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ON THE SOLUTION OF ONE PROBLEM OF THE PLATE WITH RIBS

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In the present paper one problem of the plate with ribs is solved. The technical formulation of the problem and its physical aspects are discussed in detail in [7]. Similar problems were solved in the following papers: In [2], the numerical solution of the plate with one rib which is stiff with respect to the bending is given. The problem is generalized to the case of a finite number of intersecting ribs in the paper [3]. The plate with ribs which are stiff against bending and torsion in the sense of Saint-Venant is solved in [4] and [5]. In the papers mentioned above the fini:c element technique has been used to solve the problems.

The main goal of this paper is to prove convergence of the finite element approach in the problem of the plate with ribs when the bending and torsion similar to Vlasov's torsion is taken into account. For more details of the physical meaning see [7]. In order to prove the above mentioned convergence we have to prove one density theorem. This theorem implies the convergence of finite element approximations to the solution of this important case of plate with ribs. The rate of convergence cannot be estimated because we know nothing about the apriori regularity of the solution of the problem.

1. FORMULATION

Let Ω be a bounded region in N-dimensional Euclidean space $\mathbb{R}_N$, $\bar{\Omega}$ the closure of this region. We denote by $D(\Omega)$ the space of infinitely differentiable functions with compact supports in Ω. $H^k(\Omega) = W^{k,2}(\Omega)$, $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ are the usual Sobolev's spaces with the usual algebraic and topological properties (for details see [6]), for instance

$$((u, v))_{k,\Omega} = \sum_{|i| \leq k} \int_\Omega \mathrm{D}^i u \,.\, \mathrm{D}^i v \,\mathrm{d}\Omega \quad u, v \in H^k(\Omega) \tag{1.1}$$

$$\left(\mathrm{D}^i u = \frac{\partial^{|i|} u}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_N^{i_N}} \; ; \quad |i| = \sum_{j=1}^{n} t_j\right),$$

(1.2) $$(u, v)_{k,\Omega} = \sum_{|i|=k} \int_\Omega \mathrm{D}^i u \, . \, \mathrm{D}^i v \, \mathrm{d}\Omega \quad u, v \in H_0^k(\Omega)$$

are the scalar products on $H^k(\Omega)$ and $H_0^k(\Omega)$, respectively, (k integer) and

(1.3) $$\|u\|_{k,\Omega} = ((u, u))^{1/2},$$

(1.4) $$|u|_{k,\Omega} = (u, u)^{1/2}$$

are the corresponding norms.

For a sufficiently smooth boundary of Ω it can be proved that the functions from $H_0^k(\Omega)$ are those from $H^k(\Omega)$, which satisfy the Dirichlet boundary conditions:

$$u = \partial u/\partial v = \dots = \partial^{k-1} u/\partial v^{k-1} = 0$$

on $\partial\Omega$ in the sense of traces, where v denotes the outward normal to the boundary of Ω (denoted by $\partial\Omega$). Let

$$G = (-a, a) \times (-b, b) \subset \mathbb{R}_2,$$

a and b given real numbers, and let $I = \{I_i\}_{i=1}^n$, $J = \{J_j\}_{j=1}^m$ be two systems of segments in G:

$$I_i = \{(x, y) \subset \mathbb{R}_2 \;\; x = x_i, \; y \in (-b, b)\},$$
$$J_j = \{(x, y) \subset \mathbb{R}_2 \;\; y = y_j, \; x \in (-a, a)\}$$

where x_i, y_j are given real numbers, $-a < x_i < x_j < a$, $-b < y_i < y_j < b$ for $i < j$. For the same reason as in [2] (see the definition of the trial problem below) we introduce the space

(1.5) $$V(G, n, m) = \{u; u \in H_0^2(G), \; u/_{I_i} \in H_0^2(I_i), \; u/_{J_j} \in H_0^2(J_j),$$
$$u_x//_{I_i} \in H_0^2(I_i), \; u_y//_{J_j} \in H_0^2(J_j) \quad \text{for} \quad I_i \in I,$$
$$J_j \in J, \quad i = 1, \dots, n, \quad j = 1, \dots, m\}.$$

In the definition (1.5), the symbol $u/_I$ denotes the restriction of the function $u : G \to \mathbb{R}_1$ to the segment I and $u//_I$ denotes the trace of $u : G \to \mathbb{R}_1$ to I. For the sake of simplicity we omit in the following the specification "restriction" and "trace" as there is no danger of misunderstanding. In the above definition as well as in the following the lower indices of functions will denote partial derivatives.

On $V(G, n, m)$ the scalar product is defined in the natural way:

(1.6) $$(((u, v))) = (u, v)_{2,G} + (u, v)_I + (u, v)_J,$$

where

$$(u, v)_I = \sum_{i=1}^{n} \int_{I_i} (uv + u_x v_x)_{yy} \, dy$$

$$u, v \in V(G, n, m)$$

$$(u, v)_J = \sum_{j=1}^{m} \int_{J_j} (uv + u_y v_y)_{xx} \, dx .$$

The space $V(G, n, m)$ equipped with the norm

(1.7) $$|||u||| = (((u, v)))^{1/2}$$

is a Hilbert space (see [7]).

Denote by $/V(G, n, m)/'$ the dual space to $V(G, n, m)$ and let $f \in /V(G, n, m)/'$ be given.

Problem: *Find $u \in V(G, n, m)$ such that*

(1.8) $$(((u, v))) = (f, v) \quad \textit{for all} \quad v \in V(G, n, m)$$

where $(.,.)$ is the duality $V(G, n, m)$ and $[V(G, n, m)]'$. There exists a solution of this problem and it is unique (see [7]).

Our problem is the weak formulation of the problem of the clamped plate with ribs which are stiff against bending and torsion similar to Vlasov's torsion of box girders. In (1.8), the type of bilinear form $(((.,.)))$ is taken into investigation in order to include more hypotheses of the dimensional reduction of the plate and beams (denoted by segments I and J) and, of course, for the sake of simplicity. This approach is advantageous and has been used also in the papers [2] through [5] and [7].

Now, we shall solve Problem by Ahlin's conforming elements. Our main goal will be to prove convergence of this method to the solution of Problem assuming that it is nothing known about the regularity of the solution of Problem.

We note that in the following discussion the higher order polynomials are included with the only restriction concerning the conformity of elements generated by these polynomials.

To each $h \in (0, 1)$ there is a division $G_h = \{G_{ih}\}_{i=1}^{k(h)}$ of the region G and each element of G_h is a rectangle.

Let R be a nondegenerate rectangle and let

$$Ah(R) = \{v; v = \sum_{0 \leq i,j \leq 3} a_{ij} x^i y^j\}$$

be the set of the so called Ahlin's polynomials over R. The coefficients a_{ij} are real numbers.

For $h \in (0, 1)$ fixed we introduce the space V_h which approximates $V(G, n, m)$ in the following way:

(1.9) $\quad V_h = \{v; v \in Ah(G_{ij}) \text{ on } G_{ij} \text{ for each } i = 1, \ldots, k(h)$

if A is a nodal point of the division G_h then:

a) *v, v_x, v_y, v_{xy} are continuous functions at A,*

b) *v, v_x, v_y, v_{xy} are equal to zero at $A \in \partial G\}$* .

Introducing a relative topology on V_h it is easy to verify the inclusion $V_h \subset V(G, n, m)$, $h \in (0, 1)$. Let us denote by u_h the solution of

Approximative Problem:

$$(((u_h, v_h))) = (f, v_h) \quad \text{for each} \quad v_h \in V_h . \tag{1.10}$$

Since there exists a unique solution of Trial Problem, there exists also a unique solution of Approximative Problem (1.10).

Now, we shall be interested in the question of convergence

$$|||u_h - \mathrm{u}||| \to 0 \quad \text{for} \quad h \to 0 \tag{1.11}$$

if we have no information about apriori regularity of u at our disposal. We only know that $u \in V(G, n, m)$. Suppose that the division G_h is regular (see [3]). Then the following assertion holds:

Let S be a dense subset in $V(G, n, m)$ and let r_h be the mapping from S to V_h such that

$$|||v - r_h v||| \to 0 , \quad h \to 0 , \quad v \in S . \tag{1.12}$$

Then (1.11) holds (see [11]).

Now, let $r_h : D(G) \to V_h$ be such that $r_h v$ is an Hermite interpolation of $v \in D(G)$. The functions from $D(G)$ belong to the space $H^4(G)$. According to [1] we have

$$|||v - r_h v||| \leqq Ch^2 \|v\|_{4,G}$$

where C does not depend on h. So, if $S \equiv D(G)$, then (1.12) holds. This means that when we prove the density of $D(G)$ in $V(G, n, m)$ with respect to the norm (1.4) and apply the above assertions we get the convergence needed.

The density of $D(G)$ in $V(G, n, m)$ with respect to the norm (1.4) is proved in Theorem 2.1 in the following chapter.

2. DENSITY THEOREM

The goal of this chapter is to prove the density of the space $D(G)$ in $V(G, n, m)$ with respect to the norm $|||.|||$ provided both n, m are finite integers: Without restriction of generality let G be the square $(-1, 1) \times (-1, 1) \subset \mathbb{R}^2$.

First, let

$$I_1 = \{(x, y) \in \mathbb{R}_2;\ x = 0,\ y \in (-1, 1)\} \equiv I\,, \quad J = \emptyset$$

so that $V(G, n, m) = V(G, 1, 0)$.

Lemma 1. *The space $D(G)$ is dense in $V(G, 1, 0)$ with respect to the norm of $V(G, 1, 0)$.*

Proof. Let $u \in V(G, 1, 0)$ be an arbitrary function. The existence of functions from $D(G)$ approximating u with an arbitrary precision with respect to the norm of $V(G, 1, 0)$ will be proved by turns.

Because $u \in H_0^2(I_1)$ there exist functions $\psi_i \in D(I_1)$, $\psi_i \to u$ on $H_0^2(I_1)$ for $i \to \infty$. Define $\varphi(x) \in D((-1, 1))$ with the following properties:

$$\varphi(x) = 1 \quad \text{for} \quad x \in \langle -1 + a, 1 - a \rangle\,, \quad a \in (0, 1)\,,$$

$$0 \leqq \varphi(x) \leqq 1\,.$$

Put $\Phi(x, y) = \varphi(x)$ for $(x, y) \in G$. *Furthermore, let*

$$\bar{\Psi}_i(x, y) = \psi_i(y)\,,$$
$$\bar{U}(x, y) = u(0, y)\,,$$
$$\Psi_i(x, y) = \bar{\Psi}_i(x, y)\,.\,\Phi(x, y)\,,$$
$$U(x, y) = \bar{U}(x, y)\,.\,\Phi(x, y) \quad i = 1, \ldots, ..$$

It is easy to verify that the functions Ψ_i from the above definition belong to $D(G)$ and $\Psi_i \to U$ on $V(G, 1, 0)$. Really, in the neighbourhood of the segment I_1 the first derivatives with respect to x of both $\bar{U}$ and $\bar{\Psi}_i$ vanish. Furthermore, it is clear that

$$\bar{U}_{xx} = \bar{U}_{xy} = \bar{\Psi}_{i,xx} = \bar{\Psi}_{i,xy} = 0$$

and according to the definition of $\bar{U}$ and $\bar{\Psi}_i$ and Fubini's theorem we obtain

$$\bar{\Psi}_{i,yy} \to \bar{U}_{yy} \quad \textit{for} \quad i \to \infty \quad \textit{on} \quad L_2(G)\,.$$

In virtue of the last two formulas it holds $\bar{\Psi}_i \to \bar{U}$ on $H_0^2(G)$. This together with Friedrich's inequality (see [6]) yields $\bar{\Psi}_i \to U$ on $H_0^2(G)$. Moreover, $\Psi_i = \psi_i$ and $U = u$ on I_1 so that we get $\Psi_i \to U$ on $V(G, 1, 0)$.

The trace u_x on I_1 belongs to $H_0^2(I_1)$ according to the definition of $V(G, 1, 0)$. There exist functions $\chi_i \in D(I_1)$, $\chi_i \to u_x$ on $H_0^2(I)$ Put

$$\bar{\chi}_i(x, y) = \chi_i(y)\,.\,x\,,$$
$$V(x, y) = u_x(0, y)\,.\,x\,,$$
$$\chi_i(x, y) = \bar{\chi}_i(x, y)\,.\,\Phi(x, y)$$
$$V(x, y) = V(x, y)\,.\,\Phi(x, y) \quad i = 1, \ldots, \ldots$$

The functions χ_i, V satisfy $\chi_i \in D(G)$ and $V \in H_0^2(G)$. Following the definition of the functions $\bar{V}$ and $\bar{\chi}_i$ we get

$$\bar{V}_{xx} = \bar{\chi}_{i,xx} = 0 .$$

Fubini's theorem and Schwarz's inequality give

$$\bar{\chi}_{i,xy} \to \bar{V}_{xy} ,$$

$$\bar{\chi}_{i,yy} \to \bar{V}_{yy} \quad on \quad L_2(G) , \quad i \to \infty .$$

The above discussion implies $\bar{\chi}_i \to \bar{V}$ on $H_0^2(G)$. Hence by virtue of Friedrich's inequality $\chi_i \to V$ on $H_0^2(G)$. As $\chi_i = V = 0$ on I_1 and $\chi_{i,x} = \chi_i$ and $V_x = u_x$ on I_1 we have also $\chi_i \to V$ on $V(G, 1, 0)$.

Now, we put

$$u = U + V + Z$$

so that Z is a function from $H_0^2(G)$ satisfying the identity

$$Z = Z_x = 0 \quad on \quad I_1 ,$$

which implies $Z \in H_0^2(G_1) \cap H_0^2(G_2)$ where $G_1 = (-1, 0) \times (-1, 1)$ and $G_2 = (0, 1) \times (-1, 1)$. According to the definition of spaces $H_0^2(G_i)$, $i = 1, 2$ there exist functions ξ_i such that ξ_i restricted to G_1 belong to $D(G_1)$ and ξ_i restricted to G_2 belong to $D(G_2)$, $\xi_i \to Z$ on $H_0^2(G)$ and so also on $V(G, 1, 0)$.

Setting

$$\zeta_i = \xi_i + \chi_i + \Psi_i \quad (\zeta_i \in D(G))$$

one obtains

$$|||u - \zeta_i||| \leqq |||U - \Psi_i||| + |||V - \chi_i||| + |||Z - \xi_i|||$$

according to the triangle inequality. The right hand side of the last inequality converges to zero for $i \to \infty$.

Now, we prove a similar theorem for the case $J = J_1$, $I = I_1$, $G = (-1, 1) \times (-1, 1)$, $I_1 = \{(x, y); x = 0, y \in (-1, 1)\}$, $J_1 = \{(x, y). x \in (-1, 1), y = 0\}$. It is necessary to point out the symmetry of the segments I_1 and J_1, which will be very important in the following. Trial Problem will be solved on $V(G, 1, 1)$. If S is a subset of $V(G, 1, 1)$ than $\bar{S}$ denotes the closure of S on $V(G, 1, 1)$.

Lemma 2. *The space $D(G)$ is dense in $V(G, 1, 1)$ with respect to the norm of $V(G, 1, 1)$.*

Proof. In order to continue systematically we take into account the following circumstance: Each function from $V(G, 1, 1)$ may be divided into four functions

$$w = w^s + w^x + w^y + w^a \quad w \in V(G, 1, 1)$$

where

$$w^s(x, y) = \tfrac{1}{4}(w(x, y) + w(x, -y) + w(-x, y) + w(-x, -y)),$$
$$w^x(x, y) = \tfrac{1}{4}(w(x, y) - w(x, -y) + w(-x, y) - w(-x, -y)),$$
$$w^y(x, y) = \tfrac{1}{4}(w(x, y) + w(x, -y) - w(-x, y) - w(-x, -y)),$$
$$w^a(x, y) = \tfrac{1}{4}(w(x, y) - w(x, -y) - w(-x, y) + w(-x, -y)),$$
$$(x, y) \in G.$$

It is easy to see that the following conditions are satisfied:

$$w^s(-x, y) = w^s(x, y), \quad w^s(x, -y) = w^s(x, y), \tag{2.1}$$
$$w^x(-x, y) = w^x(x, y), \quad w^x(x, -y) = -w^x(x, y), \tag{2.2}$$
$$w^y(-x, y) = -w^y(x, y), \quad w^y(x, -y) = w^y(x, y), \tag{2.3}$$
$$w^a(-x, y) = -w^a(x, y), \quad w^a(x, -y) = -w^a(x, y). \quad (x, y) \in G. \tag{2.4}$$

We introduce auxiliary spaces:

$$V^s = \{u \in V(G, 1, 1)\ u \textit{ fulfils } (2.1)\},$$
$$D^s = \{v \in D(G); v \textit{ fulfils } (2.1)\},$$
$$V^x = \{u \in V(G, 1, 1); u \textit{ fulfils } (2.2)\},$$
$$D^x = \{v \in D(G); v \textit{ fulfils } (2.2)\},$$
$$V^y = \{u \in V(G, 1, 1); u \textit{ fulfils } (2.3)\},$$
$$D^y = \{v \in D(G); v \textit{ fulfils } (2.3)\},$$
$$V^a = \{u \in V(G, 1, 1); u \textit{ fulfils } (2.4)\},$$
$$D^a = \{v \in D(G); v \textit{ fulfils } (2.4)\}.$$

Now, we have to prove

$$\bar{D}^s = V^s, \tag{2.5}$$
$$\bar{D}^x = V^x, \tag{2.6}$$
$$\bar{D}^y = V^y, \tag{2.7}$$
$$\bar{D}^a = V^a. \tag{2.8}$$

Suppose that (2.5) through (2.8) hold. Then for each $w \in V(G, 1, 1)$ the above division exists. According to (2.5) through (2.8) there exist functions ξ_i^s, ξ_i^x, ξ_i^y, ξ_i^a such that

$$\xi_i^s \to w^s, \quad \xi_i^x \to w^x, \quad \xi_i^y \to w^y, \quad \xi_i^a \to w^a,$$
$$i \to \infty \quad \textit{on} \quad V(G, 1, 1).$$

Put

$$\xi^i = \xi_i^s + \xi_i^x + \xi_i^y + \xi_i^a \quad (\xi^i \in D(G)) .$$

The triangle inequality gives

$$|||w - \xi^i||| \leqq |||w^s - \xi_i^s||| + |||w^x - \xi_i^x||| + |||w^y - \xi_i^y||| + \\ + |||w^a - \xi_i^a||| \to 0 , \quad i \to \infty .$$

The last formula proves the lemma.

It remains to prove the density assertions (2.5) through (2.8).

Proof of assertion (2.5): Let $u^s \in V^s$ be arbitrary. According to the definition this function is even with respect to both of coordinates (see (2.1)). Let us define some auxiliary functions:

$\varphi \in D^s : 0 \leqq \varphi(x, y) \leqq 1 , \quad \varphi(x, y) = 1 \quad \text{for} \quad x^2 + y^2 \leqq \frac{1}{4} ,$

$\varphi^x(x) = 1 , \quad x \in \langle -1 + a, 1 - a \rangle ,$

$a \in (0, 1)$

$\varphi^y(y) = 1 , \quad y \in \langle -1 + a, 1 - a \rangle ,$

$0 \leqq \varphi^x(x), \varphi^y(y) \leqq 1$, $\varphi^x, \varphi^y \in D((-1, 1))$ are even functions with respect to their arguments.

In virtue of the embedding theorem $(u^s \in C(\bar{G}))$ we can introduce on I_1,

$$u_I(y) = u^s(0, y) - u^s(0, 0) \,.\, \varphi(0, y)$$

and on J_1,

$$u_J(x) = u^s(x, 0) - u^s(0, 0) \,.\, \varphi(x, 0) .$$

It is easy to see that $u_I \in H_0^2(I_1)$ and $u_J \in H_0^2(J_1)$ and, moreover, $u_I(0) = u_J(0) = 0$. As u_I and u_J are even functions and their first derivatives are continuous on I and J, respectively (for embedding theorems see [4]), $u_{I,y} = u_{J,x} = 0$ holds at the point $(0, 0)$, so that $u_I \in H_0^2(I_{11}) \cap H_0^2(I_{12})$, $u_J \in H_0^2(J_{11}) \cap H_0^2(J_{12})$, where $I_{11} = \{(x, y);\ x = 0,\ y \in (0, 1)\}$, $I_{12} = \{(x, y);\ x = 0,\ y \in (-1, 0)\}$, $J_{11} = \{(x, y);\ x \in (-1, 0),\ y = 0\}$, $J_{12} = \{(x, y);\ x \in (0, 1),\ y = 0\}$. By the definition of spaces H_0^2 there exist functions χ_I^i and χ_J^i belonging to $D(I_1)$, $D(J_1)$, respectively, possessing the property $\chi_J^{i\prime}(0) = \chi_I^{i\prime}(0) = 0$, $\chi_J^{i\prime} \to u_I$, $\chi_J^i \to u_J$ for $i \to \infty$ on $H_0^2(I_1)$ and $H_0^2(J_1)$, respectively. Moreover, X_I^i and X_J^i are symmetric functions of their arguments.

We set

$$\begin{aligned} \Phi^x(x, y) &= \varphi^x(x) , \\ \Phi^y(x, y) &= \varphi^y(y) , \\ \bar{X}_I^i(x, y) &= \chi_I^i(y) , \\ X_I^i(x, y) &= \bar{X}_i^I(x, y) \,.\, \Phi^x(x, y) , \end{aligned}$$

$$
\begin{aligned}
\bar{X}^i_J(x, y) &= \chi^i_J(x)\,,\\
\bar{U}_I(x, y) &= u_I(y)\,,\\
\bar{U}_J(x, y) &= u_J(x)\,,\\
X^i_J(x, y) &= \bar{X}^I_i(x, y)\,.\,\Phi^y(x, y)\,,\\
U_I(x, y) &= \bar{U}_I(x, y)\,.\,\Phi^x(x, y)\,,\\
U_J(x, y) &= \bar{U}_J(x, y)\,.\,\Phi^y(x, y)
\end{aligned}
$$

and X^i_I, $X^i_J \in D^s$. We show that $X^i_I \to U_I$, $X^i_J \to U_J$ on $V(G, 1, 1)$. The following identities hold:

$$
\begin{aligned}
&\bar{U}_{I,xx} = \bar{U}_{I,xy} = \bar{X}^i_{I,xx} = \bar{X}^i_{I,xy} = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{on } G\,.\\
&\bar{U}_{J,yy} = \bar{U}_{J,xy} = \bar{X}^i_{J,yy} = \bar{X}^i_{J,xy} = 0
\end{aligned}
$$

From Fubini's theorem one obtains:

$$
\bar{X}^i_{I,yy} \to \bar{U}_{I,yy}\,, \quad \bar{X}^i_{J,xx} \to \bar{U}_{J,xx}\,, \quad i \to \infty \quad \textit{on} \quad L_2(G)\,,
$$

which yields together

$$
\bar{X}^i_I \to \bar{U}_I\,, \quad \bar{X}^i_J \to \bar{U}_J \quad \textit{on} \quad H^2_0(G)\,.
$$

That and Friedrich's inequality imply

$$
X^i_I \to U_I\,, \quad X^i_J \to U_J \quad \textit{on} \quad H^2_0(G)\,.
$$

On I_1 the following relations hold: $X^i_I = \chi^i_I$ and $U_I = u_I$. Moreover, the first derivatives with respect to x of X^i_I and U_I vanish all over I_1. Similarly on J_1: $X^i_J = \chi^i_J$, $U_J = u_J$. The first derivatives with respect to y of X^i_J and U_J vanish over J_1. So we have also

$$
X^i_I \to U_I \text{ on } H^2_0(I_1) \text{ and } X^i_J \to U_J \text{ on } H^2_0(J_1) \quad \textit{for} \quad i \to \infty\,,
$$

and together

$$
X^i_I \to U_I \text{ and } X^i_J \to U_J \text{ on } V(G, 1, 1) \quad \textit{for} \quad i \to \infty\,.
$$

$$
\begin{aligned}
U(x, y) &= u^s(0, 0)\,.\,\varphi(x, y) + U_I(x, y) + U_J(x, y)\,,\\
X_i(x, y) &= u^s(0, 0)\,.\,\varphi(x, y) + X^i_I(x, y) + X^i_J(x, y)\,,
\end{aligned}
$$

one gets $X_i \in D^s$ and $U \in V^s$. From Schwarz's inequality we conclude

$$
X_i \to U \quad \text{on} \quad V(G, 1, 1)\,. \tag{2.9}
$$

Moreover, the function $U(x, y)$ is defined in such manner that $U = u^s$ along I_1. Actually, we have

$$
\begin{aligned}
U(0, y) &= u^s(0, 0)\,.\,\varphi(0, y) + U_I(0, y) + U_J(0, y) =\\
&= u^s(0, 0)\,.\,\varphi(0, y) + u_I(y) = u^s(0, y)
\end{aligned}
$$

because $\varphi(0, 0) = 1$, $u_J(0) = 0$ which gives $U_J(0, y) = 0$.

Similarly, $U(x, 0) = u^s(x, 0)$ so that $U = u^s$ along J_1.

From the symmetry of the functions defined above we can easily deduce that the first derivative with respect to x is equal to zero along I_1 and the first derivative with respect to y is equal to zero along J_1, in both cases almost everywhere.

Now, divide u^s in the following manner

$$u^s = U + Z$$

where Z is a function belonging to V^s which has the restrictions on I_1 and J_1 as well as the traces of its first derivative with respect to x on I_1 and its first derivative with respect to y on J_1 equal to zero. We can, then, deduce, that $Z \in H_0^2(G_{11}) \cap H_0^2(G_{12}) \cap \cap H_0^2(G_{21}) \cap H_0^2(G_{22})$, where $G_{11} = (-1, 0) \times (0, 1)$, $G_{12} = (0, 1) \times (0, 1)$, $G_{21} = = (-1, 0) \times (-1, 0)$ and $G_{22} = (0, 1) \times (-1, 0)$. From the symmetry of the function Z and from the definition of spaces H_0^2 we obtain existence of functions from D^s, say ε_i, such that

$$\varepsilon_i \to Z \quad \text{on} \quad V(G, 1, 1) . \tag{2.10}$$

The above decomposition of u^s, relations (2.9) and (2.10) together with the triangle inequality complete the proof of (2.5).

Proof of assertion (2.6): Each function from V^x has vanishing restriction on J_1. Moreover, the trace of the first derivative with respect to x of this function on I_1 is equal to zero a. e. The trace of the first derivative with respect to y is continuous with respect to $I_1 \cap J_1$ (for the proof of this assertion see [3]). Let $u^x \in V^x$ be arbitrary. The above discussion makes the following construction possible: Following the definition, the trace of u_y^x on J_1 belongs to $H_0^2(J_1)$ and moreover, it is symmetric (even) with respect to the x-coordinate, the function

$$u_J(x) = u_y^x(x) - u_y^x(0, 0) \cdot \varphi^x(x)$$

also belongs to $H_0^2(J_1)$, $u_J(0) = u_{J,x}(0) = 0$. Then there exist functions $\omega^i \in D(J_1)$, $\omega^i(0) = \omega_x^i(0) = 0$ and the functions may be chosen in such a way that ω^i are symmetric for all i and $\omega^i \to u_J$ on $H_0^2(J_1)$ for $i \to \infty$ (see [5]). Now, put

$$\begin{aligned}
\bar{\Omega}^i(x, y) &= \omega^i(x) \cdot y , \\
\bar{V}(x, y) &= u_J(x) \cdot y , \\
\Omega^i(x, y) &= \bar{\Omega}^i(x, y) \cdot \varphi^y(y) , \\
V(x, y) &= \bar{V}(x, y) \cdot \varphi^y(y) , \\
(x, y) &\in G .
\end{aligned}$$

The functions Ω^i and V satisfy $\Omega^i \in D^x$ and $V \in V^x$. According to the definition of the functions V and Ω^i we have

$$\bar{V}_{yy} = \bar{\Omega}^i_{yy} = 0$$

and from Fubini's theorem we conclude

$$\bar{\Omega}^i_{xy} \to \bar{V}_{xy}\,, \quad \bar{\Omega}^i_{xx} \to \bar{V}_{xx} \quad on \quad L_2(G)\,, \quad i \to \infty$$

and so $\bar{\Omega}^i \to \bar{V}$ in the norm (1.4). From Friedrich's inequality we have also $\Omega^i \to V$ on $H^2_0(G)$. Further, $V(0, y) = \Omega^i(0, y) = V(x, 0) = \Omega^i(x, 0) = 0$, $V^x(0, y) = \Omega^i_x(0, y) = 0$, $V_y(x, 0) = u_J(x)$, $\Omega^i_y(x, 0) = \omega^i(x)$. The convergence is guaranteed also in $V(G, 1, 1)$.

Put

$$U(x, y) = V(x, y) + u^x_y(0, 0) \,.\, \varphi^x(x) \,.\, \varphi^y(y) \,.\, y\,,$$

$$\tau^i(x, y) = \Omega^i(x, y) + u^x_y(0, 0) \,.\, \varphi^x(x) \,.\, \varphi^y(y) \,.\, y$$

and we have $U \in V^x$, $\tau^i \in D^x$, $U_y(x, 0) = u^x_y(x)$ and $U(x, 0) = 0$ and

$$\tau^i \to U \quad on \quad V(G, 1, 1)\,. \tag{2.11}$$

Define the function Z in the following way:

$$u^x = U + Z$$

so that $Z \in V^x \cap V(G_1, 1, 0) \cap V(G_2, 1, 0)$, where $G_1 = (-1, 1) \times (0, 1)$ and $G_2 = (-1, 1) \times (-1, 0)$. We may use Lemma 1 for the subregion G_1 and then for G_2 and because of the symmetry of the problem we have the existence of functions $\sigma^i \in D^x$,

$$\sigma^i \to Z \quad on \quad V(G, 1, 1)\,, \quad i \to \infty\,. \tag{2.12}$$

The decomposition of u^x, approximative relations (2.11) and (2.12) and the triangle inequality prove the assertion (2.6).

Proof of assertion (2.7) can be carried out in a similar manner as the proof of (2.6).

Proof of assertion (2.8): Let u^a be an arbitrary function from V^a. According to the definition this function is odd with respect to both coordinates. The restrictions of u^a to I_1 and J_1 are equal to zero. The continuity of u^a_x and u^a_y with respect to $I_1 \cup J_1$ (see [5]) implies $u^a_x(0, 0) = u^a_y(0, 0) = 0$. Moreover, $u^a_x \in H^2_0(I_1)$ is an even function on I_1 and so u^a_{xy} is odd. From the embedding theorem $u^a_{xy}(0) = 0$ on I_1, so that $u^a_x \in H^2_0(I_{11}) \cap H^2_0(I_{12})$, where $I_{11} = \{(x, y); x = 0, y \in (0, 1)\}$, $I_{12} = \{(x, y\,; x = 0, y \in (-1, 0)\}$ there exist functions $\alpha^i \in D(I_1)$, α^i are even on I_1, $\alpha^i(0) = \alpha^i_y(0) = 0$ and $\alpha^i \to u^a_x$ on $H^2_0(I_1)$ for $i \to \infty$. In a similar way we obtain a sequence of functions $\beta^i \in D(J_1)$, β^i are even on J_1, $\beta^i(0) = \beta^i_x(0) = 0$ and $\beta^i \to u^a_y$ on $H^2_0(J_1)$. We introduce auxiliary functions

$$\gamma^i(x, y) = \alpha^i(y) \,.\, \varphi^x(x) \,.\, x + \beta^i(x) \,.\, \varphi^y(y) \,.\, y\,,$$

$$U(x, y) = u^a_x(0, y) \,.\, \varphi^x(x) \,.\, x + u^a_y(x, 0) \,.\, \varphi^y(y) \,.\, y$$

and, with respect to the definition, $U \in V^a$ and $\gamma^i \in D^a$. Now, we set

$$\begin{aligned} A^i(x, y) &= \alpha^i(y) \,.\, \varphi^x(x) \,.\, x\,, \\ B^i(x, y) &= \beta^i(x) \,.\, \varphi^y(y) \,.\, y\,, \\ U^{ax}(x, y) &= u^a_x(0, y) \,.\, \varphi^x(x) \,.\, x\,, \\ U^{ay}(x, y) &= u^a_y(x, 0) \,.\, \varphi^y(y) \,.\, y\,. \end{aligned}$$

According to Fubini's theorem and Schwarz's inequality we obtain $A^i \to U^{ax}$ and $B^i \to U^{ay}$ in $H^2_0(G)$ for $i \to \infty$. The triangle inequality then gives $\gamma^i \to U$ in $H^2_0(G)$. Moreover, $U_x = u^a_x$ and $\gamma^i_x = \alpha^i$ on I_1, $U_y = u^a_y$ and $\gamma^i_y = \beta^i$ on J_1 and both the functions U and γ^i for arbitrary i have restrictions on I_1 and J_1 equal to zero so that

$$\gamma^i \to U \quad on \quad V(G, 1, 1) \tag{2.13}$$

and the function Z from the decomposition

$$u^a = U + Z \tag{2.14}$$

belongs to $V^a \cap H^2_0(G_{11}) \cap H^2_0(G_{12}) \cap H^2_0(G_{21}) \cap H^2_0(G_{22})$, where the sets G_{ij}, $i, j = 1, 2$ are defined in the proof of (2.5), and so from the definition of the spaces H^2_0 and from the symmetry of the problem we have the existence of functions $\delta^i \in D_b$ which approximate Z with an arbitrary accuracy. From that and from (2.13), (2.14) and the triangle inequality we obtain the result desired. The proof of Lemma 2 is complete.

Using the partition of unity, Lemma 1 and Lemma 2, we can easily prove the following.

Theorem 2.1. *$D(C)$ is dense in $V(G, n, m)$ with respect to the $V(G, n, m)$-norm.*

3. CONCLUSION

This paper deals with a problem of one type for the plate with ribs and may be considered a continuation of [2] and [3] where a certain simpler model of the plate with ribs was studied. For numerical solution, the finite element method has been proposed. With respect to [2] and [3], it suffices to prove one density theorem (here Theorem 2.1) for the proof of convergence of the finite element approximations to the solution of Trial Problem. Since it is nothing known about apriori regularity of the solution of Trial Problem we cannot say anything about the rate of convergence mentioned.

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Souhrn

O ŘEŠENÍ JEDNOHO PROBLÉMU DESKY SE ŽEBRY

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V předloženém článku je studována konvergence aproximací ve smyslu metody konečných prvků k řešení problému desky se žebry, která jsou tuhá vzhledem k ohybu i v kroucení. Tento případ tuhosti v kroucení odpovídá Vlasovovu kroucení uzavřených profilů. Vzhledem k rozboru, provedenému např. v práci [3] stačí dokázat jednu větu o hustotě (zde Věta 2.1). Jelikož nevíme nic o apriorní regularitě řešení výchozího problému, nepodařilo se dokázat rychlost konvergence, ale pouze skutečnost, že aproximativní řešení konverguje k přesnému řešení úlohy v normě prostoru těchto přesných řešení.

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