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CONTACT BETWEEN ELASTIC BODIES — I. CONTINUOUS PROBLEMS

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INTRODUCTION

In some technical and physical regions a problem arises to determine the displacement and stress fields in two solid bodies which are in a mutual contact. The classical analysis of this problem, started by Hertz [1] in 1896 was limited to simple geometries. The age of high – speed computers brought qualitative change also into the analysis of the contact problem. On the basis of a suitable discretization – by means of finite differences or finite elements – the problem can be solved approximately even for complex geometrical situations and boundary conditions.

Many contributions are available in the literature dealing with the numerical solution of the plane contact problem. Linear finite elements on the triangulations have been applied most often and various discrete formulations proposed (see e.g. [2], [3], [4], [5]). The authors, however, do not present the formulation of the continuous problem, but start immediately with a discretized problem. As a consequence, errors can neither be defined nor analyzed.

It is the aim of the Part I of our paper to formulate the continuous contact problems and to discuss the existence and uniqueness of (variational) solutions. In Part II we present a displacement finite element model for solving the contact problems, error estimates in case of regular solution, convergence proof for the case of irregular solution and some algorithms. In Part III a dual variational approach will be discussed (a generalization of the Castigliano principle) for both the continuous problem and the finite element discretization.

Throughout the paper we restrict ourselves to the case of zero friction. (The problem involving friction will be treated in a following paper by J. Nečas.)

1. FORMULATIONS OF THE CONTACT PROBLEMS

Let us consider several kinds of contact between two elastic bodies. We start with problems without friction, which are much easier to deal with. First we present a set of "local" conditions – equations and boundary conditions, defining a "classical" solution. Then we define "global" – variational – solution and prove that the classical and variational solutions are equivalent in a certain sense.

1.1 Classical formulations

Throughout the paper, we assume for simplicity:

- plane problem,
- bounded bodies,
- small deformations,
- zero friction,
- zero initial strain and stress fields,
- a constant temperature field,
- linear generalized Hooke's law for an anisotropic, nonhomogeneous material.

Since the difference in the formulation of the contact problems and of the classical boundary value problems is only the boundary condition on the contact zone, the theory which follows, could be extended to other deformable bodies and an influence of a given temperature field or of an initial strain or stress could also be involved.

Let the two elastic bodies occupy the bounded regions $\Omega', \Omega'' \subset R^2$ with Lipschitz boundaries. In the following, one or two primes denotes that the quantity is referred to the body Ω' and Ω'', respectively.

Let $\boldsymbol{x} = (x_1, x_2)$ be Cartesian coordinates. We seek the displacement vector field $\boldsymbol{u} = (u_1, u_2)$ over $\Omega' \cup \Omega''$, i.e. $\boldsymbol{u}' = (u_1', u_2')$ on Ω' and $\boldsymbol{u}'' = (u_1'', u_2'')$ on Ω'' and the associated strain tensor field

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i = 1, 2 . \tag{1.1}$$

The stress tensor is related to the strain tensor by means of the following generalized Hooke's law

$$\tau_{ij} = c_{ijkm} e_{km} , \quad i, j = 1, 2 , \tag{1.2}$$

where a repeated index implies summation over 1,2. Assume that c_{ijkm} are bounded and measurable in $\Omega' \cup \Omega''$,

$$c_{ijkm} = c_{jikm} = c_{kmij} , \tag{1.3}$$

and a positive constant c_0 exists such that

$$c_{ijkm} e_{ij} e_{km} \geqq c_0 e_{ij} e_{ij} \tag{1.4}$$

holds for any symmetric e_{ij}, almost everywhere in $\Omega' \cup \Omega''$.

The stress tensor satisfies the following equations of equilibrium in $\Omega' \cup \Omega''$:

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \,, \quad i = 1, 2 \,, \tag{1.5}$$

where F_i are components of a body force vector.

Let the body Ω' be fixed on a part Γ_u of its boundary:

$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_u \subset \partial\Omega' \,. \tag{1.6}$$

Let the tractions be prescribed on some parts of the boundaries, i.e.,

$$\tau_{ij} n_j = P_i \,, \quad i = 1, 2 \,, \quad \text{on} \quad \Gamma'_\tau \in \partial\Omega' \quad \text{and on} \quad \Gamma''_\tau \subset \partial\Omega'' \,, \tag{1.7}$$

where $\boldsymbol{n}$ denotes the outward unit normal to $\partial\Omega'$ or to $\partial\Omega''$, respectively and P_i are given components of the surface traction.

Assume that a part Γ_0 of the boundary $\partial\Omega''$ is subject to "classical" bilateral contact conditions, i.e.,

$$u_n = 0 \,, \quad T_t = 0 \quad \text{on} \quad \Gamma_0 \subset \partial\Omega'' \,, \tag{1.8}$$

where

$$u_n = u_i n_i \,, \quad T_t = \tau_{ij} n_j t_i \,, \quad \mathbf{t} = (t_1, t_2) = (-n_2, n_1)$$

are the normal displacement and the tangential stress vector components, respectively. Such conditions may occur e.g. on the axis of symmetry, enabling us to solve one half of the whole elastic system only.

As the remaining parts of the boundaries $\partial\Omega' \cup \partial\Omega''$ are concerned, we consider that a possible contact may occur there and distinguish two classes of contact problems, as follows.

1.1.1 Bounded contact zone

First let us consider the case, when the contact zone cannot enlarge during the deformation process. Such an assertion is determined by the geometrical shape of the two bodies in a neighbourhood of the possible contact zone – see e.g. Fig. 1.

Hence we may define the contact zone

$$\Gamma_K = \partial\Omega' \cap \partial\Omega''$$

and we have the following decompositions:

$$\partial\Omega' = \bar{\Gamma}_u \cup \bar{\Gamma}'_\tau \cup \Gamma_K \,, \quad \partial\Omega'' = \bar{\Gamma}''_\tau \cup \bar{\Gamma}_0 \cup \Gamma_K \,, \tag{1.9}$$

where Γ_u, Γ'_τ, Γ''_τ and Γ_0 are mutually disjoint open parts of the boundaries;

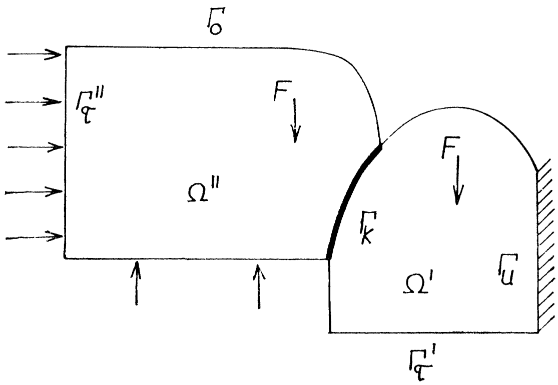

Fig. 1.

assume that Γ_u and Γ_K have positive measure. The remaining parts may be either of positive measure or empty sets.

We say that a *unilateral bounded contact* occurs on Γ_K if

$$u_n' + u_n'' \leqq 0 \tag{1.10}$$

holds on Γ_K, where $u_n' = u_i'n_i'$, $u_n'' = u_i''n_i''$.

Let us describe a derivation of the condition (1.10). Suppose that before the deformation the bodies Ω' and Ω'' were in contact along the whole arc Γ_K (see Fig. 2). Let us put the x_1 – axis into the normal $\boldsymbol{n}''$ and the x_2 – axis into the tangent $\mathbf{t}''$

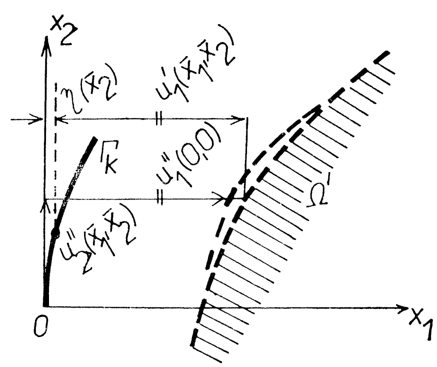

Fig.2.

at a point $O \in \Gamma_K$. During the deformation process the points $O' \in \partial\Omega'$ and $O'' \in \partial\Omega''$ will displace by a different way, in general, but always the *body Ω'' cannot penetrate into the body Ω'*. From this condition it follows that

$$u_1''(0, 0) \leqq u_1'(\bar{x}_1, \bar{x}_2) + \eta(\bar{x}_2), \tag{1.11}$$

where η is the function, describing the curve Γ_K and $\mathbf{x} = (\bar{x}_1, \bar{x}_2)$ is a point of Γ_K such that

$$u_2'(\bar{x}_1, \bar{x}_2) + \bar{x}_2 = u_2''(0, 0) .$$

The point $\bar{x}$ is unknown, of course, and (1.11) is too complicated condition. Therefore we simplify it by means of the following "natural" hypotheses:

$$\eta(\bar{x}_2) \doteq 0 , \tag{1.12}$$

$$u_1'(\bar{x}_1, \bar{x}_2) \doteq u_1'(0, 0) . \tag{1.13}$$

Obviously, (1.12) is true for a "flat" arc Γ_K; (1.13) holds if e.g. the mutual "shifts" $|u_2' - u_2''|$ and the derivatives $|\partial u_1'/\partial x_2|$ are small in a neighbourhood of the origin.

Inserting (1.12), (1.13) and $u_1'' = u_n''$, $u_1' = -u_n'$ into (1.11) we obtain the condition (1.10) at a point $O \in \Gamma_K$.

Next let us consider the contact forces. By virtue of the law of action and reaction we have

$$T_n' = T_n'' , \quad T_t' = T_t'' \quad \text{on} \quad \Gamma_K .$$

On the other hand, the tangential components vanish because of zero friction and the normal contact force cannot be tensile, i.e.

$$T_t' = T_t'' = 0 , \quad T_n' = T_n'' \leqq 0 .$$

Altogether, we define the boundary conditions on Γ_K as follows:

$$u_n' + u_n'' \leqq 0 , \quad T_n' = T_n'' \leqq 0 , \tag{1.14}$$

$$(u_n' + u_n'')\, T_n' = 0 , \tag{1.15}$$

$$T_t' = T_t'' = 0 . \tag{1.16}$$

Instead of (1.14), (1.15) we may write the following equivalent system: (1.14) and

$$u_n' + u_n'' < 0 \Rightarrow T_n' = 0 , \tag{1.16'}$$

the meaning of which is that at points without contact no contact force may occur.

Remark 1.1 Let one of the two bodies become rigid. Then the system (1.14)–(1.16) reduces to the well-known system of Signorini's conditions – cf. [8], [9], [10]. A particular geometrical case of the contact problem has been deduced in [7].

Definition 1.1 *A function* $\mathbf{u}$ *will be called a classical solution of the problem* $\mathscr{P}_1$ *with a bounded contact zone, if* $\mathbf{u}$ *satisfies the equations* (1.1), (1.2), (1.5) *in* $\Omega' \cup \Omega''$ *and the boundary conditions* (1.6) *on* Γ_u, (1.7) *on* $\Gamma_\tau' \cup \Gamma_\tau''$, (1.8) *on* Γ_0 *and* (1.14), (1.15), (1.16) *on* Γ_K.

1.1.2 Enlarging contact zone

In some important cases the contact zone can enlarge during the deformation process. Such a situation occurs if the two bodies Ω' and Ω'' have smooth boundaries in a neighbourhood of $\Gamma_K = \partial\Omega' \cap \partial\Omega''$. Then we must change the definition of the contact conditions.

Let us consider the case of Fig. 3. We place a coordinate system (ξ, η) in such a way that the ξ-axis coincides with the direction of $\boldsymbol{n}''$ and η-axis with the common

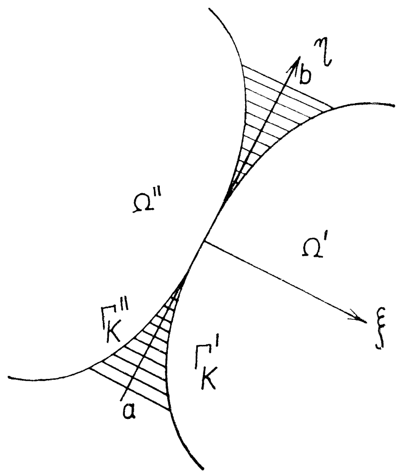

Fig. 3.

tangent of $\partial\Omega'$ and $\partial\Omega''$ at a "central" point $P \in \partial\Omega' \cap \partial\Omega''$. The figure corresponds with the situation before the deformation. The parts of $\partial\Omega'$ and $\partial\Omega''$ which come into a contact during the deformation process, can be estimated as follows:

$$\Gamma'_K = \{(\xi, \eta) \mid a \leqq \eta \leqq b,\ \xi = f'(\eta)\}$$
$$\Gamma''_K = \{(\xi, \eta) \mid a \leqq \eta \leqq b,\ \xi = f''(\eta)\},$$

where f' and f'' are continuous on $\langle a, b\rangle$. (The interval $\langle a, b\rangle$ has to be chosen a priori.)

Arguing similarly as in the derivation of the condition (1.10), we are led to the following condition:

$$u''_\xi(\eta) - u'_\xi(\eta) \leqq \varepsilon(\eta) \quad \forall\eta \in \langle a, b\rangle, \tag{1.17}$$

where

$$\varepsilon(\eta) = f'(\eta) - f''(\eta)$$

is the distance of the two boundaries before the deformation and u'_ξ, u''_ξ are projections of the displacement vectors into the direction of (positive) ξ-axis.

Using also the law of action and reaction, we come to the conditions

$$- T'_\xi (\cos \alpha')^{-1} = T''_\xi (\cos \alpha'')^{-1} \leqq 0 , \tag{1.18}$$

$$T'_\eta = T''_\eta = 0 , \tag{1.19}$$

$$T''_\xi (u''_\xi - u'_\xi - \varepsilon) = 0 , \tag{1.20}$$

which hold at all points of $\Gamma'_K \cup \Gamma''_K$ with the same coordinates $\eta \in \langle a, b \rangle$. Here

$$(\cos \alpha^M)^{-1} = [1 + (\mathrm{d}f^M/\mathrm{d}\eta)^2]^{1/2} , \quad M = ', '' ,$$

α' and α'' being the angle between η-axis and the tangent to Γ'_K and Γ''_K, respectively.

Instead of (1.17), (1.18), (1.20) we may write the following equivalent system: (1.17), (1.18) and

$$u''_\xi - u'_\xi < \varepsilon \Rightarrow T'_\xi = T''_\xi = 0 ,$$

(i.e. at points without contact no contact force may occur). The conditions (1.19) approximate the zero friction (after neglecting the projections $T'_\xi \sin \alpha'$ and $T''_\xi \sin \alpha''$).

Definition 1.2 *A function* $\mathbf{u}$ *will be called a classical solution of the problem* $\mathscr{P}_2$ *with an enlarging contact zone, if* $\mathbf{u}$ *satisfies the equations* (1.1), (1.2), (1.5) *in* $\Omega' \cup \Omega''$ *and the boundary conditions* (1.6) *on* Γ_u, (1.7) *on* $\Gamma'_\tau \cup \Gamma''_\tau$, (1.8) *on* Γ_0 *and* (1.17), (1.18), (1.19), (1.20) *on* $\Gamma'_K \cup \Gamma''_K$.

The following decompositions hold

$$\partial\Omega' = \bar{\Gamma}_u \cup \bar{\Gamma}'_\tau \cup \Gamma'_K , \quad \partial\Omega'' = \bar{\Gamma}_0 \cup \bar{\Gamma}''_\tau \cup \bar{\Gamma}''_K , \quad \Gamma^M_K \cap \operatorname{supp} \mathbf{P} = \emptyset .$$

1.2 Variational formulations

To both the problems $\mathscr{P}_1$ and $\mathscr{P}_2$ a variational formulation-principle of minimum potential energy — can be associated. Introduce the space of displacement functions with finite energy

$$\mathbf{W} = \{\mathbf{u} \mid \mathbf{u} = (\mathbf{u}', \mathbf{u}'') \in [H^1(\Omega')]^2 \times [H^1(\Omega'')]^2\} ,$$

where $H^1(\Omega')$ and $H^1(\Omega'')$ is the Sobolev space ($H^1 \equiv W^{1,2}$), and the space of virtual displacements

$$V = \{\mathbf{u} \in \mathbf{W} \mid , \ \mathbf{u}' = 0 \text{ on } \Gamma_u, \ u''_n = 0 \text{ on } \Gamma_0\} .$$

We use the functional of the potential energy

$$\mathscr{L}(\mathbf{v}) = \tfrac{1}{2} A(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) , \tag{1.21}$$

where

$$(1.22) \qquad A(\mathbf{u}, \mathbf{v}) = \int_{\Omega} c_{ijkm}\, e_{ij}(\mathbf{u})\, e_{km}(\mathbf{v})\, \mathrm{d}x\,, \quad \Omega = \Omega' \cup \Omega''\,,$$

$$(1.23) \qquad L(\mathbf{v}) = \int_{\Omega} F_i v_i \,\mathrm{d}\mathbf{x} + \int_{\Gamma_\tau' \cup \Gamma_\tau''} P_i v_i \,\mathrm{d}s\,.$$

Let us consider the problem $\mathscr{P}_1$ with a bounded contact zone. Define the set of admissible displacements

$$(1.24) \qquad K = \{\mathbf{v} \in V \mid v_n' + v_n'' \leqq 0 \text{ on } \Gamma_K\}\,.$$

Definition 1.3 *A function $\mathbf{u} \in K$ will be called a weak (variational) solution of the problem $\mathscr{P}_1$ with a bounded contact zone, if*

$$(1.25) \qquad \mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K\,.$$

Theorem 1.1 *Any classical solution of the problem $\mathscr{P}_1$ is a weak solution of $\mathscr{P}_1$. If a weak solution of the problem $\mathscr{P}_1$ is sufficiently smooth, it is a classical solution of $\mathscr{P}_1$, as well.*

Proof. 1. Let $\mathbf{u}$ be a classical solution. Then $\tau_{ij}(\mathbf{u}) = c_{ijkm}\, e_{km}(\mathbf{u})$ satisfies the equations (1.5). Multiplying (1.5) by a test function $w \in V$ and integrating over Ω' and Ω'', we obtain

$$0 = \int_{\Omega' \cup \Omega''} \left[-\tau_{ij}(\mathbf{u}) \frac{\partial w_i}{\partial x_j} + F_i w_i \right] \mathrm{d}\mathbf{x} + \int_{\partial\Omega' \cup \partial\Omega''} \tau_{ij}(\mathbf{u})\, n_j w_i \,\mathrm{d}s =$$

$$= -A(\mathbf{u}, \mathbf{w}) + \int_{\Omega' \cup \Omega''} F_i w_i \,\mathrm{d}\mathbf{x} + \int_{\Gamma_\tau' \cup \Gamma_\tau''} P_i w_i \,\mathrm{d}s \, +$$

$$+ \int_{\Gamma_0} [T_n(\mathbf{u})\, w_n + T_t(\mathbf{u})\, w_t] \,\mathrm{d}s \, +$$

$$+ \int_{\Gamma_K} [T_n'(\mathbf{u})\, w_n' + T_t'(\mathbf{u})\, w_t' + T_n''(\mathbf{u})\, w_n'' + T_t''(\mathbf{u})\, w_t''] \,\mathrm{d}s\,.$$

The integral over Γ_0 vanishes by virtue of $w_n = 0$, $T_t(\mathbf{u}) = 0$. Using also (1.14) and (1.16), we have

$$A(\mathbf{u}, \mathbf{w}) - L(\mathbf{w}) = \int_{\Gamma_K} T_n'(\mathbf{u})\, (w_n' + w_n'') \,\mathrm{d}s\,.$$

Let us put $\mathbf{w} = \mathbf{v} - \mathbf{u}$, $\mathbf{v} \in K$. At the points. where $u_n' + u_n'' < 0$, $T_n'(\mathbf{u})$ vanishes (see (1.16′)). At the points, where $u_n' + u_n'' = 0$,

$$w_n' + w_n'' = v_n' + v_n'' \leqq 0\,, \quad T_n'(\mathbf{u}) \leqq 0\,.$$

Altogether, the integral is non-negative and we obtain

$$A(\mathbf{u}, \mathbf{v} - \mathbf{u}) - L(\mathbf{v} - \mathbf{u}) \geqq 0 \quad \forall \mathbf{v} \in K . \tag{1.26}$$

On the other hand $\mathbf{u} \in K$ satisfies (1.25) if and only if (1.26) is true. This follows from the convexity of K and $\mathscr{L}$. Hence $\mathbf{u}$ is a weak solution of $\mathscr{P}_1$.

2. Let $\mathbf{u} \in K$ be sufficiently smooth weak solution. It satisfies (1.26). Integrating in (1.26) by parts, and denoting $\mathbf{v} - \mathbf{u} = \mathbf{w}$, we may write

$$0 \leqq A(\mathbf{u}, \mathbf{w}) - L(\mathbf{w}) = -\int_{\Omega' \cup \Omega''} \left(\frac{\partial \tau_{ij}(\mathbf{u})}{\partial x_j} + F_i \right) w_i \, \mathrm{d}\mathbf{x} - \tag{1.27}$$

$$- \int_{\Gamma_\tau' \cup \Gamma_\tau''} P_i w_i \, \mathrm{d}s + \int_{\partial\Omega' \cup \partial\Omega''} (T_n w_n + T_t w_t) \, \mathrm{d}s .$$

If we choose $\mathbf{v} = \mathbf{u} \pm \mathbf{w}$, where $w_i \in C_0^\infty(\Omega^M)$, $M = {}', {}''$, then $\mathbf{v} \in K$ and the equations of equilibrium (1.5) follow.

Let $\mathbf{v} = \mathbf{u} \pm \mathbf{w}$ and let the support of the traces of w_i on $\partial\Omega' \cup \partial\Omega''$ belong to $\Gamma_\tau' \cup \Gamma_\tau''$. Then we have $\mathbf{v} \in K$,

$$0 = \int_{\Gamma_\tau' \cup \Gamma_\tau''} (T_i - P_i) \, w_i \, \mathrm{d}s$$

and the boundary conditions (1.7) on $\Gamma_\tau' \cup \Gamma_\tau''$ follow. The conditions (1.6) and (1.8) for $\mathbf{u}$ and u_n, respectively, are satisfied by definition of $\mathbf{u} \in K$.

Let $\mathbf{v} = \mathbf{u} \pm \mathbf{w}$, where the support of the traces of w_i belongs to Γ_0 and $w_n = 0$ on Γ_0. Then $\mathbf{v} \in K$,

$$0 = \int_{\Gamma_0} T_t w_t \, \mathrm{d}s ,$$

and the second condition (1.8) follows.

Thus we obtained from (1.27):

$$0 \leqq \int_{\Gamma_K} (T_n' w_n' + T_t' w_t' + T_n'' w_n'' + T_t'' w_t'') \, \mathrm{d}s . \tag{1.28}$$

Let us choose w such that

$$w_n' = -w_n'' = \pm\psi , \quad w_t' = w_t'' = 0 \ \text{ on } \Gamma_K .$$

Then

$$0 = \int_{\Gamma_K} (T_n' - T_n'') \, \psi \, \mathrm{d}s \Rightarrow T_n' = T_n'' \ \text{ on } \Gamma_K . \tag{1.29}$$

Next let us choose

$$w_n' = w_n'' = 0 , \quad w_t'' = 0 , \quad w_t' = \pm\psi \ \text{ on } \Gamma_K .$$

Then

$$0 = \int_{\Gamma_K} T_t' \psi \, ds \Rightarrow T_t' = 0 \text{ on } \Gamma_K .$$

A parallel approach results in $T_t'' = 0$ on Γ_K.

It remains to verify the conditions (1.14), (1.15). To this end let us choose a function $\boldsymbol{w}$ such that

$$w_n'' = 0 , \quad w_n' = \psi \leqq 0 \text{ on } \Gamma_K .$$

Then $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} \in K$ and (1.28) implies that

$$0 \leqq \int_{\Gamma_K} T_n' \psi \, ds \quad \forall \psi \leqq 0 .$$

From there it follows that $T_n' \leqq 0$ on Γ_K. Using (1.29), we may write $T_n'' \leqq 0$ as well and (1.14) is true.

Let $u_n' + u_n'' < 0$ at a point $\boldsymbol{x} \in \Gamma_K$. Then a smooth function $\psi \geqq 0$ exists on Γ_K such that $\psi(\boldsymbol{x}) > 0$ and $u_n' + u_n'' + \psi \leqq 0$ on Γ_K. There exists a $\boldsymbol{w} \in V$ such that $w_n' = \psi$, $w_n'' = 0$ on Γ_K. Then $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} \in K$.

The conditions (1.28) and $T_n' \leqq 0$ on Γ_K result in

$$0 \leqq \int_{\Gamma_K} T_n' \psi \, ds \Rightarrow T_n'(\boldsymbol{x}) = 0 ,$$

which means that (1.15) holds. Q.E.D.

Next let us consider the problem $\mathscr{P}_2$ with an enlarging contact zone. Define the set of admissible displacements

$$K_\varepsilon = \{ \boldsymbol{v} \in V \mid v_\xi''(\eta) - v_\xi'(\eta) \leqq \varepsilon(\eta) \text{ for } a.\, a.\, \eta \in \langle a, b \rangle \} .$$

Definition 1.4 *A function $\boldsymbol{u} \in K_\varepsilon$ will be called a weak (variational) solution of the problem $\mathscr{P}_2$ with an enlarging contact zone, if*

$$\mathscr{L}(\boldsymbol{u}) \leqq \mathscr{L}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in K_\varepsilon . \tag{1.30}$$

Theorem 1.2 *Any classical solution of the problem $\mathscr{P}_2$ is a weak solution of $\mathscr{P}_2$. If a weak solution of the problem $\mathscr{P}_2$ is suficciently smooth, it is a classical solution of $\mathscr{P}_2$, as well.*

Proof. 1. Let $\boldsymbol{u}$ be a classical solution. Multiplying the equations (1.5) by a function $\boldsymbol{w} \in V$ and integrating by parts, we obtain

$$0 = -A(\boldsymbol{u}, \boldsymbol{w}) + L(\boldsymbol{w}) + \int_{\Gamma_K'} (T_\xi' w_\xi' + T_\eta' w_\eta') \, ds' + \int_{\Gamma_K''} (T_\xi'' w_\xi'' + T_\eta'' w_\eta'') \, ds'' ,$$

where the boundary conditions (1.6) on Γ_u, (1.7) on $\Gamma'_\tau \cup \Gamma''_\tau$ and (1.8) on Γ_0 have also been used. On the basis of (1.19) we may write

$$A(\mathbf{u}, \mathbf{w}) - L(\mathbf{w}) = \int_a^b [T'_\xi w'_\xi (\cos \alpha')^{-1} + T''_\xi w''_\xi (\cos \alpha'')^{-1}] \, d\eta \, .$$

Moreover, let us employ the relations

$$T''_\xi u''_\xi = T''_\xi (u'_\xi + \varepsilon) \, ,$$

$$T''_\xi w''_\xi = T''_\xi (v''_\xi - u''_\xi) = T''_\xi (v''_\xi - u'_\xi - \varepsilon) \, ,$$

which follow from (1.20) for any $\mathbf{w} = \mathbf{v} - \mathbf{u}$.

Using also (1.18), we may write for any $\mathbf{v} \in K_\varepsilon$:

$$T'_\xi (\cos \alpha')^{-1} (v'_\xi - u'_\xi) + T''_\xi (\cos \alpha'')^{-1} (v''_\xi - u'_\xi - \varepsilon) =$$
$$= T''_\xi (\cos \alpha'')^{-1} (v''_\xi - v'_\xi - \varepsilon) \geqq 0$$

on the interval $\langle a, b \rangle$. Consequently, $\mathbf{u} \in K_\varepsilon$ and

(1.31) $$A(\mathbf{u}, \mathbf{v} - \mathbf{u}) - L(\mathbf{v} - \mathbf{u}) \geqq 0 \quad \forall \mathbf{v} \in K_\varepsilon \, ,$$

which is equivalent to the condition (1.30).

2. Let $\mathbf{u} \in K_\varepsilon$ be a sufficiently smooth weak solution. Integrating in (1.31) by parts, we obtain that $\mathbf{u}$ satisfies the equations (1.5) and the boundary conditions (1.7) on $\Gamma'_\tau \cup \Gamma''_\tau$ and (1.8) on Γ_0, as previously (see the proof of Theorem 1.1).

Denoting $\mathbf{v} - \mathbf{u} = \mathbf{w}$, we have then

(1.32) $$0 \leqq \int_{\Gamma_{K'}} (T'_\xi w'_\xi + T'_\eta w'_\eta) \, ds' + \int_{\Gamma_{K''}} (T''_\xi w''_\xi + T''_\eta w''_\eta) \, ds'' \, .$$

Let us choose $\mathbf{w}$ such that

$$w'_\xi = w''_\xi = \pm\psi \, , \quad w'_\eta = w''_\eta = 0$$

holds on the interval $\langle a, b \rangle$. Then we may write

$$0 = \int_a^b \psi [T'_\xi \, (\cos \alpha')^{-1} + T''_\xi (\cos \alpha'')^{-1}] \, d\eta$$

which implies that

(1.33) $$-T'_\xi (\cos \alpha')^{-1} = T''_\xi (\cos \alpha'')^{-1} \, .$$

Next let

$$w'_\xi = w''_\xi = 0 \, , \quad w''_\eta = 0 \, , \quad w'_\eta = \pm\psi \, .$$

Then

$$0 = \int_{\Gamma_{K'}} \psi T'_\eta \, ds \Rightarrow T'_\eta = 0$$

follows. An analogous approach leads to $T''_\xi = 0$.

Let $\mathbf{w}$ be such that

$$w'_\xi = 0\,, \quad w''_\xi = \psi \leqq 0\,.$$

Then

$$0 \leqq \int_{\Gamma_{K''}} \psi T''_\xi \, ds'' \quad \forall \psi \leqq 0 \Rightarrow T''_\xi \leqq 0 \text{ on } \Gamma''_K\,,$$

and (1.18) follows, making use of (1.33).

It remains to prove (1.20). Assume that

$$u''_\xi(\bar{\eta}) - u'_\xi(\bar{\eta}) < \varepsilon(\bar{\eta})$$

at a point $\bar{\eta} \in \langle a, b \rangle$. Then a smooth function $\psi \geqq 0$ exists on $\langle a, b \rangle$ such that $\psi(\bar{\eta}) > 0$ and

$$u''_\xi - u'_\xi + \psi \leqq \varepsilon \quad \forall \eta \in \langle a, b \rangle\,.$$

A function $\mathbf{w} \in V$ exists such that $w''_\xi = \psi$ on Γ''_K, $w'_\xi = 0$ on Γ'_K. Then we have $\mathbf{v} = \mathbf{u} + \mathbf{w} \in K_\varepsilon$. Since $T''_\xi \leqq 0$ on Γ''_K, from (1.32) we obtain

$$0 \leqq \int_{\Gamma_{K''}} \psi T''_\xi \, ds'' \Rightarrow T''_\xi(\bar{\eta}) = 0\,.$$

Consequently, (1.20) is verified. Q.E.D.

2. EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS

In the present section we shall discuss the conditions, which guarantee the existence and uniqueness of weak solutions to the problems $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively.

2.1 The problem with a bounded contact zone

Let us introduce the subspace of rigid bodies displacements

$$\mathscr{R} = \{\mathbf{z} \in \mathbf{W} \mid \mathbf{z} = (\mathbf{z}', \mathbf{z}''),\ z_1^M = a_1^M - b^M x_2,\ M = {}', {}'',\ z_2^M = a_2^M + b^M x_1\}\,,$$

where $a_i^M \in \mathbb{R}^1$, $i = 1, 2$, and $b^M \in \mathbb{R}^1$ are arbitrary parameters. Obviously, $e_{ij}(\mathbf{z}) = 0$ $\forall \mathbf{z} \in \mathscr{R}$ $\forall i, j$ and therefore we have

$$A(\mathbf{v}, \mathbf{z}) = 0 \quad \forall \mathbf{z} \in \mathscr{R}\,.$$

Moreover, if $\mathbf{p} \in \mathbf{W}$, $e_{ij}(\mathbf{p}) = 0\ \forall i, j$, then $\mathbf{p} \in \mathscr{R}$. (For the proof – see [11]).

Lemma 2.1 *Let there exist a weak solution of the problem $\mathscr{P}_1$. Then it holds*

(2.1) $$L(\mathbf{y}) \leqq 0 \quad \forall \mathbf{y} \in K \cap \mathscr{R}\,.$$

Proof. The weak solution $\boldsymbol{u}$ satisfies the condition (1.26). Inserting $\boldsymbol{v} = \boldsymbol{u} + \mathbf{y}$, $\mathbf{y} \in K \cap \mathscr{R}$, we obtain $\boldsymbol{v} \in K$ and

$$0 = A(\boldsymbol{u}, \mathbf{y}) \geqq L(\mathbf{y}) .$$

Theorem 2.1 *Assume that $V \cap \mathscr{R} = \{0\}$ or*

$$L(\mathbf{z}) \neq 0 \quad \forall \mathbf{z} \in V \cap \mathscr{R} \dot{-} \{0\} . \tag{2.2}$$

Then there exists at most one weak solution of the problem $\mathscr{P}_1$.

Proof. Let $\boldsymbol{u}^1$, $\boldsymbol{u}^2$ be two weak solutions. Using (1.26), we may write

$$A(\boldsymbol{u}^1, \boldsymbol{u}^2 - \boldsymbol{u}^1) \geqq L(\boldsymbol{u}^2 - \boldsymbol{u}^1) ,$$
$$A(\boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2) \geqq L(\boldsymbol{u}^1 - \boldsymbol{u}^2) .$$

Adding these two inequalities leads to the following

$$A(\boldsymbol{u}^1 - \boldsymbol{u}^2, \boldsymbol{u}^2 - \boldsymbol{u}^1) \geqq 0 .$$

Denoting $\mathbf{z} = \boldsymbol{u}^1 - \boldsymbol{u}^2$, we have $A(\mathbf{z}, \mathbf{z}) \leqq 0$. From (1.4) it follows that $e_{ij}(\mathbf{z}) = 0$ $\forall i, j$, consequently $\mathbf{z} \in V \cap \mathscr{R}$. If $V \cap \mathscr{R} = \{0\}$, $\mathbf{z} = 0$ and the solution is unique. If $\mathbf{z} \neq 0$, let us denote $\boldsymbol{u}^2 = \boldsymbol{u}$, $\boldsymbol{u}^1 = \boldsymbol{u} + \mathbf{z}$. Then

$$A(\boldsymbol{u}, \mathbf{z}) = A(\mathbf{z}, \mathbf{z}) = 0 ,$$

$$\mathscr{L}(\boldsymbol{u}) = \mathscr{L}(\boldsymbol{u} + \mathbf{z}) \Rightarrow L(\boldsymbol{u}) = L(\boldsymbol{u} + \mathbf{z}) \Rightarrow L(\mathbf{z}) = 0 ,$$

which contradicts the assumption (2.2). Hence $\mathbf{z} = 0$ again.

Example 2.1 Let Γ_0 consists of straight segment parallel to the x_1-axis (see Fig. 1). Then we have

$$V \cap \mathscr{R} = \{\mathbf{z} \mid \mathbf{z}' = (0, 0),\ \mathbf{z}'' = (a, 0),\ a \in R^1\} .$$

Assume that $n''_1 \geqq 0$ on Γ_K (almost everywhere) and there exists $\boldsymbol{x} \in \Gamma_K$ such that $n''_1(\boldsymbol{x}) > 0$. Then

$$K \cap \mathscr{R} = \{\mathbf{y} \mid \mathbf{y}' = (0, 0),\ \mathbf{y}'' = (a, 0),\ a \leqq 0\} .$$

In fact, $\mathbf{y} \in K \cap \mathscr{R} \subset V \cap \mathscr{R}$,

$$y'_n + y''_n = a n''_1 \leqq 0 \quad \text{on} \quad \Gamma_K \Leftrightarrow a \leqq 0 .$$

From Lemma 2.1 it follows that a weak solution exists only if

$$V''_1 = \int_{\Omega''} F_1 \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_\tau''} P_1 \, \mathrm{d}s \geqq 0 .$$

Indeed, inserting $\mathbf{y} \in K \cap \mathscr{R}$ into the condition (2.1), we obtain

$$0 \geqq L(\mathbf{y}) = a V_1'' \quad \forall a \leqq 0 .$$

From theorem 2.1 it follows that if $V_1'' \neq 0$, there exists at most one weak solution. In fact, for $\mathbf{z} \in V \cap \mathscr{R} \dot{-} \{0\}$ we have

$$L(\mathbf{z}) = a V_1'' , \quad a \neq 0$$

and if $V_1'' \neq 0$, then $L(\mathbf{z}) \neq 0$.

Let us present a general result on the existence of a weak solution of the problem $\mathscr{P}_1$.

Let us introduce the set of "bilateral" admissible rigid displacements

$$\mathscr{R}^* = \{\mathbf{z} \in K \cap \mathscr{R} \mid \mathbf{z} \in \mathscr{R}^* \Rightarrow -\mathbf{z} \in \mathscr{R}^*\} .$$

It is readily seen that

$$\mathscr{R}^* = \{\mathbf{z} \in \mathscr{R} \cap V \mid z_n' + z_n'' = 0 \text{ on } \Gamma_K\} . \tag{2.3}$$

Theorem 2.2 Assume that

$$L(\mathbf{y}) \leqq 0 \quad \forall \mathbf{y} \in K \cap \mathscr{R} , \tag{2.4}$$

$$L(\mathbf{y}) < 0 \quad \forall \mathbf{y} \in K \cap \mathscr{R} \dot{-} \mathscr{R}^* . \tag{2.5}$$

Then there exists a weak solution $\boldsymbol{u}$ of the problem $\mathscr{P}_1$. Any other weak solution $\hat{\boldsymbol{u}}$ can be written in the form $\hat{\boldsymbol{u}} = \boldsymbol{u} + \mathbf{y}$, where $\mathbf{y} \in \mathscr{R} \cap V$ is such that $\boldsymbol{u} + \mathbf{y} \in K$, $L(\mathbf{y}) = 0$.

Proof. Existence can be based on an abstract theorem by Fichera ([10] – Th. 1.II).

The nonuniqueness of solution, however, is a great obstacle in the numerical analysis of the contact problem. Moreover, in the proof of convergence (Part II) also the coerciveness of the functional $\mathscr{L}$ over the set of admissible functions will be required. Therefore we restrict ourselves to cases with one-dimensional spaces of rigid virtual displacements, in what follows (see Remarks 2.1, 2.3 and 2.5). This enables us to define a contact problem, possessing a unique solution and the coerciveness property mentioned above.

Theorem 2.3 *Denote $\mathscr{R} \cap V \equiv \mathscr{R}_v$. Assume that:*

$$\mathscr{R} \cap K = \mathscr{R}_v \tag{2.6}$$

and

$$L(\mathbf{y}) = 0 \quad \forall \mathbf{y} \in \mathscr{R}_v . \tag{2.7}$$

Denote by

$$V = H \oplus \mathscr{R}_v$$

the orthogonal decomposition of the space V (with an arbitrary scalar product).

Then

(i) $\mathscr{L}$ *is coercive on* H, (*i.e.* $\mathscr{L}(\mathbf{v}) \to +\infty$ *for* $\|\mathbf{v}\| \to \infty$, $\mathbf{v} \in H$).

(ii) *there exists a unique solution* $\hat{\mathbf{u}} \in \hat{K}$ *of the problem*

$$\mathscr{L}(\hat{\mathbf{u}}) \leqq \mathscr{L}(\mathbf{z}) \quad \forall \mathbf{z} \in \hat{K}\,, \quad \hat{K} = K \cap H\,, \tag{2.8}$$

(iii) *any weak solution of* $\mathscr{P}_1$ *can be written in the form*

$$\mathbf{u} = \hat{\mathbf{u}} + \mathbf{y}\,,$$

where $\hat{\mathbf{u}} \in \hat{K}$ *is the solution of the problem* (2.8) *and* $\mathbf{y} \in \mathscr{R}_v$;

(iv) *if* $\hat{\mathbf{u}} \in \hat{K}$ *is the solution of* (2.8), *then* $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{y}$, *where* $\mathbf{y}$ *is any element of* $\mathscr{R}_v$, *represents a weak solution of the problem* $\mathscr{P}_1$.

Remark 2.1 Note that the assumption (2.6) can be satisfied only if dim $\mathscr{R}_v \leqq 1$. In fact, dim $\mathscr{R}_v \leqq 3$ (since mes $\Gamma_u > 0$) and the case dim $\mathscr{R}_v = 2$ is not possible.*) Therefore, let us consider the case dim $\mathscr{R}_v = 3$, which implies $\Gamma_0 = \emptyset$ and

$$\mathscr{R}_v = \{\mathbf{y} = (\mathbf{y}', \mathbf{y}'') |\ \mathbf{y}' = 0,\ y_1'' = a_1 - bx_2,\ y_2'' = a_2 + bx_1\}\,,$$

with a_i and b arbitrary constants. Consequently, the body Ω'' is completely free. Since the set $\mathscr{R} \cap K \subset \mathscr{R}_v$ is restricted by the condition $y_n'' \leqq 0$ on Γ_K, we have $\mathscr{R} \cap K \neq \mathscr{R}_v$, which contradicts (2.6).

An example with dim $\mathscr{R}_v = 1$, satisfying (2.6), is shown in Fig. 4. Then $a_2 = b = 0$, a_1 is arbitrary. If the force resultant

$$V_1'' = \int_{\Omega''} F_1'' \,\mathrm{d}\mathbf{x} + \int_{\Gamma_\tau''} P_1'' \,\mathrm{d}s = 0\,,$$

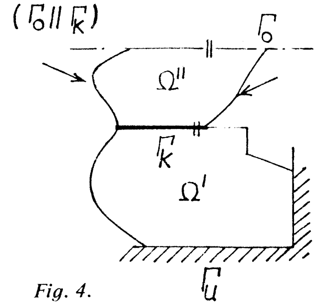

Fig. 4.

*) If $\Gamma_0 = \emptyset$ then dim $\mathscr{R}_v = 3$; if $\Gamma_0 \neq \emptyset$, dim $\mathscr{R}_v \leqq 1$.

then

$$L(\mathbf{y}) = a_1 V_1'' = 0 \quad \forall a_1 \in \mathbb{R}^1$$

and (2.7) is also true.

Another example is given if both Γ_0 and Γ_K are parts of concentric circles. Then the rigid body Ω'' may rotate, and if the moment resultant

$$M = \int_{\Omega''} (x_1 F_2'' - x_2 F_1'') \, \mathrm{d}\mathbf{x} + \int_{\Gamma_\tau''} (x_1 P_2'' - x_2 P_1'') \, \mathrm{d}s = 0$$

then

$$L(\mathbf{y}) = bM = 0 \quad \forall b \in \mathbb{R}^1$$

and (2.7) holds (provided the origin coincides with the center of the circles).

Remark 2.2 From the numerical point of view it is convenient to introduce the following types of scalar product in V (see [11] – I, Th. 2.3). For example, let $\dim \mathscr{R}_v = 1$. We set

$$(\mathbf{u}, \mathbf{v})_v = \int_{\Omega' \cup \Omega''} e_{ij}(\mathbf{u}) \, e_{ij}(\mathbf{v}) \, \mathrm{d}\mathbf{x} + p(\mathbf{u}) \, p(\mathbf{v}),$$

where p is a linear bounded functional on V such that

$$\{\mathbf{y} \in \mathscr{R}_v, \ p(\mathbf{y}) = 0\} \Rightarrow \mathbf{y} = 0 \,. \tag{2.9}$$

For instance, if

$$\mathscr{R}_v = \{\mathbf{y} = (\mathbf{y}', \mathbf{y}'') \mid \mathbf{y}' = 0, \ y_1'' = a \in \mathbb{R}^1, \ y_2'' = 0\}$$

(see Fig. 4), we can choose

$$p(\mathbf{v}) = \int_{\Gamma_1} v_1'' \, \mathrm{d}s \,, \tag{2.10}$$

where $\Gamma_1 \subset \bar{\Omega}''$, $\operatorname{mes} \Gamma_1 > 0$.

Then (cf. [11]-I, Remark 4)

$$H = V \ominus \mathscr{R}_v = \{\mathbf{v} \in V \mid p(\mathbf{v}) = 0\} \,. \tag{2.11}$$

Proof of Theorem 2.3 (i). The following inequality of Korn's type is true for any $\mathbf{v} \in H$ (cf. [11]-I, Remarks 3 and 4)

$$C_1 \|\mathbf{v}\| \leqq |\mathbf{v}| \,, \tag{2.12}$$

where $\|\cdot\|$ is the norm in $\mathbf{W}$ and

$$|\mathbf{v}|^2 = \int_{\Omega' \cup \Omega''} e_{ij}(\mathbf{v}) \, e_{ij}(\mathbf{v}) \, \mathrm{d}\mathbf{x} \,. \tag{2.13}$$

Then we have for any $\mathbf{v} \in H$

$$\mathscr{L}(\mathbf{v}) \geqq \tfrac{1}{2} c_0 |\mathbf{v}|^2 - L(\mathbf{v}) \geqq L\|\mathbf{v}\|^2 - \|L\| \, \|\mathbf{v}\|$$

and the coerciveness of $\mathscr{L}$ over H follows.

(ii) Since $\mathscr{L}$ is Gâteaux differentiable and convex, $\hat{K}$ being convex and closed, there exists a solution $\hat{\mathbf{u}}$ of the problem (2.8).

Let $\mathbf{u}^1 \in \hat{K}$ and $\mathbf{u}^2 \in \hat{K}$ be two solutions of (2.8). Arguing as in the proof of Theorem 2.1, we arrive at

$$\mathbf{z} = \mathbf{u}^1 - \mathbf{u}^2 \in \mathscr{R}_v .$$

Since $\mathbf{z} \in H$, we obtain $\mathbf{z} \in H \cap \mathscr{R}_v = \{0\}$. Therefore the solution $\hat{\mathbf{u}}$ is unique.

(iii) By virtue of (2.7) we have

$$\mathscr{L}(\mathbf{v}) = \mathscr{L}(\mathbf{v} + \mathbf{y}) \quad \forall \mathbf{y} \in \mathscr{R}_v . \tag{2.14}$$

Moreover, it holds

$$P_H(K) = K \cap H . \tag{2.15}$$

In fact, let $v \in K$. Then using (2.3) and (2.6), we obtain

$$P_H \mathbf{v} = \mathbf{v} - P_{\mathscr{R}_v} \mathbf{v} , \quad \mathscr{R}^* = \mathscr{R}_v ,$$

$$(P_H v)'_n + (P_H v)''_n = v'_n + v''_n \leqq 0 \quad \text{on} \quad \Gamma_K ,$$

consequently $P_H \mathbf{v} \in K \cap H$.

The inclusion $K \cap H = P_H(K \cap H) \subset P_H(K)$ is obvious.

Let $\mathbf{u}$ be a weak solution of the problem $\mathscr{P}_1$. By virtue of (2.14) we may write

$$\mathscr{L}(P_H \mathbf{v}) = \mathscr{L}(P_H \mathbf{v} + P_{\mathscr{R}_v} \mathbf{v}) = \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in V ,$$

furthermore, $P_H \mathbf{u} \in K \cap H$,

$$\mathscr{L}(P_H \mathbf{u}) = \mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) = \mathscr{L}(P_H \mathbf{v}) \quad \forall \mathbf{v} \in K$$

and from (2.15) it follows that $P_H \mathbf{u}$ is a solution of (2.8). The uniqueness implies that $P_H \mathbf{u} = \hat{\mathbf{u}}$, $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{y}$, $\mathbf{y} \in \mathscr{R}_v$.

(iv) Let $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{y}$, where $\mathbf{y} \in \mathscr{R}_v$. Then we have $\mathbf{u} \in K$, using (2.3), and

$$\mathscr{L}(\mathbf{u}) = \mathscr{L}(\hat{\mathbf{u}}) \leqq \mathscr{L}(\mathbf{z}) \quad \forall \mathbf{z} \in \hat{K} . \tag{2.16}$$

Let $\mathbf{v} \in K$. Making use of (2.14) and of the decomposition

$$\mathbf{v} = P_H \mathbf{v} + P_{\mathscr{R}_v} \mathbf{v} ,$$

we obtain for $\mathbf{z} = P_H \mathbf{v} \in P_H(K) = K \cap H = \hat{K}$

$$\mathscr{L}(\mathbf{z}) = \mathscr{L}(\mathbf{v}) . \tag{2.17}$$

Finally, (2.16) and (2.17) lead to the relation

$$\mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K .$$

Theorem 2.4 *Assume that*

$$\mathscr{R}^* = \{0\} , \quad \mathscr{R}_v \neq \{0\} , \tag{2.18}$$

$$L(\mathbf{y}) \neq 0 \quad \forall \mathbf{y} \in \mathscr{R}_v \dot{-} \{0\} \tag{2.19}$$

and either $K \cap \mathscr{R} = \{0\}$ *or*

$$K \cap \mathscr{R} \neq \{0\} , \tag{2.20}$$

$$L(\mathbf{y}) < 0 \quad \forall \mathbf{y} \in K \cap \mathscr{R} \dot{-} \{0\} . \tag{2.21}$$

Then $\mathscr{L}$ is coercive on K and there exists a unique weak solution of the problem $\mathscr{P}_1$.

Remark 2.3 The assumption (2.19) cannot be satisfied unless $\dim \mathscr{R}_v \leqq 1$. In fact, for $\Gamma_0 = \emptyset$, $\dim \mathscr{R}_v = 3$ (cf. Remark 2.1) and

$$L(\mathbf{y}) = a_1 V_1 + a_2 V_2 + bM = 0$$

for any vector (a_1, a_2, b) orthogonal to (V_1, V_2, M) in the space R^3. An example with $\dim \mathscr{R}_v = 1$, satisfying (2.18) and (2.20), is shown in Fig. 5.

Another example with $\dim \mathscr{R}_v = 1$, satisfying (2.18) and $K \cap \mathscr{R} = \{0\}$, is presented in Fig. 6.

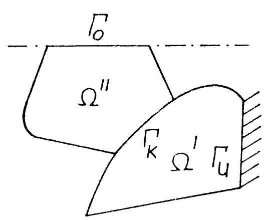

Fig. 5.

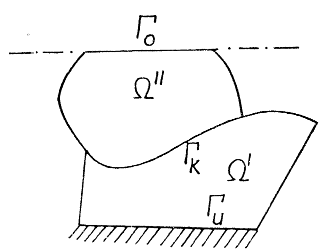

Fig. 6.

Let Γ_0 be parallel to x_1 – axis and let $V_1'' > 0$. Then

$$\mathbf{y} \in \mathscr{R}_v \dot{-} \{0\} \Rightarrow L(\mathbf{y}) = a_1 V_1'' \neq 0 ,$$

consequently (2.19) is satisfied. It is also easy to verify (2.21) in case of Fig. 5.

Proof of Theorem 2.4

(i) Let us consider the case $K \cap \mathscr{R} = \{0\}$. We shall employ the following abstract result ([12] – Th. 2.2):

Proposition 1. Let $|u|$ be a seminorm in a Hilbert space H with the norm $\|u\|$. Assume that if we introduce the subspace

$$R = \{u \in H \mid |u| = 0\},$$

then $\dim R < \infty$ and it holds

$$C_1\|u\| \leqq |u| + \|P_R u\| \leqq C_2\|u\| \quad \forall u \in H, \tag{2.22}$$

where P_R is the orthogonal projection onto R.

Let K be a convex closed subset of H, containing the origin, $K \cap R = \{0\}$, $\beta : H \to \mathbb{R}^1$ a penalty functional with a differential, which is 1-positively homogeneous, i.e.

$$D\beta(tu, v) = t\, D\beta(u, v) \quad \forall t > 0, \quad u, v \in H,$$

and such that

$$\beta(u) = 0 \Leftrightarrow u \in K.$$

Then it holds

$$|u|^2 + \beta(u) \geqq C\|u\|^2 \quad \forall u \in H. \tag{2.23}$$

The Proposition 1 can be applied with:

$$H = V, \quad R = \mathscr{R} \cap V \equiv \mathscr{R}_v, \quad |v| \text{ defined as in (2.13)}$$

$$\beta(\boldsymbol{u}) = \frac{1}{2}\int_{\Gamma_K} ([u_n' + u_n'']^+)^2 \, ds.$$

To verify (2.22), we make use of the inequality of Korn's type [11] and of the decomposition $V = Q \oplus \mathscr{R}_v$ to obtain

$$\|\boldsymbol{u}\|^2 = \|P_Q\boldsymbol{u}\|^2 + \|P_{\mathscr{R}_v}\boldsymbol{u}\|^2 \leqq C|P_Q\boldsymbol{u}|^2 + \|P_{\mathscr{R}_v}\boldsymbol{u}\|^2 = C|\boldsymbol{u}|^2 + \|P_{\mathscr{R}_v}\boldsymbol{u}\|^2.$$

From (2.23) it follows that

$$|\boldsymbol{u}|^2 \geqq C\|\boldsymbol{u}\|^2 \quad \forall \boldsymbol{u} \in K.$$

Then we can deduce easily that $\mathscr{L}$ is coercive on K and the existence of a weak solution $\boldsymbol{u}$ of $\mathscr{P}_1$.

If $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ are two weak solutions of $\mathscr{P}_1$, using the same approach as in the proof of Theorem 2.1, we obtain

$$\boldsymbol{y} = \boldsymbol{u}^1 - \boldsymbol{u}^2 \in \mathscr{R}_v.$$

Moreover,

$$\mathscr{L}(\boldsymbol{u}^1) = \mathscr{L}(\boldsymbol{u}^2) \Rightarrow L(\boldsymbol{u}^1) = L(\boldsymbol{u}^2) \Rightarrow L(\boldsymbol{y}) = 0$$

and from the assumption (2.19) we conclude that $\boldsymbol{y} = 0$.

(ii) Let us consider the case (2.20), (2.21). We shall employ the following abstract result ([12] – Th. 2.3):

Proposition 2. Let the assumptions of Proposition 1 be satisfied with the only exception that $K \cap R \neq \{0\}$.

Moreover, let f be a linear bounded functional on H such that

$$f(y) < 0 \quad \forall y \in K \cap R \dot{-} \{0\} .$$

Then

$$|u|^2 + \beta(u) - f(u) \geqq C_1 \|u\| - C_2 \quad \forall u \in H . \tag{2.24}$$

The Proposition 2 can be applied with the same H, R, $|\cdot|$, β as previously and with

$$f(\mathbf{v}) = L(\mathbf{v}) .$$

Then (2.24) implies that $\mathscr{L}$ is coercive over K. The existence and uniqueness of the weak solution can be deduced in the same way as in the previous case (i).

Remark 2.4 The simplest is the "coercive" case, i.e. the case $V \cap \mathscr{R} = \{0\}$. Then we have the inequality of Korn's type

$$\|\mathbf{v}\| \leqq C|\mathbf{v}| \quad \forall \mathbf{v} \in V,$$

so that $\mathscr{L}$ is coercive on the whole space V. The existence and uniqueness of the solution of $\mathscr{P}_1$ is readily seen.

2.2 Problems with enlarging contact zone

Let us consider the cases of one-dimensional spaces of rigid virtual displacements. First we obtain a theorem analogous to Theorem 2.3.

Theorem 2.5 *Let us denote*

$$K_0 = \{\mathbf{v} \in V \mid v''_\xi - v'_\xi \leqq 0 \text{ for } a.\,a.\ \eta \in \langle a, b \rangle\} .$$

Assume that

$$\mathscr{R}_v = K_0 \cap \mathscr{R} , \tag{2.25}$$

$$L(\mathbf{y}) = 0 \quad \forall \mathbf{y} \in \mathscr{R}_v . \tag{2.26}$$

Let $V = H \oplus \mathscr{R}_v$ be an orthogonal decomposition of the space V (with an arbitrary scalar product).

Then

(i) *$\mathscr{L}$ is coercive on H,*

(ii) *there exists a unique solution $\hat{\mathbf{u}} \in \hat{K}_\varepsilon$ of the problem*

$$\mathscr{L}(\hat{\mathbf{u}}) \leqq \mathscr{L}(\mathbf{z}) \quad \forall \mathbf{z} \in K_\varepsilon \cap H \equiv \hat{K}_\varepsilon ; \tag{2.27}$$

(iii) *any weak solution of* $\mathscr{P}_2$ *can be written in the form*

$$\mathbf{u} = \hat{\mathbf{u}} + \mathbf{y} ,$$

where $\hat{\mathbf{u}} \in \hat{K}$ *is the solution of* (2.27) *and* $\mathbf{y} \in \mathscr{R}_v$,

(iv) *if* $\hat{\mathbf{u}} \in \hat{K}$ *is the solution of* (2.27), *then* $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{y}$, *where* $\mathbf{y}$ *is any element of* $\mathscr{R}_v$, *represents a weak solution of the problem* $\mathscr{P}_2$.

Remark 2.5 Arguing in the same way as in Remark 2.1, one can prove that (2.25) can be satisfied only if $\dim \mathscr{R}_v \leqq 1$. An example, when the assumption (2.25) is satisfied, is shown in Fig. 7. Then

$$\mathscr{R}_v = \{\mathbf{y}' = 0,\ \mathbf{y}'' = (a, 0),\ a \in R^1\} ,$$

and if $V_1'' = 0$, (2.26) is true.

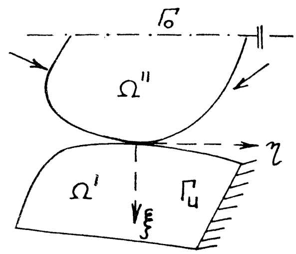

Fig. 7.

Remark 2.6 For the choice of a suitable scalar product in V, the approach of Remark 2.2 can be applied.

Proof of Theorem 2.5 is quite analogous to that of Theorem 2.3.

Theorem 2.6 *Assume that* Γ_0 *consists of straight segments parallel to the* x_1*-axis,* $\cos(\xi, x_1) > 0$ (*see Fig.* 8) *and*

(2.28)
$$\int_{\Omega''} F_1'' \,\mathrm{d}\mathbf{x} + \int_{\Gamma_\tau''} P_1'' \,\mathrm{d}s > 0 .$$

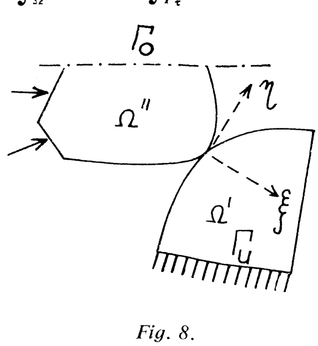

Fig. 8.

Then $\mathscr{L}$ is coercive on K_ε and a unique solution of the problem $\mathscr{P}_2$ exists.

Proof. Let us define

$$p_0(v) = \int_a^b (v''_\xi - v'_\xi)\,\mathrm{d}\eta\,,$$

$$V_p = \{\mathbf{v} \in V \mid p_0(\mathbf{v}) = 0\}\,.$$

Then it holds

(2.29) $$\mathscr{R} \cap V_p = \{0\}\,.$$

In fact, $\mathscr{R} \cap V_p \subset \mathscr{R}_v = \{\mathbf{z}' = 0,\ \mathbf{z}'' = (c, 0),\ c \in \mathbb{R}^1\}$. If $p_0(\mathbf{z}) = 0$, then

$$0 = \int_a^b z''_\xi\,\mathrm{d}\eta = c\int_a^b \cos(\xi, x_1)\,\mathrm{d}\eta \Rightarrow c = 0\,.$$

Using (2.29), we can prove the following inequality of Korn's type (cf. [11])

(2.30) $$|\mathbf{v}| \geqq C\|\mathbf{v}\| \quad \forall \mathbf{v} \in V_p\,.$$

Let $\mathbf{v} \in V$ and define $\mathbf{y} \in \mathscr{R}_v$ as follows

$$\mathbf{y}' = 0\,,\quad y''_1 = p_0(\mathbf{v})\,d^{-1}\,,\quad y''_2 = 0\,,$$

where

$$d = \int_b^a \cos(\xi, x_1)\,\mathrm{d}\eta\,.$$

It is easy to verify that for $P\mathbf{v} = \mathbf{v} - \mathbf{y}$ it holds

$$p_0(P\mathbf{v}) = p_0(\mathbf{v}) - p_0(\mathbf{y}) = p_0(\mathbf{v}) - \int_a^b p_0(\mathbf{v})\,d^{-1}\cos(\xi, x_1)\,\mathrm{d}\eta = 0\,,$$

consequently $P\mathbf{v} \in V_p$.

Using (2.30), we may write

(2.31) $$\mathscr{L}(\mathbf{v}) = \tfrac{1}{2}A(P\mathbf{v}, P\mathbf{v}) - L(P\mathbf{v}) - L(\mathbf{y}) \geqq C_1\|P\mathbf{v}\|^2 - C_2\|P\mathbf{v}\| - y''_1 V''_1\,,$$

where

$$V''_1 = \int_{\Omega''} F_1\,\mathrm{d}x + \int_{\Gamma_\tau''} P_1\,\mathrm{d}s\,.$$

If $\|\mathbf{v}\| \to \infty$, at least one of the norms $\|P\mathbf{v}\|$ and $\|\mathbf{y}\|$ tends to infinity. Moreover, we have

(2.32) $$\mathbf{v} \in K_\varepsilon \Rightarrow p_0(\mathbf{v}) \leqq \int_a^b \varepsilon\,\mathrm{d}\eta < +\infty\,,$$

(2.33) $$\|\mathbf{y}\| = |y''_1|\left(\int_{\Omega''}\mathrm{d}x\right)^{1/2} = |p_0(\mathbf{v})|\,d^{-1}(\operatorname{mes}\Omega'')^{1/2}\,.$$

1° Let $\|\mathbf{y}\| \to +\infty$. Then (2.32) and (2.33) imply $-p_0(\mathbf{v}) \to +\infty$, and consequently $-y_1'' \to +\infty$. Since

$$C_1\|P\mathbf{v}\|^2 - C_2\|P\mathbf{v}\| \geqq C_3 > -\infty,$$

(2.31) and (2.38) lead to

$$\mathscr{L}(\mathbf{v}) \to +\infty.$$

2° Let $\|P\mathbf{v}\| \to +\infty$. Then

$$\mathscr{L}_1(P\mathbf{v}) = C_1\|P\mathbf{v}\|^2 - C_2\|P\mathbf{v}\| \to +\infty,$$

$$\mathscr{L}_2(\mathbf{y}) = -y_1''V_1'' = -p_0(\mathbf{v})\, d^{-1}V_1'' \geqq -d^{-1}V_1'' \int_a^b \varepsilon \, \mathrm{d}\eta > -\infty$$

holds, by virtue of (2.32) and (2.28). Finally, (2.31) yields

$$\mathscr{L}(\mathbf{v}) \geqq \mathscr{L}_1(P\mathbf{v}) + \mathscr{L}_2(\mathbf{y}) \to +\infty.$$

Thus we have proved that $\mathscr{L}$ is coercive over K_ε.

Since K_ε is closed and convex, $\mathscr{L}$ convex and continuously differentiable, the solution of $\mathscr{P}_2$ exists.

The uniqueness follows from (2.28). In fact, we prove that any two solutions $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ differ by an element $\mathbf{z} \in \mathscr{R}_v$, such that $L(\mathbf{z}) = 0$ (see the proof of Theorem 2.1). On the other hand,

$$L(\mathbf{z}) = cV_1'', \quad c \in \mathbb{R}^1.$$

Hence (2.28) implies that $c = 0$, i.e., $\mathbf{z} = 0$.

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Souhrn

KONTAKT PRUŽNÝCH TĚLES – I. SPOJITÉ PROBLÉMY

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V práci je provedena podrobná analýza kontaktní úlohy v rovinné pružnosti. Je zkoumána situace, kdy v závislosti na geometrii úlohy nemůže dojít k rozšíření kontaktní zóny při deformaci a rovněž úloha, kdy zóna styku se může rozšířit během deformace.

Od heuristických „klasických" formulací se přechází k formulacím ve tvaru variačních nerovnic. Pro ně se pak dokazuje existence řešení metodami konvexní analýzy, s důrazem na jednoznačnost řešení a na koercivitu energetických funkcionálů.

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