t)}{\tau} + A_{h\tau}(t)\,y_{h\tau}(t) = f_{h\tau}(t)\,,$$

$$0 \leqq t = m\tau < t_0\,;$$

$$y_{h\tau}(0) = y_{0h\tau} \in H_h\,;$$

where $y_{0h\tau}$ denotes the starting approximation for the iterative scheme (1.2). For the sake of convenience we shall sometimes drop the subscripts h and τ. Note that a multilevel iteration scheme can be reduced to the form (1.5) by introducing new variables.

Let $(.,.)_h$ and $\|y\|_h = (y, y)_h^{1/2}$ be the scalar product and norm in H_h, and let E be the identity operator. An operator A is selfadjoint ($A = A^*$) if $(A\ u, v) = (u, Av)$ for all $u, v \in H$; A is positive ($A > 0$) if $(Au, u) > 0$, $u \neq 0$, $u \in H$; A is positive definite ($A \geqq \delta E$) if $(Au, u) \geqq \delta(u, u)$, $\delta > 0$, $u \in H$; $A \geqq B$ if $(Au, u) \geqq (Bu, u)$ for all $u \in H$. If $B = B^* > 0$, the square root $B^{1/2}$ exists [3] and $B^{1/2} = (B^{1/2})^* > 0$.

A positive operator $B = B(t)$, dependent upon $t \in \omega_\tau$, is said to be Lipschitz continuous in t [9] if

$$(1.6) \qquad |(B(t)\, u, u) - (B(t - \tau)\, u, u)| \leqq \tau c_1(B(t - \tau)\, u, u), \quad t \in \omega_\tau,$$

where c_1 is a positive number independent of t, τ and h. The operator norm is defined as $\|A\| = \sup_{\|x\|=1} \|Ax\|$, $x \in H$. If A is a selfadjoint operator, then $\|A\| = \sup_{\|x\|=1} |(Ax, x)|$, $x \in H$. If $D(t)$ is a positive linear operator on H, then an energy norm can be defined:

$$(1.7) \qquad \|y\|_{d(t)} = (D(t)\, y, y)^{1/2},$$

where the lower case letter $d(t)$ relates to the operator $D(t)$.

We remark here that we consider a real Hilbert space in order to study the nonselfadjoint positive operators.

2. STABILITY

The initial value problem under consideration is

$$(2.1) \qquad B(t) \frac{y(t + \tau) - y(t)}{\tau} + A(t)\, y(t) = f(t),$$

$$0 \leqq t = m\tau < t_0;$$

$$y(0) = y_0 \in H_h.$$

This problem is properly posed [8] if there exists $\tau_0 = \tau_0(h)$ such that for $\tau \leqq \tau_0$, a solution of (2.1) exists for arbitrary $y_0 \in H_h$ and $f(t) \in H_h$, $t \in \omega_\tau$. The scheme (2.1) is stable if there are positive constants M_1 amd M_2, independent of t, τ and h, such that the following inequality is satisfied for $\tau \leqq \tau_0$:

$$(2.2) \qquad \|y(t)\|_{(1,t)} \leqq M_1 \|y(0)\|_{(1,0)} + M_2 \max_{0 \leqq t' \leqq t} \|f(t)\|_{(2,t')}$$

where $\|\,.\,\|_{(1,t)}$ and $\|\,.\,\|_{(2,t)}$ are certain norms defined on H_h. Examples of these norms, which are functions of t, are the energy norms related to the operators of the scheme (2.1):

$$(2.3) \qquad \|y(t + \tau)\|_{(1,t)} = \|y(t + \tau)\|_{a(t)} = (A(t)\, y(t + \tau),\ y(t + \tau))^{1/2};$$

$$\|y(t + \tau)\|_{(1,t)} = \|y(t + \tau)\|_{b(t)}.$$

First of all we discuss the stability of (2.1) with respect to the initial data. The corresponding definition of stability is obtained by putting $f(t) \equiv 0$ in (2.2). We assume that $B(t)$ is a selfadjoint and positive operator that satisfies a Lipschitz condition in t and $A(t)$ is a positive operator. Since $B(t) = B^*(t) > 0$, the square root $B^{1/2}(t)$ exists for each $t \in \omega_\tau$ and the equation (2.1) can be written in the following form [2, 10]:

$$x(t + \tau) = S(t)\,\bar{x}(t)\,, \quad S(t) = E - \tau\,C(t) \tag{2.4}$$

where

$$x(t + \tau) = B^{1/2}(t)\,y(t + \tau)\,, \quad \bar{x}(t) = B^{1/2}(t)\,y(t)$$

and

$$C(t) = B^{-1/2}(t)\,A(t)\,B^{-1/2}(t)\,, \quad \bar{x}(0) = x(0)\,;$$

$$\|x(t + \tau)\| = \|B^{1/2}(t)\,y(t + \tau)\| = \|y(t + \tau)\|_{b(t)}\,.$$

The following lemma from [2] provides a stability criterion:

Lemma 2.1. *The initial value problem* (2.1) *is stable if*

$$\|S_{h\tau}(t)\| \leqq e^{c_0\tau} = \varrho\,, \quad t \in \omega_\tau\,, \tag{2.5}$$

where c_0 *is a real number independent of* t, τ *and* h.

The bound (2.5) also gives a necessary condition of stability when the operator $S_{h\tau}$ is independent of t. In this case the condition (2.5) is similar to the stability condition given by Lax and Wendroff [5]. We are interested in obtaining stability conditions in terms of the original operators A and B.

The rest of this section deals with the stability of (2.1) with respect to the initial data when A and B are independent of t. In the following section we consider the stability when A and B are time-dependent operators. Section 4 deals with the stability with respect to the right hand side $f(t)$ (which includes the boundary conditions associated with the original partial differential equation) according to the definition (2.2).

When A and B are independent of t and $B = B^* > 0$, $A > 0$, the equation (2.4) becomes

$$x(t + \tau) = S_{h\tau}\,x(t)\,, \quad 0 \leqq t = m\tau < t_0\,, \quad x(0) = x_0 \in H_h\,, \tag{2.6}$$

where

$$S = E - \tau C\,, \quad C = B^{-1/2}AB^{-1/2}\,, \quad x = B^{1/2}y\,;$$

and

$$\|x\| = (B^{1/2}y, B^{1/2}y)^{1/2} = (By, y)^{1/2} = \|y\|_b\,. \tag{2.7}$$

The necessary and sufficient condition of stability becomes

$$\|S_{h\tau}\| \leqq \varrho = e^{c_0\tau}\,, \quad c_0 \text{ independent of } t, \tau \text{ and } h\,. \tag{2.8}$$

Samarskii [10] obtained the following equivalent necessary and sufficient conditions when $A = A^*$:

$$(2.9) \qquad (1 - \varrho) B \leqq \tau A \leqq (1 + \varrho) B, \quad \varrho = e^{c_0 \tau}$$

with any constant c_0. The following conditions were obtained when $A \neq A^*$:

$$(2.10) \qquad C^{-1} \geqq (\tau/2) E \Rightarrow A^{-1} \geqq (\tau/2) B^{-1} \Rightarrow \|S\| \leqq 1,$$

where E is the identity operator. This condition involves the inverse operators A^{-1}, B^{-1} and C^{-1} which are rarely known *a priori*. Consequently, the condition (2.10) is not suitable for testing.

If the iterative scheme (2.6) is stable, then

$$(2.11) \qquad \|S\| \leqq \varrho \Rightarrow \tau \|C\| \leqq 1 + \varrho, \quad \varrho = e^{c_0 \tau}$$

which gives a necessary condition of stability. A sufficient condition is obtained as follows:
Since

$$\|S\| = \|E - \tau C\| \leqq 1 + \tau \|C\|,$$

$$(2.12) \qquad \tau \|C\| \leqq \varrho - 1 \Rightarrow \|S\| \leqq \varrho, \quad \varrho = e^{c_0 \tau}.$$

This condition, however, is restrictive but can be improved if more information about the operator C is available. If C is a selfadjoint positive semidefinite operator ($C = C^* \geqq 0$), then the condition (2.11) is also sufficient for stability. If C is a nonselfadjoint positive definite operator ($C \neq C^*$. $C \geqq \delta E$, $\delta > 0$), then

$$\|Sx\|^2 \leqq (1 - 2\tau\delta + \tau^2 \|C\|^2) \|x\|^2,$$

and

$$(2.13) \qquad \|S\| \leqq \varrho \quad \text{if} \quad 0 < \tau \leqq \tilde{\tau}, \tilde{\tau} = \frac{\delta + (\delta^2 + \|C\|^2 (\varrho^2 - 1))^{1/2}}{\|C\|^2}.$$

Frequently the operator A, and consequently the operator C, is the sum of a selfadjoint and a nonselfadjoint operator. In such a case further improvements can be carried out and a better stability range obtained:

Theorem 2.1. *Let C be a positive semidefinite operator such that*

$$(2.14) \qquad C = C_0 + C_1, \quad C_0 = C_0^* \geqq 0; \quad C_1 \neq C_1^*, \quad C_1 \geqq \delta E, \quad \delta \geqq 0.$$

The operator scheme (2.6) is stable ($\|S_{h\tau}\| \leqq 1$) if the iteration parameter τ is chosen such that

$$(2.15) \qquad 0 < \tau \leqq \tau_0 \tilde{\theta}, \ \tilde{\theta} = \frac{2(1 + b - \varrho_0)}{1 + 2b - \varrho_0^2 + a^2},$$

where

$$a = \tau_0 \|C_1\| , \quad b = \tau_0 \delta , \quad \varrho_0 = \|E - \tau_0 C_0\| ;$$

and τ_0 is any real number satisfying

$$\varrho_0 \leqq 1 + b , \quad (1 - \varrho_0)^2 \leqq a^2 . \tag{2.16}$$

Proof.

$$S = E - \tau C = E - \tau C_0 - \tau C_1 = (\theta E - \tau C_0) + ((1 - \theta) E - \tau C_1) , \quad 0 \leqq \theta \leqq 1 .$$

By the triangle inequality

$$\|S\| \leqq \theta \|E - \tau/\theta C_0\| + \|(1 - \theta) E - \tau C_1\| .$$

We rescale τ in terms of τ_0 and θ by writting $\tau = \tau_0 \theta$, $0 \leqq \theta \leqq 1$. Moreover,

$$\|\{(1 - \theta) E - \tau C_1\} x\|^2 = (1 - \theta)^2 \|x\|^2 - 2\tau(1 - \theta)(C_1 x, x) + \tag{2.17}$$
$$+ \tau^2 \|C_1 x\|^2 \leqq \{(1 - \theta)^2 - 2\tau\delta(1 - \theta) + \tau^2 \|C_1\|^2\} \|x\|^2 .$$

This is true for all $x \in H$, so that

$$\|(1 - \theta) E - \tau C_1\| \leqq \{(1 - \theta)^2 - 2\tau\delta(1 - \theta) + \tau^2 \|C_1\|^2\}^{1/2} ,$$

and $\|S\| \leqq f(\theta)$ where

$$f(\theta) = \theta \|E - \tau_0 C_0\| + \{(1 - \theta)^2 - 2\tau_0 \theta \delta(1 - \theta) + \tau_0^2 \theta^2 \|C_1\|^2\}^{1/2} = \tag{2.18}$$
$$= \theta \varrho_0 + \{(1 - \theta)^2 - 2b\theta(1 - \theta) + a^2 \theta^2\}^{1/2} ,$$

where $a = \tau_0 \|C_1\|$, $b = \tau_0 \delta$ and $\varrho_0 = \|E - \tau_0 C_0\|$.

The value of $f(\theta)$ is less than unity if

$$0 \leqq \theta \leqq \tilde{\theta} , \quad \tilde{\theta} = \frac{2(1 + b - \varrho_0)}{1 + 2b - \varrho_0^2 + a^2} .$$

The value of $\tilde{\theta}$ lies in the interval $[0, 1]$ if $\varrho_0 \leqq 1 + b$ and $(1 - \varrho_0)^2 \leqq a^2$. It follows that

$$\|S\| \leqq f(\theta) \leqq 1 \quad \text{if} \quad 0 < \tau = \tau_0 \theta \leqq \tau_0 \tilde{\theta}$$

where τ_0 is a real number satisfying (2.16).

Sometimes the operator A and hence the operator C is the sum of a positive selfadjoint operator and a skew operator:

$$C = C_0 + C_1 , \quad C_0 = C_0^* > 0 , \quad C_1 = -C_1^* . \tag{2.19}$$

In this case $(C_1x, x) = 0$ for all $x \in H$ and the corresponding stability condition is obtained by putting $\delta = 0$ in Theorem 2.1. It may be noted that the stability range given in (2.15) is an increasing function of δ.

The next theorem [2] obtains the value of τ_0 that maximizes the stability range (2.15). The proof is elementary and is omitted.

Theorem 2.2. *Let the operators C_0 and C_1 satisfy the following conditions:*

$$\gamma_1 E \leqq C_0 = C_0^* \leqq \gamma_2 E\,, \quad C_1 \geqq \delta E\,, \quad \|C_1\| \leqq \gamma_3\,; \tag{2.20}$$
$$0 \leqq \gamma_1 \leqq \gamma_2\,, \quad 0 \leqq \delta \leqq \gamma_3\,.$$

The stability range (2.15) is the largest when $\tau_0 = 2/(\gamma_1 + \gamma_2)$. The computations defined by the iterative scheme (2.6) are stable if τ is chosen such that

$$0 < \tau \leqq \frac{2(\gamma_1 + \delta)}{(\gamma_1 + \gamma_2)(\gamma_1 + \delta) + (\gamma_3^2 - \gamma_1^2)}\,. \tag{2.21}$$

If we ignore the positive definiteness of C_1 in (2.14) and assume δ to be zero, then a smaller stability range is obtained in (2.21).

Corollary 1. If $C = C_0 + C_1$, then $\|C\| \leqq \gamma_2 + \gamma_3$ *and a necessary condition of stability is obtained from* (2.11) as $0 < \tau \leqq 2/(\gamma_2 + \gamma_3)$. *This becomes a necessary and sufficient condition when $\gamma_3 \leqq \gamma_1$. When C is a selfadjoint operator, then $\gamma_3 = 0$ and this condition becomes $0 < \tau \leqq 2/\gamma_2$ which is equivalent to (2.9) with $\varrho = 1$.*

Corollary 2. *The function $f(\theta)$ of (2.18) assumes a minimum in the interval $(0, \tilde{\theta})$. This minimum is achieved for $\theta = \hat{\theta}$ given by*

$$\hat{\theta} = \frac{(1 + b)(d - \varrho_0^2)^{1/2} - \varrho_0(a^2 - b^2)^{1/2}}{d(d - \varrho_0^2)^{1/2}}\,, \quad d = a^2 + 2b + 1\,. \tag{2.22}$$

The iteration parameter $\hat{\tau}$ for the optimum rate of convergence of the iterative scheme (2.6) is given by $\hat{\tau} = \tau_0\hat{\theta}$, $\tau_0 = 2/(\gamma_1 + \gamma_2)$. The norm of the transition operator $S_{h\tau}$ satisfies

$$\|S_{h\tau}\| \leqq \hat{\varrho}\,, \quad \hat{\varrho} = \frac{\varrho_0(1 + b) + (a^2 - b^2)^{1/2}(d - \varrho_0^2)^{1/2}}{d} < 1\,. \tag{2.23}$$

When $\delta = 0$, then $b = 0$ and we obtain the values of $\hat{\tau}$ and $\hat{\varrho}$ for the optimum convergence as obtained by Samarskii [11].

The value of $\hat{\varrho}$ given in (2.23) with $b \neq 0$ is smaller than that obtained by Samarskii [11] with $b = 0$. Thus, the iterative scheme (2.6) has a faster rate of convergence if the optimum value $\tau = \tau_0\hat{\theta}$ is used. These rates of convergence (with $b = 0$ or $b \neq 0$) are faster than that obtained by Gunn [1] who studied a class of semi-explicit iterative schemes used for solving an elliptic differential equation of the second order. The value of $\hat{\tau}$ for the optimal rate of convergence is also larger than that given by Gunn. This would result in smaller rounding errors. Gunn proved his results in a complex finite dimensional inner product space and could not deal with the non-selfadjoint operators which also may be positive. Moreover, the results obtained by Gunn for selfadjoint operators can be reproduced from Corollary 2 by putting $a = b = 0$ and without the assumption of finite dimensionality of the space H_h.

The conditions of Theorem 2.2 may be verified in terms of the norms of A and B by using the following lemma [2].

The proof is trivial.

Lemma 2.2. *Let* $A = A_0 + A_1$, $A_0 = A_0^* > 0$, $A_1 > 0$ *and* $B = B^* > 0$ *such that*

$$\gamma_1 B \leqq A_0 \leqq \gamma_2 B\,, \quad \alpha_1 E \leqq B \leqq \alpha_2 E\,, \quad \|A_1\| \leqq \beta\,, \quad A_1 \geqq \alpha E\,; \tag{2.24}$$

then $\gamma_1 E \leqq C_0 \leqq \gamma_2 E$, $C_1 \geqq \delta E$ *and* $\|C_1\| \leqq \gamma_3$
where $\delta = \alpha/\alpha_2$, $\gamma_3 = \beta/\alpha_1$.

Lemma 2.1 implies that the sufficient condition of stability is $\|S_{h\tau}\| \leqq e^{c_0\tau} \approx \approx 1 + 0(\tau)$ for small τ. In the results obtained above, we have used the criterion $\|S\| \leqq 1$ which may give a conservative stability range. The above results, however, ensure the convergence of the iterative procedure. A wider stability range is obtained if the condition $\|S\| \leqq 1 + K\tau$ is satisfied, where K is constant.

Theorem 2.3. *Let the operators* C_0 *and* C_1 *satisfy the conditions of Theorem* 2.2 *and let* $\gamma_3 \geqq \gamma_1$. *Let* τ *be chosen such that*

$$0 < \tau \leqq \tau_0 \tilde{\tilde{\theta}}\,, \tilde{\tilde{\theta}} = \frac{2(1 + b + c - \varrho_0)}{1 + 2b + a^2 - (c - \varrho_0)^2}\,, \quad c = K\tau_0\,, \tag{2.25}$$

where $\tau_0 = 2/(\gamma_1 + \gamma_2)$ *and* K *is a real number, independent of* t, τ *and* h *such that*

$$0 \leqq K \leqq \gamma_3 - \gamma_1\,. \tag{2.26}$$

Such a value of K *always exists. The iterative scheme* (2.6) *is stable and satisfies the condition*

$$\|S_{h\tau}\| \leqq 1 + K\tau \leqq e^{K\tau}\,. \tag{2.27}$$

The proof is similar to those of Theorems 2.1 and 2.2. This result is useful if we can find a nonzero value of K which satisfies (2.26) and is independent of the mesh parameters. The stability range given in (2.25) increases with K and its maximum is attained when K has its maximum value, subject to the condition (2.26).

In the case when the operators $A(t)$ and $B(t)$ are time-dependent, we assume that $A(t) > 0$, $B(t) = B^*(t) > 0$, $t \in \omega_\tau$; and that $B(t)$ satisfies a Lipschitz condition in t. The equation (2.1) with $f(t) \equiv 0$ reduces to (2.4) and a sufficient condition of stability is given by Lemma 2.1:

$$\|S_{h\tau}(t)\| \leqq e^{c_0 \tau} = \varrho\,, \quad t \in \omega_\tau\,, \tag{3.1}$$

where c_0 is a real number independent of t, τ and h.

When $A(t) = A^*(t)$, $t \in \omega_\tau$, Samarskii [10] obtained the following sufficient condition of stability:

$$(1 - \varrho)\,B(t) \leqq \tau\,A(t) \leqq (1 + \varrho)\,B(t)\,, \quad \varrho = e^{c_0 \tau}\,, \quad t \in \omega_\tau\,. \tag{3.2}$$

When $A(t)$ is nonselfadjoint, the analysis of Samarskii produces a condition of the form (2.10) which involves inverses of $A(t)$ and $B(t)$ and is not suitable for testing. However, our analysis of Theorems 2.1–2.3 is valid in this case and the stability results (2.11)–(2.27) hold. We rewrite Theorems 2.1 and 2.2 for the present case while the other results can be extended to the case of time-dependent operator in a similar manner [2].

Theorem 3.1. *Let $C_0(t)$ and $C_1(t)$ be linear operators on H_h such that*

$$C(t) = C_0(t) + C_1(t)\,, \quad C_0(t) = C_0^*(t)\,; \tag{3.3}$$

$$C_1(t) \geqq \delta E\,, \quad \delta > 0\,, \quad t \in \omega_\tau\,.$$

Then, the corresponding initial value problem is stable ($\|S(t)\| \leqq 1$, $t \in \omega_\tau$) provided

$$0 < \tau \leqq \tau_0 \tilde{\theta}\,, \quad \tilde{\theta} = \frac{2(1 + b - \varrho_0)}{1 + 2b - \varrho_0^2 + a^2}\,, \tag{3.4}$$

$$a = \tau_0 \|C_1(t)\|\,, \quad b = \tau_0 \delta\,, \quad \varrho_0 = \|E - \tau_0\, C_0(t)\|\,, \quad t \in \omega_\tau$$

and τ_0 is a real number satisfying the conditions

$$\varrho_0 \leqq 1 + b\,, \quad (1 - \varrho_0)^2 < a^2\,.$$

Theorem 3.2. *Let the operators $C_0(t)$ and $C_1(t)$ satisfy the conditions*

$$\gamma_1 E \leqq C_0(t) = C_0^*(t) \leqq \gamma_2 E\,; \quad C_1(t) \geqq \delta E\,, \tag{3.5}$$

$$\|C_1(t)\| \leqq \gamma_3\,, \quad t \in \omega_\tau\,;$$

$$0 \leqq \gamma_1 \leqq \gamma_2\,, \quad 0 \leqq \delta \leqq \gamma_3\,.$$

In this case $\|S(t)\| \leqq 1$, $t \in \omega_\tau$ *and the stability range is largest when* $\tau_0 = 2/(\gamma_1 + \gamma_2)$ *and the parameter* τ *satisfies the condition* (2.21).

The proofs of these theorems follow their counterparts in the case of time-independent operators. These theorems ensure the uniform boundedness of $\|S_{h\tau}(t)\|$ for each $t \in \omega_\tau$ which, from (3.1), proves the stability of the problem (2.1) with $f(t) \equiv 0$.

4. STABILITY WITH RESPECT TO THE RIGHT HAND SIDE

We now discuss the stability of (2.1) with respect to the right hand side $f(t)$ which includes the boundary data for the original boundary value problem. The following result from [2] shows that the iterative scheme (2.1) is stable with respect to the right hand side if the sufficient conditions of stability with respect to the initial data are satisfied. The estimate (4.2) proves the stability of the iterative scheme according to the definition (2.2). It also provides an *a priori* error estimate.

Theorem 4.1. *Let* $B(t)$ *be a positive selfadjoint operator satisfying a Lipschitz condition in* t. *Let* $A(t)$ *be a positive nonselfadjoint operator and let*

$$\|S_{h\tau}(t)\| \leqq \varrho = e^{c_0 \tau}, \quad t \in \omega_\tau . \tag{4.1}$$

Then the solution of the problem (2.1) *satisfies the estimate*

$$\|y(t + \tau)\|_{b(t)} \leqq \tilde{\varrho}^{m+1} \|y(0)\|_{b(0)} + \sum_{m'=0}^{m} \tau \, \tilde{\varrho}^{m-m'} \|f(t')\|_{b^{-1}(t')} , \tag{4.2}$$

where $\tilde{\varrho} = \exp(\tilde{c}_0 \tau)$, $\tilde{c}_0 = c_0 + c_1/2$, $t' = m'\tau$.

Proof. Since $B(t) = B^*(t) > 0$, $B^{1/2}(t)$ exists and from (2.1) we get

$$B^{1/2}(t)\, B^{1/2}(t) \, \frac{y(t + \tau) - y(t)}{\tau} + A(t)\, y(t) = f(t) .$$

Writing $x(t + \tau) = B^{1/2}(t)\, y(t + \tau)$ and $\bar{x}(t) = B^{1/2}(t)\, y(t)$ we get

$$x(t + \tau) = S(t)\, \bar{x}(t) + \tau\, B^{-1/2}(t) f(t) , \tag{4.3}$$

where $S(t) = E - \tau\, C(t)$, $C(t) = B^{-1/2}(t)\, A(t)\, B^{-1/2}(t)$.

Since $B(t)$ is Lipschitz continuous in t, it satisfies the condition (1.6) with a constant c_1. We substitute $B^{1/2}(t - \tau)\, u = z$ in (1.6) and obtain

$$|(B^{-1/2}(t - \tau)\, B(t)\, B^{-1/2}(t - \tau)\, z, z) - (z, z)| \leqq c_1 \tau (z, z) , \quad z \in H_h$$

or

$$|(B^{-1/2}(t - \tau)\, B(t)\, B^{-1/2}(t - \tau)\, z, z)| \leqq (1 + c_1 \tau)\,(z, z) \leqq e^{c_1 \tau}(z, z)$$

which yields

$$\|B^{1/2}(t)\, B^{-1/2}(t-\tau)\| \leqq e^{c_1\tau/2}\,, \quad t \in \omega_\tau\,. \tag{4.4}$$

Now,

$$\begin{aligned}\bar{x}(t) &= B^{1/2}(t)\, y(t) \\ &= B^{1/2}(t)\, B^{-1/2}(t-\tau)\, x(t)\,.\end{aligned}$$

Using (4.4) we get

$$\|\bar{x}(t)\| \leqq e^{c_1\tau/2} \|x(t)\|\,. \tag{4.5}$$

It follows that

$$\begin{aligned}\|x(t+\tau)\| &\leqq \|S(t)\| \,.\, \|\bar{x}(t)\| + \tau \|B^{-1/2}(t) f(t)\| \leqq \\ &\leqq e^{c_0\tau} \,.\, e^{c_1\tau/2} \|x(t)\| + \tau \|f(t)\|_{b^{-1}(t)} = \\ &= \tilde{\varrho} \|x(t)\| + \tau \|f(t)\|_{b^{-1}(t)}\,, \quad t > 0\,;\end{aligned}$$

$$\begin{aligned}\|x(\tau)\| &\leqq \|S(0)\| \,.\, \|\bar{x}(0)\| + \tau \|f(0)\|_{b^{-1}(0)} \leqq \\ &\leqq \exp(c_0\tau)\, \|y(0)\|_{b(0)} + \tau \|f(0)\|_{b^{-1}(0)} \leqq \\ &\leqq \tilde{\varrho} \|y(0)\|_{b(0)} + \tau \|f(0)\|_{b^{-1}(0)}\,.\end{aligned}$$

Using the above inequalities for $t' = \tau, 2\tau, \ldots$ we get

$$\begin{aligned}\|x(t+\tau)\| &\leqq \tilde{\varrho}^m \|x(\tau)\| + \tau \sum_{m'=1}^{m} \tilde{\varrho}^{m-m'} \|f(t')\|_{b^{-1}(t')} \leqq \\ &\leqq \tilde{\varrho}^{m+1} \|y(0)\|_{b(0)} + \tau \sum_{m'=0}^{m} \tilde{\varrho}^{m-m'} \|f(t')\|_{b^{-1}(t')}\,; \\ & t' = m'\tau\,, \quad t = m\tau\,.\end{aligned}$$

This yields the estimate (4.2) since

$$\|x(t+\tau)\| = \|y(t+\tau)\|_{b(t)}\,.$$

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Souhrn

STABILITA ITERAČNÍHO SCHEMATU PRO NESAMOADJUNGOVANÉ ROVNICE

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Nechť A je nesamoadjungovaný kladný operátor v reálném Hilbertově prostoru. V článku se zkoumá stabilita třídy iteračních schemat, užívaných při řešení operátorové rovnice $Au = f$. S použitím týchž iteračních schemat je možno řešit také odpovídající třídu parabolických rovnic. Dokazuje se několik postačujících podmínek stability, které jsou vyjádřeny pomocí známých operátorů a mohou být použity a priori. Výsledky lze aplikovat na problémy s proměnnými koeficienty a na smíšené úlohy.

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