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CONTACT BETWEEN ELASTIC BODIES – II. FINITE ELEMENT ANALYSIS

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INTRODUCTION

In Part I of our paper [1] the existence and uniqueness of continuous contact problems have been discussed. In the present Part we apply the simplest finite element technique, i.e., the piecewise linear triangular elements, to the solution of the contact problems. Some error estimates are deduced, assuming that the exact solution is regular enough and using a modified method of Falk. If the solution is not regular, we prove the convergence itself. For problems with enlarging contact zone, the element of Zlámal's type [3] with one curved side along the boundaries, are employed.

1. APPROXIMATION OF VARIATIONAL INEQUALITIES

Let H be a real Hilbert space, $\mathscr{K} \neq \emptyset$ a closed convex subset of H and $J : H \to R_1$ be a given quadratic functional:

$$J(v) = 1/2\, A(v, v) - f(v),$$

where $A(u, v)$ is a symmetric, positive – semidefinite bilinear form defined on $H \times H, f \in H'$ a given linear continuous functional on H.

We shall consider the following problem:

$$(\mathscr{P}) \qquad \begin{cases} \text{find} \quad u \in \mathscr{K} \quad \text{such that} \\ J(u) = \min\limits_{v \in \mathscr{K}} J(v) . \end{cases}$$

Let $\{\mathscr{K}_h\}$, $h \in (0, 1)$ be a system of finite – dimensional approximations of $\mathscr{K}$, i.e. $\mathscr{K}_h \neq \emptyset$ are closed convex subset of H for $h \in (0, 1)$, contained in finite – dimensional subspaces $S_h \subset H$. Let us define an element $u_h \in \mathscr{K}_h$ such that

$$(\mathscr{P}_h) \qquad J(u_h) = \min_{v \in \mathscr{K}_h} J(v) .$$

Lemma 1.1. *Let u, u_h be solutions of $(\mathscr{P})$, $(\mathscr{P}_h)$, respectively. Then it holds:*

$$(1.1)\qquad A(u - u_h, u - u_h) \leqq \{f(u - v_h) + f(u_h - v) + A(u_h - u, v_h - u) + \\ + A(u, v - u_h) + A(u, v_h - u)\}$$

for any $v \in \mathscr{K}$, $v_h \in \mathscr{K}_h$.

Proof. As J is convex on H then $u \in \mathscr{K}$, $u_h \in \mathscr{K}_h$ solve $(\mathscr{P})$, $(\mathscr{P}_h)$ respectively if and only if

$$A(u, v - u) \geqq f(v - u) \quad \forall v \in \mathscr{K}\,,$$

$$A(u_h, v_h - u_h) \geqq f(v_h - u_h) \quad \forall v_h \in \mathscr{K}_h\,.$$

Hence

$$A(u - u_h, u - u_h) = A(u, u) + A(u_h, u_h) - 2A(u, u_h) \leqq$$

$$\leqq A(u, v) + f(u - v) + A(u_h, v_h) + f(u_h - v_h) - 2A(u, u_h) =$$

$$= f(u - v_h) + f(u_h - v) + A(u, v - u_h) + A(u_h - u, v_h - u) +$$

$$+ A(u, v_h - u) \quad \forall v \in \mathscr{K}\,, \quad \forall v_h \in \mathscr{K}_h\,.$$

Remark 1.1. If $\mathscr{K}_h \subset \mathscr{K}$, the inequality (1.1) yields

$$(1.2)\qquad A(u - u_h, u - u_h) \leqq \{f(u - v_h) + A(u_h - u, v_h - u) + A(u, v_h - u)\} \\ \forall v_h \in \mathscr{K}_h\,.$$

Proof. Inserting $v = u_h$ in (1.1), we obtain (1.2).

Remark 1.2. Let $\|\ \|$ and $|\ |$ be a norm and a seminorm in H, respectively. If there exists a constant $\gamma < 0$ such that

$$(1.3)\qquad A(v, v) \geqq \gamma\|v\|^2 \quad \forall v \in H\,,$$

then $(\mathscr{P})$, $(\mathscr{P}_h)$ have unique solutions u and u_h, respectively and

$$(1.4)\qquad \gamma\|u - u_h\|^2 \leqq \{f(u - v_h) + f(u_h - v) + A(u_h - u, v_h - u) + \\ + A(u, v - u_h) + A(u, v_h - u)\} \quad \forall v \in \mathscr{K}\,, \quad \forall v_h \in \mathscr{K}_h\,.$$

If

$$(1.5)\qquad A(v, v) \geqq \gamma|v|^2 \quad \forall v \in H\,,$$

then $|u - u_h|$ instead of $\|u - u_h\|$ can be written in the left hand side of (1.4).

Remark 1.3. As $A(u, v)$ is symmetric and positive – semidefinite, we have

$$A(u, v) \leqq 1/2\, A(u, u) + 1/2\, A(v, v) \quad \forall u, v \in H$$

and (1.1) can be written as follows:

$$1/2\, A(u - u_h, u - u_h) \leqq \{f(u - v_h) + f(u_h - v) + 1/2\, A(v_h - u, v_h - u) + \\ + A(u, v - u_h) + A(u, v_h - u)\} \quad \forall v \in \mathscr{K}\,, \quad \forall v_h \in \mathscr{K}_h\,.$$

Theorem 1.1. *Let us assume that*

(1.6) $$\forall v \in \mathscr{K} \quad \exists\{v_h\} \in \{\mathscr{K}_h\} : \|v - v_h\| \to 0 , \quad h \to 0+ ;$$

(1.7) $$v_h \in \mathscr{K}_h , \quad v_h \rightharpoonup v \quad \text{(weakly) in } H \text{ implies} \quad v \in \mathscr{K} .$$

Let the form A satisfy (1.3). *Then*

$$\|u - u_h\| \to 0 , \quad h \to 0+ .$$

For the proof, see e.g. [2] – chpt. 4.

Remark 1.4. If $\mathscr{K}_h \subset \mathscr{K}$ $\forall h \in (0, 1)$, then (1.7) is satisfied, since $\mathscr{K}$ is weakly closed.

We shall need also a slight modification of Theorem 1.1.

Theorem 1.2. *Let us suppose that*

(1.8) $$\|v_h\| \to \infty , \quad v_h \in \mathscr{K}_h \quad \textit{implies} \quad J(v_h) \to +\infty$$

and let (1.5), (1.6), (1.7) *be satisfied. Moreover, let us suppose that* ($\mathscr{P}$) *has precisely one solution. Then*

$$|u - u_h| \to 0 , \quad h \to 0+ .$$

Proof. (1.6) ensures the existence of $\{v_h\}$, $v_h \in \mathscr{K}_h$ such that

$$\|v_h - u\| \to 0 , \quad h \to 0+ .$$

Hence

(1.9) $$J(v_h) \to J(u) , \quad h \to 0+$$

and from the definition of $(\mathscr{P}_h) : J(u_h) \leqq J(v_h)$.

From (1.8), (1.9) the boundedness of $\{u_h\}$ follows:

$$\exists c = \text{const.} > 0 : \|u_h\| \leqq c \quad \forall h \in (0, 1) .$$

Then there exists an element $u^* \in H$ and a subsequence of $\{u_h\}$ (let us denote it by $\{u_h\}$ again) such that:

$$u_h \rightharpoonup u^* \quad \text{in} \quad H .$$

From (1.7) it follows that $u^* \in \mathscr{K}$. As J is weakly lower – semicontinuous on H, we obtain:

$$J(u^*) \leqq \liminf_{h \to 0+} J(u_h) \leqq \lim_{h \to 0+} J(v_h) = J(u) .$$

The uniqueness of the solution of ($\mathscr{P}$) implies $u = u^*$ and even the whole sequence $\{u_h\}$ converges weakly to u. Furthermore, we may write

$$J(u_h) = J(u) + A(u, u_h - u) - f(u_h - u) + \tfrac{1}{2} A(u - u_h, u - u_h) .$$

Hence

$$\gamma/2 \, |u - u_h|^2 \leqq J(u_h) - J(u) + A(u, u - u_h) + f(u_h - u) \to 0 \quad \text{if} \quad h \to 0+ .$$

Remark 1.5. If $\mathscr{K}_h \subset \mathscr{K}\ \forall h \in (0,1)$ then (1.8) can be replaced by the coerciveness of J on $\mathscr{K}$, i.e. by the assumption

$$\|v\| \to \infty\,, \quad v \in \mathscr{K} \Rightarrow J(v) \to +\infty\,.$$

Remark 1.6. Let $\mathscr{H}$ be another Hilbert space with the norm $\|\ \|_{\mathscr{H}}$, $H \subset \mathscr{H}$ with completely continuous imbedding. Assume that a constant $c > 0$ exists such that

$$\|v\|^2 \leqq c(\|v\|^2_{\mathscr{H}} + |v|^2) \quad \forall v \in H \tag{1.10}$$

(for example $H = H^1(\Omega)$, $\mathscr{H} = L^2(\Omega)$, $|v| = (\int_\Omega |\text{grad } v|^2 \, dx)^{1/2}$, Ω a domain with a continuous boundary). Let all assumptions of Theorem 1.2 be satisfied. Then

$$\|u - u_h\| \to 0\,, \quad h \to 0+\,.$$

Proof. From Theorem 1.2 and its proof it follows

$$|u - u_h| \to 0\,, \quad h \to 0+$$

and

$$u_h \rightharpoonup u \quad \text{in} \quad H \quad \text{if} \quad h \to 0+\,.$$

Since the imbedding of H into $\mathscr{H}$ is completely continuous, $u_h \to u$ in $\mathscr{H}$. The assertion is a consequence of (1.10).

2. APPROXIMATION OF CONTACT PROBLEMS BY FINITE ELEMENT METHOD

In this Section we describe the construction of finite-dimensional approximations of $\mathscr{K}$, $\mathscr{K}_\varepsilon$, i.e. of closed convex sets of admissible displacements in the problems $(\mathscr{P}_1)$, $(\mathscr{P}_2)$. For definitions – see [1].

First we introduce some notations. Let Ω', Ω'' be two bounded disjoint domains with Lipschitz boundaries $\partial\Omega'$, $\partial\Omega''$ and let us set $\Omega = \Omega' \cup \Omega''$. By $\mathscr{H}^k(\Omega)$, $k \geqq 0$ integer, we denote the space, isomorphe with $[H^k(\Omega')]^2 \times [H^k(\Omega'')]^2$, i.e.

$$u \in \mathscr{H}^k(\Omega) \Leftrightarrow u|_{\Omega'} \equiv u' \in [H^k(\Omega')]^2$$
$$u|_{\Omega''} \equiv u'' \in [H^k(\Omega'')]^2\,,$$

where $H^k(\Omega^M)$, $M = {}'\,,\,{}''$ denote the Sobolev spaces. The norm and the seminorm in $\mathscr{H}^k(\Omega)$ is defined as follows:

$$\|u\|^2_{k,\Omega} = \|u'\|^2_{k,\Omega'} + \|u''\|^2_{k,\Omega''} \tag{2.1}$$
$$|u|^2_{k,\Omega} = |u'|^2_{k,\Omega'} + \|u''|^2_{k,\Omega''}\,,$$

where $\|u^M\|_{k,\Omega^M}$ and $|u^M|_{k,\Omega^M}$ $(M = {}'\,,\,{}'')$ are the usual norms and seminorms, respectively in $[H^k(\Omega^M)]^2$. In what follows, we shall consider the problems $(\mathscr{P}_1)$, $(\mathscr{P}_2)$ separately.

2.1. APPROXIMATION OF CONTACT PROBLEMS WITH A BOUNDED CONTACT ZONE

Let us consider the following decompositions of $\partial\Omega'$, $\partial\Omega''$ (see [1] – Section 1.):

$$\partial\Omega' = \bar{\Gamma}_u \cup \bar{\Gamma}'_\tau \cup \Gamma_K\,, \quad \partial\Omega'' = \bar{\Gamma}_0 \cup \bar{\Gamma}''_\tau \cup \Gamma_K\,,$$

where Γ_u, Γ_K are non-empty parts of $\partial\Omega'$. Let us recall that

$$V = \{v \in \mathscr{H}^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_u,\ v_n = 0 \quad \text{on} \quad \Gamma_0\}$$

$$\mathscr{K} = \{v \in V \mid v'_n + v''_n \leqq 0 \quad \text{on} \quad \Gamma_K\}\,.$$

A. First let us suppose that both Ω', Ω'' are *polygonal domains*, $\bar{\Gamma}_K = \sum_{i=1}^{m} \bar{\Gamma}_{K,i}$, where $\bar{\Gamma}_{K,i}$ denotes a straight-line segment A_iA_{i+1}. Let $\mathscr{T}'_h$ and $\mathscr{T}''_h$ be triangulations of Ω' and Ω'', respectively, having common nodes on Γ_K and such that $A_1, \ldots, A_m$ as well as all boundary points of Γ_u, Γ_0, Γ'_τ, Γ''_τ, Γ_K belong to nodes of $\mathscr{T}'_h$ and $\mathscr{T}''_h$. We set $\mathscr{T}_h = \mathscr{T}'_h \cup \mathscr{T}''_h$. Let h denote the maximal side and ϑ the minimal interior angle of all triangles $T_i \in \mathscr{T}_h$. Assume that a system of $\{\mathscr{T}_h\}$, $h \to 0+$ is *regular*, i.e. a constant $\alpha > 0$ exists such that $\vartheta \geqq \alpha$ if $h \to 0+$.

We define

$$V_h = \{v \in [C(\bar{\Omega}')]^2 \times [C(\bar{\Omega}'')]^2 \cap V \mid v|_T \in [P_1(T)]^2 \quad \forall T \in \mathscr{T}_h\}\,, \tag{2.2}$$

where $P_1(T)$ is the set of linear polynomials, defined on T.

Let a^i_j, $j = 1, \ldots, m_i$ be the nodes of $\mathscr{T}_h$, lying on $\bar{\Gamma}_{K,i}$ ($a^i_1 \equiv A_i$, $a^i_{m_i} \equiv A_{i+1}$), $i = 1, \ldots, m$ and n^i be the outward unit normal of the side $\Gamma_{K,i}$, related to $\partial\Omega'$. Let us define:

$$\mathscr{K}_h = \{v \in V_h \mid n^i \,.\, (v' - v'')(a^i_j) \leqq 0\,, \quad i = 1, \ldots, m\,; \quad j = 1, \ldots, m_i\}\,. \tag{2.3}$$

It is readily seen that $\mathscr{K}_h$ are finite-dimensional approximations of $\mathscr{K}$. Moreover, it holds:

Lemma 2.1. $\mathscr{K}_h \subset \mathscr{K}$ *for every* $h \in (0, 1)$.

Proof. Let $v \in \mathscr{K}_h$. Then $n^i \,.\, (v' - v'')|_{\Gamma_{K,i}}$ is piecewise linear function on $\bar{\Gamma}_{K,i}$. Hence

$$n^i \,.\, (v' - v'') \leqq 0 \quad \text{on} \quad \bar{\Gamma}_{K,i} \Leftrightarrow n^i \,.\, (v' - v'')(a^i_j) \leqq 0\,, \quad j = 1, \ldots, m_i\,.$$

B. Next we shall consider the case, when Ω', Ω'' are domains with more *general boundaries*. For the sake of simplicity we restrict ourselves to the case, when only Γ_K is curved. Let ψ be a continuous concave (or convex) function, defined on $\langle a, b\rangle$ (see Fig. 1), the graph of which is $\bar{\Gamma}_K$. We choose $(m + 1)$ points $A_1, \ldots, A_{m+1}$ on Γ_K in such a way that A_1 and A_{m+1} are boundary points of $\bar{\Gamma}_K$. Let A_i, $A_{i+1} \in \bar{\Gamma}_K$, $S \in \Omega^M$,

$M = '\ ,\ ''$. By a curved element T we call a closed set bounded by the straight – lines SA_i, SA_{i+1} and the arc $\widehat{A_iA_{i+1}}$ such that $T \subset \bar{\Omega}^M$, $M = '\ ,\ ''$. The minimal interior angle of the triangle $A_iA_{i+1}S$ is called the minimal angle of the curved element T. A triangulation $\mathscr{T}_h$ of $\Omega = \Omega' \cup \Omega''$ contains curved elements along Γ_K and internal triangular elements. By the symbols h and ϑ we denote the maximal of diameter and the minimal interior angle, respectively, of all elements $T \in \mathscr{T}_h$. Analogously as in the previous case we define a regular system of triangulations.

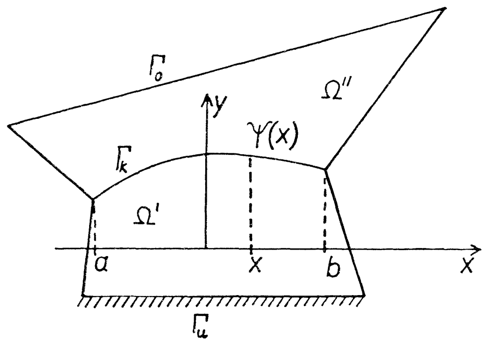

Fig. 1.

Define

$$V_h = \{v \in [C(\bar{\Omega}')]^2 \times [C(\bar{\Omega}'')]^2 \cap V | v_{|T} \in [P_1(T)]^2 \quad \forall T \in \mathscr{T}_h\} \tag{2.4}$$

$$\mathscr{K}_h = \{v \in V_h \mid n \,.\, (v' - v'')(A_i) \leqq 0\,, \quad i = 1, \ldots, m + 1\}\,. \tag{2.5}$$

It is easy to see that also in this case $\mathscr{K}_h$ represents a finite – dimensional approximation of $\mathscr{K}$, but $\mathscr{K}_h \notin \mathscr{K}$, in general.

Let

$$\mathscr{L}(v) = \tfrac{1}{2} \int_\Omega \tau_{ij}(v)\, \varepsilon_{ij}(v)\, \mathrm{d}x - \int_\Omega F \,.\, v \,\mathrm{d}x - \int_{\Gamma_\tau' \cup \Gamma_\tau''} P \,.\, v \,\mathrm{d}s \tag{2.6}$$

be the functional of the total potential energy, $F \in \mathscr{H}^0(\Omega)$, $P \in [L^2(\Gamma_\tau')]^2 \times [L^2(\Gamma_\tau'')]^2$, $\varepsilon_{ij}(v) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$, $\tau_{ij}(v) = c_{ijkl}\varepsilon_{kl}(v)$; the coefficients c_{ijkl} are bounded and measurable in Ω,

$$c_{ijkl} = c_{jikl} = c_{klij} \quad \text{a.e.} \quad \text{in } \Omega$$

and a positive constant c_0 exists such that

$$c_{ijkl}e_{ij}e_{kl} \geqq c_0 e_{ij}e_{ij} \quad \text{a.e.} \quad \text{in } \Omega^{1)} \tag{2.7}$$

holds for any symmetric e_{ij}.

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[1]) A repeated Latin index implies summation over the range 1,2.

An approximation of the contact problem with a bounded contact zone is defined as the solution of the following problem:

$(\mathscr{P}_{1h})$ *find* $u_h \in \mathscr{K}_h$ *such that*

$$\mathscr{L}(u_h) = \min_{v \in \mathscr{K}_h} L(v) ,$$

where $\mathscr{K}_h$ is given by (2.3) or (2.5).

2.2. APPROXIMATION OF CONTACT PROBLEMS WITH AN ENLARGING CONTACT ZONE

Let Ω', Ω'' be bounded domains with the following decomposition of the boundaries $\partial\Omega'$, $\partial\Omega''$:

$$\partial\Omega' = \bar{\Gamma}_u \cup \bar{\Gamma}'_\tau \cup \Gamma'_K , \quad \partial\Omega'' = \bar{\Gamma}_0 \cup \bar{\Gamma}''_\tau \cup \Gamma''_K ,$$

where

$$\Gamma'_K = \{(\xi, \eta) \mid a \leqq \eta \leqq b , \quad \xi = f'(\eta)\}$$

$$\Gamma''_K = \{(\xi, \eta) \mid a \leqq \eta \leqq b , \quad \xi = f''(\eta)\} ,$$

f', f'' are continuous functions on $\langle a, b\rangle$ (for details see [1]). The space V is defined as in the previous case. Define

$$\mathscr{K}_\varepsilon = \{v \in V \mid v''_\xi - v'_\xi \leqq \varepsilon \quad \forall \eta \in \langle a, b\rangle\} ,$$

where $\varepsilon(\eta) = f'(\eta) - f''(\eta)$ is the distance of Γ'_K, Γ''_K before the deformation and v'_ξ, v''_ξ are projections of v', v'' into the *fixed* direction ξ.

For simplicity we restrict ourselves to the case when only Γ'_K, Γ''_K are *curved* and the functions f', f'', describing these arcs are twice continuously differentiable on $\langle a, b\rangle$. Curved element T is defined in the same way as in the previous case-Part B. For the construction of finite – dimensional spaces on T, we use the technique, developed in [3].

Let $\hat{T}$ be the triangle with the vertices: [0, 0], [1, 0], [0, 1]. Let $A_i, A_{i+1} \in \Gamma'_K$, $S \in \Omega'$ (for example), and let $x = \varphi(s)$, $y = \psi(s)$, $s \in \langle 0, 1\rangle$, $\varphi, \psi \in C^2(\langle 0, 1\rangle)$ be a parametric representation of the arc $\widehat{A_i A_{i+1}}$ and T the curved element, determined by A_i, A_{i+1}, S. Then we can construct the mapping $F_T : R_2 \to R_2$, which is C^1-diffeomorphism $\hat{T}$ onto T. Let $\hat{P} = P_1$ be a set of linear polynomials defined on $\hat{T}$. Then we set

$$P(T) = \{p \mid \exists \hat{p} \in \hat{P} : p = \hat{p} \circ F_T^{-1}\} , \tag{2.8}$$

where $F_T^{-1}(T) = \hat{T}$.

The triangulation $\mathscr{T}_h = \mathscr{T}'_h \cup \mathscr{T}''_h$ of Ω consists of curved elements along Γ'_K, Γ''_K and interior triangles.

Elements along Γ'_K, Γ''_K are constructed in the following way: let $\{C_j\}_{j=1}^m$ be a division of $\langle a, b\rangle$, $C_1 \equiv a$, $C_m \equiv b$, A_j, B_j being the intersections of perpendicular lines at C_j with Γ'_K, Γ''_K, respectively. Points A_j and B_j coincide with the nodes of $\mathcal{T}_h$ on Γ'_K and Γ''_K, respectively (cf. Fig. 2).

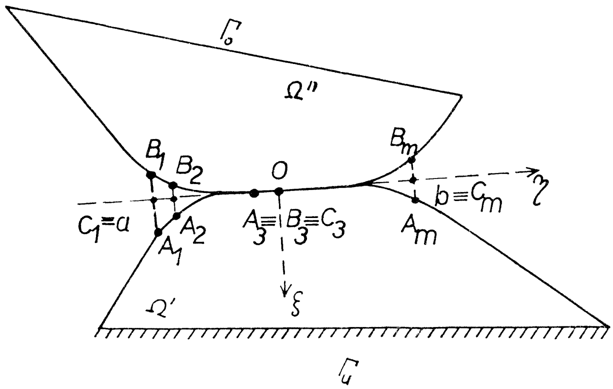

Fig. 2.

Let us define

$$V_h = \{v \in V \mid v_{|T} \in [P(T)]^2 \quad \forall T \in \mathcal{T}_h\}, \tag{2.9}$$

where $P(T) = P_1(T)$ if T is a triangle or $P(T)$ is defined by (2.8) if T is a curved element. Let

$$\mathcal{K}_{\varepsilon h} = \{v \in V_h \mid v''_\xi(B_j) - v'_\xi(A_j) \leqq \varepsilon(C_j),\quad j = 1, \dots, m\}. \tag{2.10}$$

It is easy to see that $\mathcal{K}_{\varepsilon h}$ is the finite – dimensional approximation of $\mathcal{K}_\varepsilon$ and $\mathcal{K}_{\varepsilon h} \not\subset \mathcal{K}_\varepsilon$, in general.

An approximation of the contact problem with *an enlarging contact* zone is defined as the solution of the following problem:

$(\mathcal{P}_{2h})$ *find* $u_h \in \mathcal{K}_{\varepsilon h}$ *such that*

$$\mathcal{L}(u_h) = \min_{v \in \mathcal{K}_{\varepsilon h}} \mathcal{L}(v)$$

3. ERROR ESTIMATES

In this Section we establish the rate of convergence of approximate solutions u_h (defined in the previous Section), provided the exact solution is smooth enough. We shall analyze the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ separately. The results of Section 1 will be used with

$$A(u, v) = \int_\Omega \tau_{ij}(u)\, \varepsilon_{ij}(v)\, dx, \tag{3.1}$$

(3.2) $$f(v) = \int_\Omega F \,.\, v \,\mathrm{d}x + \int_{\Gamma_{j'} \cup \Gamma_{j''}} P \,.\, v \,\mathrm{d}s\,.$$

First we recall the well-known Green's formula. Let $Q \subset R_2$ be a domain with a Lipschitz boundary ∂Q. Let us introduce

$$Y(Q) = \{\tau \in [L^2(Q)]^4 \mid \tau_{ij} = \tau_{ji} \quad \text{a.e. in} \quad Q\}$$
$$\hat{Y}(Q) = \{\tau \in Y(Q) \mid \tau_{ij,j} \in L^2(Q), \quad i = 1, 2\}\,,$$

where $\tau_{ij,j} = \partial\tau_{ij}/\partial x_j$ is taken in the distribution sense. Then there exists a unique $T \in \mathscr{L}(Y(Q), (H^{-1/2}(\partial Q))^2)$ such that[1])

(3.3) $$\int_\Omega \tau_{ij}\, \varepsilon_{ij}(v) \,\mathrm{d}x = -\int_\Omega \tau_{ij,j} v_i \,\mathrm{d}x + \langle T, v\rangle \quad \forall \tau \in \hat{Y}(Q)\,, \quad \forall v \in [H^1(Q)]^2\,,$$

where $\langle\ ,\ \rangle$ denotes the duality between $[H^{-1/2}(\partial Q)]^2$ and $[H^{1/2}(\partial Q)]^2$. If $T \in [L^2(\partial Q)]^2$ then

$$\langle T, v\rangle = \int_{\partial Q} T_i v_i \,\mathrm{d}s$$

Using (3.3) with $Q = \Omega'$ and $Q = \Omega''$, respectively, we obtain

(3.4) $$\int_\Omega \tau_{ij}\, \varepsilon_{ij}(v) \,\mathrm{d}x = -\int_\Omega \tau_{ij,j} v_i \,\mathrm{d}x + \langle T', v'\rangle_{\partial\Omega'} + \langle T'', v''\rangle_{\partial\Omega''}$$
$$\forall \tau \in \hat{Y}(\Omega)\,, \quad \forall v \in \mathscr{H}^1(\Omega)\,,$$

where $T^M \in [H^{-1/2}(\partial\Omega^M)]^2$ and $\langle\ ,\ \rangle_{\partial\Omega^M}$ denotes the duality between $[H^{-1/2}(\partial\Omega^M)]^2$ and $[^{1/2}(\partial\Omega^M)]^2$, $M = {}'\,,\ {}''$. Henceforth *we assume for simplicity* that $T' \in [L^2(\partial\Omega')^2$, $T'' \in [L^2(\partial\Omega'')]^2$. Finally, let us denote

$$|v|^2 = \int_\Omega \varepsilon_{ij}(v)\, \varepsilon_{ij}(v) \,\mathrm{d}x\,.$$

3.1. CONTACT PROBLEMS WITH A BOUNDED CONTACT ZONE

Let the weak solution u of the problem $(\mathscr{P}_1)$ be such that $\tau^M(u) \in \hat{Y}(\Omega^M)$, $M = {}'\,,\ {}''$. Then using the definition of $(\mathscr{P}_1)$ and (3.4) we obtain (cf. [1] – Th. 1.1):

(3.5) $$\tau_{ij,j}(u) + F_i = 0 \quad \text{in} \quad \Omega\,, \quad i = 1, 2;$$

(3.6) $$u = 0 \quad \text{on} \quad \Gamma_u\,;$$

(3.7) $$\tau_{ij}(u)\, n_j = P_i \quad \text{on} \quad \Gamma_\tau \equiv \Gamma'_\tau \cup \Gamma''_\tau\,, \quad i = 1, 2;$$

(3.8) $$u_n = 0\,, \quad T_t(u) = 0 \quad \text{on} \quad \Gamma_0\,;$$

[1]) $H^{1/2}(\partial Q)$ is the space of traces of all functions belonging to $H^1(Q)$, $H^{-1/2}(\partial Q)$ denotes the dual space to $H^{1/2}(\partial Q)$. We shall write simply $\langle T, v\rangle$ instead of $\langle T(\tau), v\rangle$.

(3.9) $$u'_n + u''_n \leqq 0 \quad \text{on} \quad \Gamma_K\,,$$

(3.10) $$T'_n(u') = T''_n(u'') \equiv T_n(u) \leqq 0 \quad \text{on} \quad \Gamma_K\,,$$

(3.11) $$T_t(u') = T_t(u'') \quad = 0 \quad \text{on} \quad \Gamma_K\,,$$

(3.12) $$T_n(u) \,.\, (u'_n + u''_n) = 0 \quad \text{on} \quad \Gamma_K\,,$$

where $T_n(u)$ and $T_t(u)$ are the normal and tangential components of the stress vector, respectively.

A. Let us suppose that Ω', Ω'' are *polygonal domains*, $\mathscr{K}_h$ is defined by (2.3).

Theorem 3.1. *Let there exist solutions u, u_h* of $(\mathscr{P}_1)$, $(\mathscr{P}_{1h})$, *respectively, such that* $u \in \mathscr{H}^2(\Omega) \cap \mathscr{K}$, $\tau^M(u) \in \hat{Y}(\Omega^M)$, $M = {}'\,,\ {}''$, $u', u'' \in [W^{1,\infty}(\Gamma_{K,i})]^2$, $i = 1, \ldots, m$. *Moreover, let us suppose that the number of points on Γ_K, where the contact changes from binding to nonbinding, is finite. Then*

(3.13) $$|u - u_h| \leqq ch\{|u|^2_{2,\Omega} + \sum_{i=1}^{m} \|T_n(u)\|_{\infty,\Gamma_{K,i}} (|u'|_{1,\infty,\Gamma_{K,i}} + |u''|_{1,\infty,\Gamma_{K,i}})\}^{1/2}$$

if the system $\{\mathscr{T}_h\}$, $h \to 0+$ is regular.

Proof. From (1.1), Remarks 1.1, 1.3 and Lemma 2.1 we obtain

(3.14) $$\tfrac{1}{2} A(u - u_h, u - u_h) \leqq f(u - v_h) + \tfrac{1}{2} A(u - v_h, u - v_h) + A(u, v_h - u)$$
$$\forall v_h \in \mathscr{K}_h\,.$$

The definition of $(\mathscr{P}_1)$ and the Green's formula, (3.5)–(3.8), (3.10) and (3.11) yields:

(3.15) $$f(u - v_h) + A(u, v_h - u) = \int_{\Gamma_K} T_n(u) \{(v'_{hn} - u'_n) - (v''_{hn} - u''_n)\} \, ds$$
$$\forall v_h \in \mathscr{K}_h\,.$$

(In the following the normal n will be related to Ω' only.) To prove (3.13) we make a special choice of v_h, namely $v_h = u_I$, where u_I is a *piecewise linear Lagrange interpolate* of u over the triangulation $\mathscr{T}_h$. It is readily seen that $u_I \in \mathscr{K}_h$. In fact,

$$n^i \,.\, (u'_I - u''_I) \,.\, (a^i_j) = n^i \,.\, (u' - u'')\,(a^i_j) \leqq 0\,.$$

Using the well – known approximative properties of u_I, we deduce

(3.16) $$|A(u_I - u, u_I - u)| \leqq c|u - u_I|^2 \leqq \bar{c}h^2|u|^2_{2,\Omega}$$

Let

$$\Gamma^-_{K,i} = \{x \in \Gamma_{K,i} \mid (u'_n - u''_n)(x) < 0\}$$

$$\Gamma^0_{K,i} = \{x \in \Gamma_{K,i} \mid (u'_n - u''_n)(x) < 0\}\,.$$

Denoting $\mathscr{U}_I \equiv (u_I - u''_I) \,.\, n^i = [(u' - u'') \,.\, n^i]_I$, we have

$$(3.17) \qquad \mathscr{U}_I \equiv 0$$

on every side $s^i_j = a^i_j a^i_{j+1} \subseteq \Gamma^0_{K,i}$. Moreover, $T_n(u) = 0$ a.e.on $\Gamma^-_{K,i}$ follows from (3.12). From this and (3.17) we obtain

$$f(u - u_I) + A(u, u_I - u) = \sum_{i=1}^{m} \sum_{s_j{}^i \in \mathscr{J}_i} \int_{s_j{}^i} T_n(u) \left[(u'_n - u''_n)_I - (u'_n - u''_n)\right] \mathrm{d}s \,,$$

where $\mathscr{J}_i$ is the system of all $s^i_j \subseteq \Gamma_{K,i}$ containing both points of $\Gamma^0_{K,i}$ and $\Gamma^-_{K,i}$. Using the assumptions of the Theorem, we have

$$\int_{s_i{}^j} T_n(u) \left[\mathscr{U}_I - (u'_n - u''_n)\right] \mathrm{d}s \leqq h \|T_n(u)\|_{\infty, s_j{}^i} \,.\, \|\mathscr{U}_I - (u'_n - u''_n)\|_{\infty, s_j{}^i} \leqq$$

$$\leqq ch^2 \|T_n(u)\|_{\infty, s_j{}^i} \left|u'_n - u''_n\right|_{1,\infty, s_j{}^i} \leqq ch^2 \|T_n(u)\|_{\infty, s_j{}^i} \,.$$

$$.\, \left(|u'|_{1,\infty, s_j{}^i} + |u''|_{1,\infty, s_j{}^i}\right) .$$

Due to the assumptions, the number of all $s^i_j \in \mathscr{J}_i$, $i = 1, \ldots, m$ can be bounded from above independently of h. Using (3.14) and (3.16), the assertion (3.13) follows.

Remark 3.1. Some sufficient conditions for the existence and uniqueness of solution of the problem $(\mathscr{P}_1)$ are given in [1]. Since $\mathscr{K}_h \subset \mathscr{K}$, the same conditions are true for the problem $(\mathscr{P}_{1h})$. The previous Theorem 3.1 however, doesn't require the uniqueness of the solutions.

Remark 3.2. The same rate of convergence $O(h)$ can be obtained for example if $u \in \mathscr{H}^2(\Omega) \cap \mathscr{K}$, $\tau^M(u) \in \hat{Y}(\Omega)^M)$, $M = {}'\,,\,{}''$ and $u'_n, u''_n \in H^2(\Gamma_{K,i})$ $\forall i = 1, \ldots, m$. Under the single assumptions $u \in \mathscr{H}^2(\Omega) \cap \mathscr{K}$, $\tau^M(u) \in \hat{Y}(\Omega)^M)$, $M = {}'\,,\,{}''$ we obtain

$$|u - u_h| = O(h^{3/4}) \,, \quad h \to 0+ \,.$$

Remark 3.3. Let on $\Gamma_{K0} \subset \Gamma_K$ (given a priori) Ω' and Ω'' are in a bilateral contact i.e. $u'_n + u''_n = 0$ on Γ_{K0} and let Γ_{K0} be such that the rigid virtual displacements, satisfying the bilateral contact, reduce to zero field, i.e.

$$v \in \mathscr{R} \cap V, \quad v'_n + v''_n = 0 \quad \text{on} \quad \Gamma_{K0} \Leftrightarrow v \equiv 0 \,.$$

Then using the inequality of Korn's type, we obtain the rate of convergence in $\mathscr{H}^1(\Omega)$ – norm.

In the above error estimates we needed very strong regularity assumptions, concerning the solution u. Unfortunately, there are no reasons to expect such a great smoothness in a general case. This is why we are going to prove the convergence of u_h to u without estimating the rate of convergence, using no regularity hypothesis. To this end we need the following.

Lemma 3.1. *Let us suppose that* $\Gamma_K \cap \bar{\Gamma}_u = \emptyset$, $\Gamma_K \cap \bar{\Gamma}_0 = \emptyset$ and *there exists only a finite number of boundary points* $\bar{\Gamma}_\tau \cap \Gamma_K$, $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_\tau \cap \bar{\Gamma}_0$. *Then the set*

$$\mathfrak{M} = \mathscr{K} \cap [C^\infty(\bar{\Omega}')]^2 \times [C^\infty(\bar{\Omega}'')]^2$$

is dense in $\mathscr{K}$ *in* $\mathscr{H}^1(\Omega)$ *– norm.*

Proof. Let $u \in \mathscr{K}$ be a fixed arbitrary function. Consider a system of open domains $\{B_i\}_{i=0}^r$, which cover $\Omega' \cup \Omega''$, such that: $\bar{\bar{B}}_0 \subset \Omega'$, $\bar{\bar{B}}_1 \subset \Omega''$,

$$\Gamma_K \subset \bigcup_{j=2}^{k} B_j\,, \quad (k < r)$$

$$\Gamma_K \cap B_i \neq \emptyset \Leftrightarrow 2 \leqq i \leqq k\,.$$

We say that a point $P \in \partial\Omega' \cup \partial\Omega''$ is a *singular point*, if P is either a vertex of the polygonal boundary or a point of

$$\Gamma_K \cap \bar{\Gamma}_\tau\,, \quad \bar{\Gamma}_u \cap \bar{\Gamma}_\tau \quad \text{or} \quad \bar{\Gamma}_0 \cap \bar{\Gamma}_\tau\,.$$

We may assume that each B_j contains at most one singular point. Denote $\{\varphi_i\}$, $i = 0, 1, \ldots, r$, the corresponding decomposition of unity (i.e. $\varphi_i \in C_0^\infty(B_i)$, $0 \leqq \varphi_i \leqq 1$, $\sum_{i=0}^r \varphi_i(x) = 1\ \forall x \in \bar{\Omega}' \cup \bar{\Omega}''$). Introducing

$$u^j = u\varphi_j\,, \quad j = 0, 1, \ldots, r\,,$$

we have supp $u^j \in B_j$, $u^j \in \mathscr{H}^1(\Omega)$, $\sum_{j=0}^r u^j = u$. For each u^j we shall construct infinitely differentiable and close functions satisfying the boundary conditions. To this end we divide the system $\{B_i\}$ into several groups.

1. group. Let $j \leqq k$ and $\{B_j\}$ do not contain any singular point. Introduce local Cartesian coordinates (ξ, η), where ξ-axis coincides with the tangent and η-axis with the normal n' with respect to Γ_K. Then (omitting the indeces j), we have

$$\Gamma_K \cap B = \{(\xi, \eta) |\ |\xi| < \xi_0,\ \eta = 0\}\,,$$

$$u^M = u_\xi^M e_\xi + u_\eta^M e_\eta\,, \quad M = '\,,\,''\,,$$

(e_ξ, e_η are unit basis vectors),

$$u_n' + u_n'' = u_\eta' - u_\eta'' \leqq 0 \quad \text{on} \quad \Gamma_K\,. \tag{3.18}$$

Let us extend u_η' into $B \cap \Omega''$ and u_η'' into $B \cap \Omega'$ in such a way that the extension is even with respect to η. Regularizing the extended function Eu_η' by means of the kernel $\omega(x, \varkappa)$, we obtain

$$R_\varkappa E\, u_\eta'(x) = \int_B \omega(x - x')\, E\, u_\eta'(x')\, \mathrm{d}x'\,, \quad x' = (\xi', \eta')\,. \tag{3.19}$$

There exists a function $v \in H^1(B)$ such that

$$v \leqq 0 \quad \text{in} \quad B\,, \quad \operatorname{supp} v \subset B\,,$$

$$v = u'_\eta - u''_\eta \leqq 0 \quad \text{on} \quad \Gamma_K$$

(see e.g. [7] – chpt. 2, the proof of Th. 5.7). Then

$$Eu'_\eta - Eu''_\eta = v + z\,, \tag{3.20}$$

where $z \in H^1(B)$, the restriction $z|_{\Omega^M} \in H^1_0(B \cap \Omega^M)$, $M = {}'\,,\,{}''$.

Obviously, it holds

$$R_\varkappa v \leqq 0 \quad \text{on} \quad \Gamma_K\,, \quad R_\varkappa v \in C_0^\infty(B)\,,$$

$$R_\varkappa v \to v \quad \text{in} \quad H^1(B) \quad \text{for} \quad \varkappa \to 0$$

and aproximations $z^M_\varkappa \in C_0^\infty(B \cap \Omega^M)$ exist such that $z^M_\varkappa \to z|_{\Omega^M}$ in $H^1(B \cap \Omega^M)$. Consequently, we have

$$R_\varkappa v + z_\varkappa \leqq 0 \quad \text{on} \quad \Gamma_K\,, \quad \text{setting} \quad z_\varkappa = (z'_\varkappa, z''_\varkappa) \in C_0^\infty(B)\,,$$

$$R_\varkappa v + z_\varkappa \to v + z \quad \text{in} \quad H^1(B)\,. \tag{3.21}$$

Let us set

$$u'_{\eta\varkappa} = R_\varkappa E u'_\eta|_{\Omega'}\,,$$

$$u''_{\eta\varkappa} = [R_\varkappa E u'_\eta - (R_\varkappa v + z_\varkappa)]|_{\Omega''}\,.$$

By virtue of (3.19), (3.20) and (3.21) it holds

$$u^M_{\eta\varkappa} \to u^M_\eta \quad \text{in} \quad H^1(B \cap \Omega^M) \quad \text{for} \quad \varkappa \to 0\,. \tag{3.22}$$

Moreover, we have

$$u'_{\eta\varkappa} - u''_{\eta\varkappa} = R_\varkappa v + z_\varkappa \leqq 0 \quad \text{on} \quad \Gamma_K\,. \tag{3.23}$$

The components u^M_ξ can be regularized by an arbitrary way.

2. group. Let $j \leqq k$ and B_j contain a vertex $P \in \Gamma_K$. We use a skew coordinate basis for the components of u. Let e^1, e^2 be unit tangential and n^1, n^2 unit outward

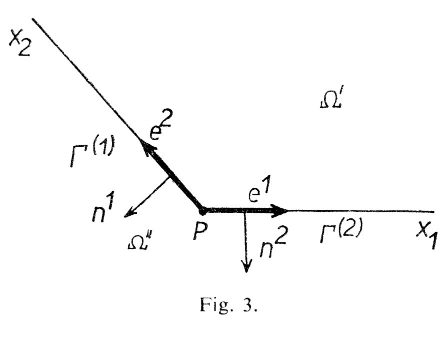

Fig. 3.

normal vectors, with respect to $\partial\Omega'$ – see Fig. 3. Then we may write (omitting indeces j)

$$u' = \sum_{p=1}^{2} \frac{u'^{(p)}}{e^p \, . \, n^p} e^p ,$$

where $u'^{(p)} = u' \, . \, n^p$, i.e.

$$u'^{(p)} = u'_n \quad \text{on} \quad \Gamma^{(p)} , \quad p = 1, 2 .$$

The same decomposition is valid for u'' and

$$u''^{(p)} = - u''_n \quad \text{on} \quad \Gamma^{(p)} , \quad p = 1, 2 .$$

Altogether we have

$$u'_n + u''_n = u'^{(p)} - u''^{(p)} \leqq 0 \quad \text{on} \quad \Gamma^{(p)} .$$

First let us consider the components $u^{M(2)}$. We map the angular domain $B \cap \Omega'$ into the upper halfplane $\{(\xi, \eta) \mid \eta > 0\}$ by means of a Lipschitz mapping T such that $T\Gamma^{(2)}$ is the positive and $T\Gamma^{(1)}$ the negative ξ-axis. Let us extend the functions

$$\hat{u}^{M(2)}(\xi, \eta) = u^{M(2)}(T^{-1}(\xi, \eta))$$

across the ξ-axis to get functions even in η. Set

$$E\, u^{M(2)}(x) = E\, \hat{u}^{M(2)}(T(x)) .$$

Regularizing, we obtain

$$R_\varkappa E\, u'^{(2)} \in C_0^\infty(B) .$$

Let us extend the function

$$\mathscr{U} = \hat{u}'^{(2)} - \hat{u}''^{(2)} \leqq 0 \quad \text{on} \quad T\Gamma^{(2)}$$

from the positive onto the negative ξ-axis in such a way that the extension $E\mathscr{U}$ is even in ξ. Then a function $\hat{v} \in H^1(TB)$ exists such that $\hat{v} = E\mathscr{U}$ on the ξ-axis, $\hat{v} \leqq 0$ in TB, supp $\hat{v} \subset TB$. Defining

$$v(x) = \hat{v}(Tx) ,$$

then $v \in H^1(B)$, supp $v \subset B$, $v \leqq 0$ in B, $v = u'^{(2)} - u''^{(2)}$ on $\Gamma^{(2)}$.

Consequently, we may write

$$Eu'^{(2)} - Eu''^{(2)} = v + z ,$$

where

$$z \in H^1(B) , \quad \text{supp}\, z \subset B , \quad z = 0 \quad \textbf{on} \quad \Gamma^{(2)} .$$

Regularizing, we obtain

(3.24) $$R_\varkappa v \leqq 0 \quad \text{on} \quad \Gamma , \quad R_\varkappa v \to v \quad \text{in} \quad H^1(B) , \quad \varkappa \to 0 .$$

There exists a function $w \in H^1(B)$, supp $w \subset B$, such that $w = z$ on $\Gamma^{(1)} \cup \Gamma^{(2)}$ and $w = 0$ in an "angular neighbourhood" $|\vartheta| < \vartheta_0$ of the part $\Gamma^{(2)}$. Defining a "shifted" function

$$w_\lambda(x) = w(x + \lambda e^1), \quad \lambda \in \mathbb{R}^1, \quad \lambda > 0,$$

then for $\varkappa < C\lambda$ it holds $R_\varkappa w_\lambda = 0$ on $\Gamma^{(2)}$,

$$(3.25) \qquad \|R_\varkappa w_\lambda - w\| \leqq \|R_\varkappa w_\lambda - w_\lambda\| + \|w_\lambda - w\| \to 0 \quad \text{for} \quad \lambda \to 0,$$

with the norms in $H^1(B)$.

Furthermore, since we have

$$z = w + z_0, \quad z_0|_{\Omega^M} \in H_0^1(B \cap \Omega^M),$$

$$(3.26) \qquad R_\varkappa w_\lambda + z_{0\varkappa} = 0 \quad \text{on} \quad \Gamma^{(2)}, \quad z_{0\varkappa}^M \in C_0^\infty(B \cap \Omega^M),$$

$$R_\varkappa w_\lambda + z_{0\varkappa} \to w + z_0 = z \quad \text{in} \quad H^1(B).$$

Let us set $u_\varkappa'^{(2)} \equiv R_\varkappa E u'^{(2)}|_{\Omega'}$

$$u_\varkappa''^{(2)} = [R_\varkappa E u'^{(2)} - (R_\varkappa v + R_\varkappa w_\lambda + z_{0\varkappa})]|_{\Omega''}$$

Then we obtain on the basis of (3.24), (3.25), (3.26) that

$$(3.27) \qquad u_\varkappa^{M(2)} \to u^{M(2)} \quad \text{in} \quad H^1(\Omega^M \cap B)$$

and

$$u_\varkappa'^{(2)} - u_\varkappa''^{(2)} = R_\varkappa v + R_\varkappa w_\lambda + z_{0\varkappa} \leqq 0 \quad \text{on} \quad \Gamma^{(2)}.$$

The component $u^{M(1)}$ can be treated in a parallel way. Since the Cartesian components w_k of an arbitrary vector w can be written as

$$w_k = a_1 w^{(1)} + a^2 w^{(2)}$$

with fixed constants a_1, a_2, it holds

$$\|w_k\|^2 \leqq C \sum_{p=1}^{2} \|w^{(p)}\|^2, \quad k = 1, 2.$$

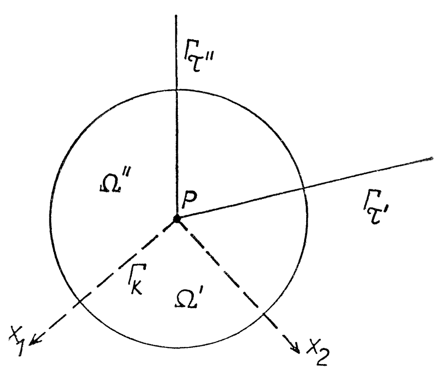

Fig. 4.

Hence it follows that defining

$$u_\varkappa^M = \sum_{p=1}^{2} u_\varkappa^{M(p)} e^p / e^p \, . \, n^p \, , \quad M = {}' \, , {}'' \, ,$$

the relations (3.27) imply

$$\|u_\varkappa^j - u^j\|_{1,\Omega} \to 0 \, , \quad \text{for} \quad \varkappa \to 0 \, , \quad \lambda \to 0 \, , \quad \varkappa < C\lambda \, .$$

3. group. Let $j \leqq k$ and let B_j contain a point $P \in \bar{\Gamma}_K \cap \bar{\Gamma}_\tau$. We place the cartesian system into the point P so that x_1-axis coincides with Γ_K. (see Fig. 4). Then we have

$$u_n' + u_n'' = -u_2' + u_2'' \leqq 0 \quad \text{on} \quad \Gamma_K \, .$$

We can proceed in a way analogous to that of the 2. group, replacing the components $u^{M(2)}$ by u_2^M and the ray $\Gamma^{(2)}$ by Γ_K.

4. group. Let B_j contain a point $P \in \bar{\Gamma}_0 \cap \bar{\Gamma}_\tau$, which can also be a vertex. If the cartesian local system is such that x_1-axis coincides with Γ_0, then $u_n'' = \pm u_2'' = 0$ holds on Γ_0. There exists a function $v \in H^1(B_j \cap \Omega'')$ such that supp $v \subset B_j$, $v = u_2''$ on $\partial\Omega''$, $v = 0$ in an angular neighbourhood of the ray Γ_0 (with the vertex at P).

If we shift v to get $v_\lambda(x) = v(x + \lambda)$ with a suitably chosen vector λ, then $R_\varkappa v_\lambda = 0$ on Γ_0, $R_\varkappa v_\lambda \to v$ in $H^1(B_j)$ for $\varkappa < C|\lambda|$, $|\lambda| \to 0$. Since $u_2'' = v + z$, where $z \in H_0^1(B_j \cap \Omega'')$, we have $u_{2\varkappa}'' = 0$ on Γ_0 and

$$u_{2\varkappa}'' \equiv R_\varkappa v_\lambda + z_\varkappa \to u_2'' \quad \text{in} \quad H^1(B_j \cap \Omega'') \, .$$

5. group. Let B_j contain a point $P \in \bar{\Gamma}_u \cap \bar{\Gamma}_\tau$. The approach, used for u_2'' in the 4. group, can be applied to both comonents u_k', $k = 1, 2$.

The cases of $B_j \cap \partial\Omega' \subset \Gamma_u$, $B_j \cap \partial\Omega'' \subset \Gamma_0$ and $B_j \cap \partial\Omega^M \subset \Gamma_\tau$, as well as of B_0, B_1 are easy.

Finally, defining

$$u_\varkappa^M = \sum_{j=0}^{r^M} u_\varkappa^{Mj} \, , \quad M = {}' \, , {}'' \, ,$$

we obtain

$$\|u_\varkappa - u\|_{1,\Omega} \to 0 \quad \text{for} \quad \varkappa \to 0 \, , \quad \lambda \to 0 \, , \quad \varkappa < C|\lambda| \, .$$

Moreover, the functions $u_\varkappa$ are infinitely differentiable and satisfy all the boundary condition involved in the definition of K.

Theorem 3.2. *Let $\mathscr{L}$ be coercive on $\mathscr{K}$ and let $(\mathscr{P}_1)$ have precisely one solution. Let the assumptions of Lemma 3.1 be satisfied. Then for any regular system of triangulations $\{\mathscr{T}_h\}$ we have*

$$\|u - u_h\|_{1,\Omega} \to 0 \, , \quad h \to 0+ \, .$$

Proof. It follows from Theorem 1.2, Remarks 1.4, 1.5 and 1.6. It remains to verify (1.6). Since the Lagrange interpolate v_I of any $v \in \mathfrak{M}$ belongs to $\mathscr{K}_h$ and $\mathfrak{M}$ is dense in $\mathscr{K}$, the condition (1.6) is true.

B. Assume that Γ_k is a *curved arc*. Let $\mathscr{K}_h$ be defined by means of (2.5). The main difficulty consists in the fact that $\mathscr{K}_h \not\subset \mathscr{K}$, in general. First we present several auxiliary results.

Lemma 3.2. *Let $Q \subset R_2$ be a bounded convex domain, the boundary of which is twice continuously differentiable and let $\{\mathscr{T}_h\}$, $h \to 0+$ be a strongly $\alpha-\beta$-regular system of triangulations,*[1]) where $\beta = 2$ and *the maximal straight side in $\mathscr{T}_h$ is not greater than the maximal chord* of ∂Q. Then

$$\|u - u_I\|_{0,\partial Q} \leqq c h^{3/2} \|u\|_{2,Q} \quad \forall u \in H^2(Q), \tag{3.28}$$

where u_I denotes the piecewise linear Lagrange interpolate of u, $c > 0$ *is independent of u, h.*

Proof – see [5].

Lemma 3.3. *Let p be a linear function, defined on the interval $\langle a, b\rangle$. Then*

$$\|p\|_{1,\langle a,b\rangle} \leqq c(b - a)^{-1/2} \|p\|_{1/2,\langle a,b\rangle} . \tag{3.29}$$

Lemma 3.4. *Assume that the arc $\widehat{A_i A_{i+1}}$ is the curved side of a curved triangular element. Let $v \in P_1(T)$ and T_h be the triangle generated by replacing the curved side by its chord. Then*

$$\|v\|^2_{1,\Delta(T,T_h)} \leqq ch\|v\|^2_{1,T_h} ,$$

where $\Delta(T, T_h) = (T - T_h) \cup (T_h - T)$, $c > 0$ *is independent of h and of v.*

Proof – see [4], p. 199.

Theorem 3.3. *Let the problems* $(\mathscr{P}_1)$ and $(\mathscr{P}_{1h})$ *have solutions u and u_h, respectively. Let $u \in \mathscr{H}^2(\Omega) \cap \mathscr{K}$, $\tau(u^M) \in \hat{Y}(\Omega^M)$, $M = {}'\,,\, {}''$, $T_n(u) \in L^2(\Gamma_K)$ and let the norms $\|u_h\|_{1,\Omega}$ remain bounded. Assume that the system $\{\mathscr{T}_h\}$, $h \to 0+$ is strongly $\alpha-\beta$-regular with $\beta = 2$ and the maximal straight side in $\mathscr{T}_h$ is not greater than the maximal chord. Let the function ψ, describing Γ_K, be three times continuously differentiable. Then*

$$|u - u_h| \leqq c(u)\, h^{3/4} , \quad h \to 0+ . \tag{3.30}$$

Proof. Using the definition of $(\mathscr{P}_1)$, (1.1), Remark 1.3 and the Green's formula, we deduce

$$\tfrac{1}{2} A(u - u_h, u - u_h) \leqq \tfrac{1}{2} A(v_h - u; v_h - u) + \int_{\Gamma_K} T_n(u) \left[(v'_n - u'_{hn}) - \right.$$
$$\left. - (v''_n - u''_{hn})\right] \mathrm{d}s + \int_{\Gamma_K} T_n(u) \left[(v'_{hn} - u'_n) - (v''_{hn} - u''_n)\right] \mathrm{d}s$$
$$\forall v_h \in \mathscr{K}_h , \quad \forall v \in \mathscr{K} .$$

---

[1]) I.e. regular system, such that the ratio of any straight sides (or chords) in $\mathscr{T}_h$ is bounded above by some constant β, independent of h.

Let $v_h = u_I$, where u_I is the piecewise linear Lagrange interpolate of u on Ω. It is easy to see that $u_I \in \mathscr{K}_h$ and we obtain

$$(3.31) \qquad |A(u_I - u; u_I - u)| \leqq M\|u_I - u\|_{1,\Omega}^2 \leqq ch^2\|u\|_{2,\Omega}^2 ,$$

$$\int_{\Gamma_K} T_n(u) \left[(u_I' - u') . n - (u_I'' - u'') . n\right] \mathrm{d}s \leqq$$

$$\leqq c(\|u_I' - u'\|_{0,\Gamma_K} + \|u_I'' - u''\|_{0,\Gamma_K}) \leqq ch^{3/2}\|u\|_{2,\Omega} ,$$

as follows from (3.28).[1]).

The most difficult is to estimate the term

$$(3.32) \qquad \int_{\Gamma_K} T_n(u) \left[(v_n' - v_n'') - (u_{hn}' - u_{hn}'')\right] \mathrm{d}s , \quad v \in \mathscr{K} .$$

In what follows we shall construct a function $v \in \mathscr{K}$ such that (3.32) is small.

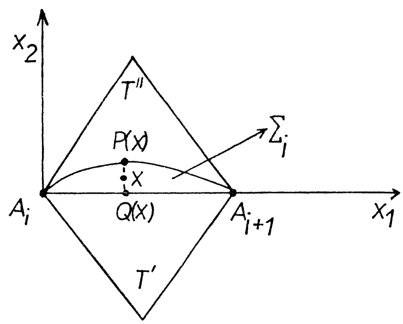

Fig. 5.

We identify the origin of the coordinate system (x_1, x_2) with the point A_i (see Fig. 5). Let Σ_i be a closed set bounded with the arc $\widehat{A_iA_{i+1}} \equiv s_i \subset \Gamma_K$, connecting A_i with A_{i+1}, and the chord A_iA_{i+1}. Let $x \in \Sigma_i$. By the symbol $P(x)$ and $Q(x)$, respectively, we denote the intersection of the perpendicular line through the point x with s_i and A_iA_{i+1}, respectively. Let $T_i \subset \Omega'$, $T_i'' \subset \Omega''$, $\partial T_i' \cap \partial T_i'' = s_i$ be the two adjacent curved elements. We extend any function $v \in [P_1(T_i'')]^2$ on $T_i'' \cup \Sigma_i$ as follows:

$$Ev \in [P_1(T_i'' \cup \Sigma_i)]^2 ;$$

$$Ev\big|_{T_i''} = v .$$

[1]) In order to aply Lemma 3.2, we can proceed as follows. Let $\tilde{\Omega}'$ be a convex set with twice continuously differentiable boundary $\partial\tilde{\Omega} \subset \Gamma_K$. Let $Eu \in [H^2(E_2)]^2$ denotes the continuous Calderon extension (see [7]) of $u' \in [H^2(\Omega')]^2$. Then, according to Lemma 3.2

$$\|u' - u_I'\|_{0,\Gamma_K} \leqq \|Eu' - (Eu')_I\|_{0,\partial\tilde{\Omega}'} \leqq ch^{3/2}\|Eu'\|_{2,\tilde{\Omega}'} \leqq ch^{3/2}\|u'\|_{2,\Omega} .$$

Analogously one can estimate $\|u'' - u_I''\|_{0,\Gamma_K}$.

For simplicity of notations, we use again the symbol v instead of Ev. Let us define functions $\mathscr{U}_h$, $\hat{U}_h$ again on $\bigcup_i \Sigma_i$ by means of the following relations:

$$\mathscr{U}_h(x) = (u'_h(x) - u''_h(x)) \,.\, n(P(x))$$

$$\hat{U}_h(x) = (u'_h(Q(x)) - u''_h(Q(x))) \,.\, n(P(x)) = (\hat{u}'_h - \hat{u}''_h)(x) \,.\, n(P(x)) \,,$$

where

$$\hat{u}'_h(x) = u'_h(Q(x)), \;\; \hat{u}''_h(x) = u''_h(Q(x)), \;\; x \in \Sigma_i .$$

Clearly

$$\mathscr{U}_h(x) = \hat{U}_h(x), \;\; x \in \bigcup_{i=1}^{m} A_iA_{i+1} .$$

Let $\Phi_i(x)$, $x \in A_iA_{i+1}$ be the linear Lagrange interpolate of $\mathscr{U}_h$ on A_iA_{i+1} and let us define the function Φ on $\bigcup_i \Sigma_i$ as follows:

$$\Phi(x) = \Phi_i(Q(x)) \,, \quad x \in \Sigma_i \,, \quad i = 1, \ldots, m \,.$$

It is readily seen that $\Phi \leqq 0$ on Γ_K. Let us estimate $\|\Phi - \mathscr{U}_h\|_{0,\Gamma_K}$. We may write:

$$\|\Phi - \mathscr{U}_h\|_{0,\Gamma_K} \leqq \|\Phi - \hat{U}_h\|_{0,\Gamma_K} + \|\hat{U}_h - \mathscr{U}_h\|_{0,\Gamma_K} \,, \tag{3.33}$$

$$\|\hat{U}_h - \mathscr{U}_h\|^2_{0,\Gamma_K} = \sum_{i=1}^{m} \|\hat{U}_h - \mathscr{U}_h\|^2_{0,s_i} \leqq$$

$$\leqq 2\Big(\sum_{i=1}^{m} \|u'_h - \hat{u}'_h\|^2_{0,s_i} + \sum_{i=1}^{m} \|u''_h - \hat{u}''_h\|^2_{0,s_i}\Big) .$$

Let τ be the arcs' parameter of the point $P(x) = (P_1(x), P_2(x))$ and denote $Q_1(x) = x_1$. Then for $M = {}' \,,\, {}''$ we have

$$u^M_{hj} - \hat{u}^M_{hj} = \int_0^{P_2(x)} \frac{\partial}{\partial x_2} (u^M_{hj} - \hat{u}^M_{hj}) \,\mathrm{d}x_2 = \int_0^{P_2(x)} \frac{\partial}{\partial x_2} u^M_{jh} \,\mathrm{d}x_2 \,, \quad j = 1, 2 \,.$$

Integrating and using Fubini's theorem we obtain

$$\|u^M_{hj} - \hat{u}^M_{hj}\|^2_{0,s_i} \leqq ch^2 |u^M_{jh}|^2_{1,\Sigma_i} \,, \quad j = 1, 2 \,.$$

From this and Lemma 3.4 we have

$$\|\hat{U}_h - \mathscr{U}_h\|^2_{0,\Gamma_K} \leqq ch^2 \Big(\sum_{i=1}^{m} |u'_h|^2_{1,\Sigma_i} + \sum_{i=1}^{m} |u''_h|^2_{1,\Sigma_i}\Big) \leqq ch^3 \|u_h\|^2_{1,\Omega} \,. \tag{3.34}$$

Let us estimate $\|\Phi - \hat{U}_h\|^2_{0,\Gamma_K}$.

$$\|\Phi - \hat{U}_h\|^2_{0,\Gamma_K} = \sum_{i=1}^{m} \|\Phi - \hat{U}_h\|^2_{0,s_i} \,.$$

$$\Phi(\tau) - \hat{U}_h(\tau) = \int_0^{Q_1(\tau)} \frac{\mathrm{d}}{\mathrm{d}x_1} [\Phi_i(x_1, 0) - \hat{U}_h(x_1, 0)] \,\mathrm{d}x_1 +$$

$$+\int_0^{P_2(x)} \frac{\mathrm{d}}{\mathrm{d}x_2} [\Phi_i(Q_1(x), x_2) - \hat{U}_h(Q_1(\tau), x_2)] \, \mathrm{d}x_2 =$$

$$= \int_0^{Q_1(\tau)} \frac{\mathrm{d}}{\mathrm{d}x_1} [\Phi_i(x_1, 0) - \hat{U}_h(x_1, 0] \, \mathrm{d}x_1 .$$

Since $\psi \in C^3(\langle a, b\rangle)$, we have $\hat{U}_h \in H^2(A_iA_{i+1})$. Hence

$$|\Phi(\tau) - \hat{U}_h(\tau)|^2 \leqq ch|\Phi_i - \hat{U}_h|^2_{1,A_iA_{i+1}} \leqq ch^3|\hat{U}_h|^2_{2,A_iA_{i+1}} . \tag{3.35}$$

As $\hat{U}_h(x) = (\hat{u}'_h - \hat{u}''_h)(x) \, . \, n(P(x))$ and $\hat{u}'_h, \hat{u}''_h \in P_1(A_iA_{i+1})$, we may write:

$$|\hat{U}_h|^2_{2,A_iA_{i+1}} \leqq c[\|u'_h\|^2_{1,A_iA_{i+1}} + \|u''_h\|^2_{1,A_iA_{i+1}}] .$$

Thus, (3.35), Lemma 3.3 on A_iA_{i+1} and the definition of strong regularity of $\{\mathcal{T}_h\}$ yield:

$$\int_{S_i} |\Phi(\tau) - \hat{U}_h(\tau)|^2 \, \mathrm{d}\tau \leqq ch^4[\|u'_h\|^2_{1,A_iA_{i+1}} + \|u''_h\|^2_{1,A_iA_{i+1}}] \leqq \tag{3.36}$$

$$\leqq ch^3\{\|u'_h\|^2_{1/2,A_iA_{i+1}} + \|u''_h\|^2_{1/2,A_iA_{i+1}}\} .$$

Adding (3.36) for $i = 1, \ldots, m$ we obtain

$$\|\Phi - \hat{U}_h\|^2_{0,\Gamma_K} \leqq ch^3\{\|u'_h\|^2_{1/2,\Gamma_h} + \|u''\|^2_{1/2,\Gamma_h}\} , \tag{3.37}$$

where $\Gamma_h = \bigcup_{i=1}^{m} A_iA_{i+1}$ is the polygonal approximation of Γ_K. Using the trace theorem and Lemma 3.4, we obtain

$$\|u'_h\|^2_{1/2,\Gamma_h} \leqq c\|u'_h\|^2_{1,\Omega'\setminus\bigcup_i\Sigma_i} \leqq c\|u'_h\|^2_{1,\Omega'}$$

$$\|u''_h\|^2_{1/2,\Gamma_h} \leqq c\|u''_h\|^2_{1,\Omega''\cup\bigcup_i\Sigma_i} \leqq c\|u''_h\|^2_{1,\Omega''} \text{ [1])}$$

Using these estimates, (3.33), (3.34) and (3.37) we deduce

$$\|\Phi - \mathcal{U}_h\|_{0,\Gamma_K} \leqq ch^{3/2}\|u_h\|_{1,\Omega} . \tag{3.38}$$

Next let $v \in V$ be such that $v'' = 0$ on Ω'' and v' such that $v' \, . \, n = \Phi$ on Γ_K. Then

$$v' \, . \, n - v'' \, . \, n = v' \, . \, n = \Phi \leqq 0 \quad \text{on} \quad \Gamma_K ,$$

consequently $v \in \mathcal{K}$. Finally we may write

$$\int_{\Gamma_K} T_n(u) \, [(v'_n - u'_{hn}) - (v''_n - u''_{hn})] \, \mathrm{d}s = \tag{3.39}$$

---

[1]) The constant c such that $\|u'_h\|_{1/2,\Gamma_h} \leqq c\|u'_h\|_{1,\Omega'\setminus\bigcup\Sigma_i}$ and $\|u''_h\|_{1/2,\Gamma_h} \leqq c\|u''_h\|_{1,\Omega''\cup\bigcup_i\Sigma_i}$ respectively, depends on h, in general. From the definition of the norm of traces however, (see [7], p. 88) it follows that c can be estimated independently on h for $h > 0$ sufficiently small.

$$= \int_{\Gamma_K} T_n(u) [\Phi - \mathscr{U}_h] \, ds \leqq c h^{3/2} \|u_h\|_{1,\Omega} .$$

The assertion (3.30) follows from (3.31) and (3.39).

Remark 3.4. If (1.6) and (1.8) hold, the norms $\|u_h\|_{1,\Omega}$ are bounded (see the proof of Theorem 1.2). The condition (1.8) follows from the coerciveness of $\mathscr{L}$ over $\mathscr{K}$, provided $\mathscr{K}_h \subset \mathscr{K}$ for all h. Unfortunately, this is not our case and therefore we have to assume the boundedness of norms in Theorem 3.3 explicitely.

Remark 3.5. The assumption on the boundedness of the norms $\|u_h\|_{1,\Omega}$ is satisfied if e.g. on $\Gamma_{K0} \subset \Gamma_K$ (chosen a priori) the bilateral contact of Ω' and Ω'' is considered (see Remark 3.3). Then $\mathscr{L}$ is coercive on V, hence (1.8) holds. The condition (1.6) follows from Lemma 3.1, which can be proved also for curved domains, modifying slightly the proof.

Remark 3.6. In [1] (Theorem 3.4) some sufficient conditions for the coerciveness of $\mathscr{L}$ on $\mathscr{K}$ have been presented. Let Γ_K contain a straight-line segment I. Let us define

$$\mathscr{K}_I = \{ v \in V \mid v'_n - v''_n \leqq 0 \quad \text{on} \quad I \} .$$

Assume that the sufficient conditions, mentioned above, are also satisfied, if $\mathscr{K}$ is replaced by $\mathscr{K}_I$. (For instance, in the situation of Fig. 5 – [1], it holds $\mathscr{K} \cap \mathscr{R} = \mathscr{K}_I \cap \mathscr{R}$ and the sufficient conditions become identical). Then $\mathscr{L}$ is coercive on $\mathscr{K}_I$, as follows from the proof of Th. 2.4 [1], substituting only $\mathscr{K}$ by $\mathscr{K}_I$ and Γ_K by I everywhere. Since

$$\mathscr{K}_h \subset \mathscr{K}_I \quad \forall h \in (0, 1)$$

(see the proof of Lemma 2.1), we have

$$\|v_h\|_{1,\Omega} \to \infty , \quad v_h \in \mathscr{K}_h \Rightarrow \mathscr{L}(v_h) \to +\infty ,$$

i.e. (1.8). From this and (1.6) the boundedness of norms $\|u_h\|_{1,\Omega}$ follows.

Remark 3.7. Let us consider the case, solved by Theorem 2.3 in [1]. Let us restrict ourselves to the problem $(\mathscr{P}_1)$, with Γ_0 and Γ_K parallel to x_1-axis. Then (see [1], Fig. 4)

$$\mathscr{R}_V = V \cap \mathscr{R} = \mathscr{K} \cap \mathscr{R} = \{ z = (z', z'') \mid z' = (0, 0), z'' = (a, 0), a \in R^1 \} ,$$

where $\mathscr{R}$ is the subspace of rigid bodies displacements. Let $V = H \oplus \mathscr{R}_V$ be the orthogonal decomposition of V with respect to the following scalar product:

$$(u, v)_V = [u, v] + p(u) . p(v) ,$$

where $[u, v] = \int_\Omega \varepsilon_{ij}(u) \, \varepsilon_{ij}(v) \, dx$,

$$p(v) = \int_{\Gamma''} v''_1 \, ds , \quad \Gamma'' \subset \partial\Omega'' , \quad \text{mes} \, \Gamma'' > 0 .$$

Then (see [1]) we have

$$H = \{v \in V \mid p(v) = 0\} .$$

From Th. 2.3 in [1] it follows that it is sufficient to solve the problem $(\mathscr{P}_1)$ only on the set $\hat{K} = \mathscr{K} \cap H$. Then the problem

$$(\hat{P}_1) \qquad \textit{find} \quad \hat{u} \in \hat{K} \quad \textit{such that}$$

$$\mathscr{L}(\hat{u}) \leqq \mathscr{L}(v) \quad \forall v \in \hat{K}$$

has a unique solution. At the same time, the closed convex sets $\hat{K}_h = \mathscr{K}_h \cap H$ can easily be realized numerically. In fact,

$$\hat{K}_h = \{v \in \mathscr{K}_h \mid \int_{\Gamma''} v_1'' \, ds = 0\} \subset \hat{K} ,$$

because Γ_K is a straight-line segment. It means that only one supplementary condition

$$\int_{\Gamma''} v_1'' \, ds = 0$$

must be added to the definition of $\mathscr{K}_h$. The problem

$$\textit{find} \quad \hat{u}_h \in \hat{K}_h \quad \textit{such that}$$

$$(\hat{P}_{1h}) \qquad \mathscr{L}(\hat{u}_h) \leqq \mathscr{L}(v) \quad \forall v \in \hat{K}_h$$

has a unique solution $\hat{u}_h$. The element $u_h \in \mathscr{K}_h$ represents a solution of $(\mathscr{P}_{1h})$ if and only if

$$u_h = \hat{u}_h + y , \quad y \in \mathscr{R}_V .$$

As the inequality of the Korn's type holds on H, i.e.

$$|v| \geqq c\|v\|_{1,\Omega} \quad \forall v \in H ,$$

the seminorm $| \; |$ in the error estimate (3.13) can be replaced by the norm $\| \; \|_{1,\Omega}$, if $(\mathscr{P}_1)$ and $(\mathscr{P}_{1h})$ are solved on $\hat{K}$ and $\hat{K}_h$, respectively (i.e. if we solve the problems $(\hat{P}_1)$ and $(\hat{P}_{1h})$). However, the proof of rate of convergence requires a slight modification: $v_h = u_I$ in the proof of Th. 3.1 must be replaced by $v_h = P_H \hat{u}_I$, where $\hat{u}_I$ is the linear Lagrange interpolate of $\hat{u}$ and P_H is the projection of V onto H (see [6], Th. 2.1).

3.2. CONTACT PROBLEMS WITH AN ENLARGING CONTACT ZONE

Let the weak solution u of the problem $(\mathscr{P}_2)$ be such that $\tau^M(u) \in \hat{Y}(\Omega^M)$, $M = {}'\,,\,{}''$. Then using the definition of $(\mathscr{P}_2)$ and (3.4) we obtain (cf. [1] – Th. 1.2) the same group of conditions (3.5)–(3.8) as in Section 3.1 and

$$(3.9') \qquad u_\xi'' - u_\xi' \leqq \varepsilon$$

$$(3.10') \qquad - T_\xi'(\cos \alpha')^{-1} = T_\xi''(\cos \alpha'')^{-1} \leqq 0$$

$$(3.11') \qquad T'_\eta = T''_\eta = 0$$

$$(3.12') \qquad T''_\xi(u''_\xi - u'_\xi - \varepsilon) = 0$$

for almost all $\eta \in \langle a, b\rangle$.

Theorem 3.4. *Let $(\mathscr{P}_2)$ and $(\mathscr{P}_{2h})$ have solutions u and u_h, respectively, let $u \in \mathscr{H}^2(\Omega) \cap \mathscr{K}_\varepsilon$, $\tau^M(u) \in \hat{Y}(\Omega^M)$, $M = ', ''$, $u'_\xi \in W^{1,\infty}(\Gamma'_K)$, $u''_\xi \in W^{1,\infty}(\Gamma''_K)$, and $f', f'' \in C^2(\langle a, b\rangle)$. Moreover, let us suppose that the number of points on Γ'_K, Γ''_K, where the contact changes from binding to nonbinding, is finite. Then*

$$|u - u_h| \leqq c(u)\, h\,,$$

if the system $\{\mathscr{T}_h\}$, $h \to 0+$ is regular.

Proof. Analogously as in the proof of Theorem 3.3, using the Green's formula, we obtain

$$1/2\, A(u - u_h, u - u_h) \leqq 1/2\, A(v_h - u, v_h - u) +$$
$$+ \int_{\Gamma_{K'}} T'_\xi(u)\,(v'_{h\xi} - u'_\xi)\,\mathrm{d}s + \int_{\Gamma_{K''}} T''_\xi(u)\,(v''_{h\xi} - u''_\xi)\,\mathrm{d}s +$$
$$+ \int_{\Gamma_{K'}} T'_\xi(u)\,(v'_\xi - u'_{h\xi})\,\mathrm{d}s + \int_{\Gamma_{K''}} T''_\xi(u)\,(v''_\xi - u''_h)\,\mathrm{d}s$$
$$\forall v_h \in \mathscr{K}_{\varepsilon h}\,, \quad \forall v \in \mathscr{K}_\varepsilon\,.$$

Let $v_h = u_I$ be the P-interpolate of u, constructed by means of the isoparametric technique, i.e.

$$u_I|_T = \hat{\Pi}(u|_T \circ F_T) \circ F_T^{-1}\,,$$

where $\hat{\Pi}$ denotes the operator of linear Lagrange interpolation on $\hat{T}$. It is easy to see that $u_I \in \mathscr{K}_{\varepsilon h}$. Using the approximative properties of u_I (cf. [3]) we can estimate $A(u_I - u, u_I - u)$ as follows:

$$(3.40) \qquad |A(u_I - u, u_I - u)| \leqq M\|u_I - u\|^2_{1,\Omega} \leqq ch^2\|u\|^2_{2,\Omega}\,.$$

We may write

$$\int_{\Gamma_{K'}} T'_\xi(u)\,(u'_{I\xi} - u'_\xi)\,\mathrm{d}s + \int_{\Gamma_{K''}} T''_\xi(u)\,(u''_{I\xi} - u''_\xi)\,\mathrm{d}s =$$
$$= \int_a^b T_\xi(u)\,[(u''_{I\xi} - u'_{I\xi}) - (u''_\xi - u'_\xi)]\,\mathrm{d}\eta\,,$$

where $T_\xi(u) \stackrel{\text{def}}{=} T''_\xi(u)\,(\cos\alpha_{II})^{-1} = -T'_\xi(u)\,(\cos\alpha_I)^{-1}$. Set

$$W_h(\eta) = u''_{I\xi}(f''(\eta), \eta)) - u'_{I\xi}(f'(\eta), \eta)$$
$$\mathscr{U}(\eta) = u''_\xi(f''(\eta), \eta) - u'_\xi(f'(\eta), \eta) \quad \eta \in \langle a, b\rangle\,.$$

From the definition of $u'_I(f'(\eta), \eta)$, $u''_I(f''(\eta), \eta)$ it follows that these are piecewise linear functions of η-variable on $\langle a, b\rangle$ with the nodes of C_j (see the construction of $\mathscr{K}_{\varepsilon h}$, where suitable multiples of η are chosen as the parameters in the arc representation). Since ξ is a fixed direction, W_h is also a piecewise linear function on $\langle a, b\rangle$.

Let

$$\Gamma^0 = \{\eta \in \langle a, b\rangle \mid u''_\xi - u'_\xi = \varepsilon\}$$
$$\Gamma^- = \{\eta \in \langle a, b\rangle \mid u''_\xi - u'_\xi < \varepsilon\}\,.$$

If $\langle C_i, C_{i+1}\rangle \subseteq \Gamma^0$ (see Fig. 2) then W_h is the linear Lagrange interpolate of ε on $\langle C_i, C_{i+1}\rangle$ and

$$\text{(3.41)} \qquad \int_{C_i}^{C_{i+1}} T_\xi(u)\,(W_h(\eta) - \mathscr{U}(\eta))\,\mathrm{d}\eta = \int_{C_i}^{C_{i+1}} T_\xi(u)\,(W_h - \varepsilon)\,\mathrm{d}\eta \leqq$$
$$\leqq ch^2|\varepsilon|_{2,<C_iC_{i+1}>}\,.$$

If $\langle C_i, C_{i+1}\rangle \subseteq \Gamma^-$, then $T_\xi(u) \equiv 0$ on $\langle C_i, C_{i+1}\rangle$. Hence

$$\text{(3.42)} \qquad \int_{C_i}^{C_{i+1}} T_\xi(u)\,(W_h(\eta) - \mathscr{U}(\eta))\,\mathrm{d}\eta = 0\,.$$

Let $\mathscr{J}$ be the system of all $\langle C_i, C_{i+1}\rangle \subset \langle a, b\rangle$, containing both points of Γ^0 and Γ^-. Using the assumption of the Theorem, we have

$$\text{(3.43)} \qquad \int_{C_i}^{C_{i+1}} T_\xi(u)(W_h(\eta) - \mathscr{U}(\eta))\,\mathrm{d}\eta \leqq h\|T_\xi(u)\|_{\infty,<C_i,C_{i+1}>}\,\|W_h - \mathscr{U}\|_{\infty,<C_i,C_{i+1}>} \leqq$$
$$\leqq ch^2\|T_\xi(u)\|_{\infty,<C_i,C_{i+1}>}\,|\mathscr{U}|_{1,\infty,<C_i,C_{i+1}>}\,.$$

Due to the assumptions, the number of all $\langle C_i, C_{i+1}\rangle \in \mathscr{J}$ can be bounded from above independently of h. From (3.41)–(3.43) we obtain

$$\text{(3.44)} \qquad \int_a^b T_\xi(u)\,[(u''_{I\xi} - u'_{I\xi}) - (u''_\xi - u'_\xi)]\,\mathrm{d}\eta \leqq c(u)\,h^2\,.$$

It remains to estimate

$$\int_{\Gamma_K'} T'_\xi(u)\,(v'_\xi - u'_{h\xi})\,\mathrm{d}s + \int_{\Gamma_K''} T''_\xi(u)\,(v''_\xi - u''_{h\xi})\,\mathrm{d}s =$$
$$= \int_a^b T_\xi(u)\,[(v''_\xi - v'_\xi) - (u''_{h\xi} - u'_{h\xi})]\,\mathrm{d}\eta\,, \quad v \in \mathscr{K}_\varepsilon\,.$$

Let us denote

$$\mathscr{U}_h(\eta) = u''_{h\xi}(\eta, f''(\eta)) - u'_{h\xi}(\eta, f'(\eta))\,, \quad \eta \in \langle a, b\rangle$$

and define the function W_h as follows

$$W_h(\eta) = \inf_{\eta \in <a,b>}\,[\mathscr{U}_h(\eta), \varepsilon(\eta)]\,.$$

It is readily seen that $W_h \in H^1(\langle a, b\rangle)$ and $W_h \leqq \varepsilon$ on $\langle a, b\rangle$. Since

$$W_h - \mathscr{U}_h = \left\langle \begin{array}{ll} 0 & \text{if } \mathscr{U}_h \leqq \varepsilon \\ \varepsilon - \mathscr{U}_h & \text{if } \mathscr{U}_h > \varepsilon \end{array} \right.$$

we can write

$$\left|\int_a^b T_\xi(u)\,(\mathscr{U}_h - W_h)\,\mathrm{d}\eta\right| \leqq c\|\mathscr{U}_h - \varepsilon\|_{0,\delta}\,, \tag{3.45}$$

where $\delta \subset \langle a, b\rangle$ is the set of points, where $\mathscr{U}_h > \varepsilon$. As $\mathscr{U}_h(C_j) \leqq \varepsilon(C_j)$, $j = 1, \ldots, m$ by the definition of $\mathscr{K}_\varepsilon$, we have also $\mathscr{U}_h(C_j) \leqq \varepsilon_I(C_j)$, where ε_I is the piecewise linear interpolate of ε on $\langle a, b\rangle$. $\mathscr{U}_h$ and ε_I are piecewise linear on $\langle a, b\rangle$, therefore $\mathscr{U}_h \leqq \varepsilon_I$ on $\langle a, b\rangle$. Hence (3.45) can be written in the following form:

$$\left|\int_a^b T_\xi(u)\,(\mathscr{U}_h - W_h)\,\mathrm{d}\eta\right| \leqq C\|\varepsilon_I - \varepsilon\|_{0,\delta} \leqq C\|\varepsilon_I - \varepsilon\|_{0,\langle a,b\rangle} \leqq ch^2|\varepsilon|_{2,\langle a,b\rangle}\,.$$

The rest of the proof can be accomplished in the same manner as that for Theorem 3.3. There exists a $v \in V$ such that $v'' = 0$ on Ω'' and $-v'_\xi = W_h$. Then $v \in \mathscr{K}_\varepsilon$ and it holds:

$$\int_a^b T_\xi(u)\,[(v''_\xi - v'_\xi) - (u''_{h\xi} - u'_{h\xi})]\,\mathrm{d}\eta =$$

$$= \int_a^b T_\xi(u)\,[W_h - \mathscr{U}_h]\,\mathrm{d}\eta \leqq ch^2|\varepsilon|_{2,\langle a,b\rangle}\,.$$

Using also (3.40), (3.44), the assertion of the Theorem now follows.

As in the case of contact problems with a bounded contact zone, we shall prove the convergence of approximate solutions u_h to the solution u of the problem $(\mathscr{P}_2)$ without any regularity assumptions. To this end, we need some auxiliary lemmas.

Lemma 3.5. *Suppose that* $f^M \in C^M$, $M = '\,,\,''$, $m \geqq 3$, $\bar{\Gamma}^M_K \cap \bar{\Gamma}_u = \emptyset$, $\bar{\Gamma}^M_K \cap \bar{\Gamma}_0 = \emptyset$ *and there exists only a finite number of points* $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_\tau$. *Let* $u \in \mathscr{K}_\varepsilon$ *satisfy the condition* $u''_\xi - u'_\xi \leqq f' - f''$ *in* $(a - \delta, b + \delta)$, *with some* $\delta > 0$.

Then u belongs to the closure (in W) of the set

$$\mathscr{K}_\varepsilon \cap [C^m(\bar{\Omega}')]^2 \times [C^m(\bar{\Omega}'')]^2\,.$$

Proof. Consider a system of open domains $\{B_i\}_{i=0}^r$ covering $\bar{\Omega}' \cup \bar{\Omega}''$ and such that $\bar{B}_0 \subset \Omega'$, $\bar{B}_1 \subset \Omega''$, $\bar{\Gamma}'_K \cup \bar{\Gamma}''_K \subset \bigcup_{j=2}^k B_j$, $(\bar{\Gamma}'_K \cup \bar{\Gamma}''_K) \cap B_i \neq \emptyset \Leftrightarrow 2 \leqq i \leqq k$. Assume that the union of arcs (see Fig. 6)

$$\widehat{PQ'} \cup \widehat{PQ''}\,, \quad Q^M = (f^M(b), b)\,, \quad M = '\,,\,''$$

is contained in one and only one domain B_j; the other domains contain at most one singular point (vertex or point of $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_\tau$).

We use the decomposition of unity as in the proof of Lemma 3.1. and construct smooth approximations of each $u_j = u\varphi_j$. In general, we can proceed like in the proof of Lemma 3.1, except for the situation of Fig. 6., where we argue as follows.

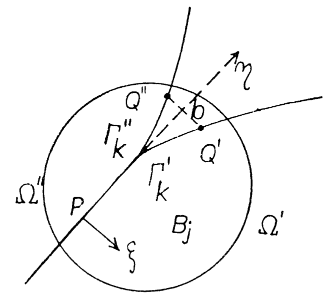

Fig. 6.

Note that $\varphi_j \equiv 1$ on $\Gamma'_K \cup \Gamma''_K$ due to the assumption. First we map $\Omega' \cap B$ (the indeces j will be omitted) into the right halfplane ($\hat\xi > 0$) and $\Omega'' \cap B$ into the left halfplane ($\hat\xi < 0$) by means of the two mappings

$$\hat{x} = T^M x : \{\hat\xi^M = \xi - f^M(\eta)\,, \quad \hat\eta^M = \eta\}\,, \quad M = '\,,\,''\,.$$

$$\hat{x} = (\hat\xi, \hat\eta)\,, \quad x = (\xi, \eta)\,.$$

Denote $\hat B = T'(\bar\Omega' \cap B) \cup T''(\bar\Omega'' \cap B)$ and $\hat u^M(\hat x) = u^M((T^M)^{-1}\,\hat x)$. Since

(3.46) $$u''_\xi(f''(\eta), \eta) - u'_\xi(f'(\eta), \eta) - \varepsilon(\eta) \leqq 0,\ \eta_0 < \eta \leqq b\,,$$

we have

$$\mathcal{U}(\eta) \equiv (\hat u''_\xi - \hat u'_\xi - \hat\varepsilon) \leqq 0 \quad \text{for} \quad \hat\xi + 0\,, \quad \eta_0 < \hat\eta \leqq b\,.$$

Let us extend $\hat\varepsilon$ onto the interval $b < \hat\eta$ in such a way that the extension $E\hat\varepsilon \in C^m$, $E\mathcal{U}$ remains non-positive, $E\mathcal{U} \subset H^{1/2}$ and supp $E\mathcal{U} \subset \hat B$.[1]) Then there exists a function $\hat v \in H^1(\hat B)$ such that $\hat v \leqq 0$ in $\hat B$, $\hat v = E\mathcal{U}$ for $\hat\xi = 0$, supp $\hat v \subset \hat B$.

If we extend $\hat u^M_\xi$ across the $\hat\eta$-axis to get functions Eu^M_ξ even in $\hat\xi$, we may write

$$E\hat u''_\xi - E\hat u'_\xi - E\hat\varepsilon = \hat v + \hat z\,,$$

$$\hat z \in H^1(\hat B)\,, \quad \hat z|_{\hat\xi=0} = 0\,.$$

Regularizing $\hat v$ and $\hat z$, we obtain

$$(R_\varkappa \hat v + \hat z)|_{\hat\xi=0} \leqq 0\,, \quad R_\varkappa \hat v + \hat z_\varkappa \to \hat v + \hat z \quad \text{in} \quad H^1(\hat B)\,.$$

Define

$$\hat u''_{\xi\varkappa} = R_\varkappa E\hat u''_\xi|_{T''\Omega''}\,,$$

$$\hat u'_{\xi\varkappa} = [R_\varkappa E\hat u''_\xi - R_\varkappa \hat v - \hat z_\varkappa]|_{T'\Omega'} - \hat\varepsilon\,,$$

$$u''_{\xi\varkappa} = \hat u''_{\xi\varkappa}(\xi - f''(\eta), \eta)\,, \quad u'_{\xi\varkappa} = \hat u'_{\xi\varkappa}(\xi - f'(\eta), \eta)\,, \quad (\hat\varepsilon = \varepsilon)\,.$$

1) We can take $E\hat\varepsilon = \varphi_j(f' - f'')$, provided that the point $(0, b + \delta)$ is outside $\bar B$.

Then the condition (3.46) is satisfied by $u_{\xi\varkappa}$, $u^M_\varkappa \in C^m$ and $\|u^M_{\xi\varkappa} - u^M_\xi\|_{1,\Omega^M} \to 0$, since both T^M and $(T^M)^{-1}$ are Lipschitz mappings.

Lemma 3.6. *Let φ be a continuous function defined on $\langle a, b\rangle$, $(-\infty < a < b < \infty)$, $D_n : a = x^n_0 < x^n_1 < \ldots < x^n_n = b$ a division of $\langle a, b\rangle$, $\nu(D_n) = \max_{i=1,\ldots,n} |x^n_i - x^n_{i-1}| \to 0$ for $n \to \infty$. Let $\{\psi_n\}_{n=1}^\infty$ be a sequence of piecewise linear functions with nodes at x^n_i such that $\psi_n(x^n_i) \leqq \varphi(x^n_i)$ $\forall i = 0, \ldots, n$; $n = 1, 2, \ldots$. Let $\psi_n \to \psi$ a.e. in $\langle a, b\rangle$. Then $\psi \leqq \varphi$ a.e. in $\langle a, b\rangle$.*

Proof. see [8] – Lemma A.2.

Theorem 3.5. *Let the problem $(\mathscr{P}_2)$ have precisely one solution u and the norms $\|u_h\|$ of solutions of the problems $(\mathscr{P}_{2h})$ remain bounded. Let all assumptions of Lemma 3.5 be satisfied. Then for any regular system of triangulations $\{\mathscr{T}_h\}$* we have

$$\|u - u_h\|_{1,\Omega} \to 0\,, \quad h \to 0+\,.$$

Proof. We must verify (1.6'), (1.7) and use Remark 1.6. From Lemma 3.5, using the same arguments as in the proof of Theorem 3.2, we obtain (1.6'). It remains to verify (1.7). Let $v_h \in \mathscr{K}_{\varepsilon h}$ be such that $v_h \rightharpoonup v$ in $\mathscr{H}^1(\Omega)$. By virtue of the complete continuity of the trace mapping we obtain

$$v'_h \to v' \text{ in } L^2(\Gamma'_K), \quad v''_h \to v'' \quad \text{in} \quad L^2(\Gamma''_K) \quad \text{(strongly)}.$$

Hence subsequences of v'_h and v''_h exist such that

$$V_h(\eta) = v''_{h\xi}(f''(\eta), \eta) - v'_{h\xi}(f'(\eta), \eta) \to v''_\xi(f''(\eta), \eta) - v'_\xi(f'(\eta), \eta) \equiv V(\eta)$$

a.e. in $\langle a, b\rangle$. Since $V_h(\eta)$ is piecewise linear on $\langle a, b\rangle$ and $V_h(C_i) \leqq \varepsilon(C_i)$, $i = 1, 2, \ldots$ $\ldots, m$, Lemma 3.6 implies $V \leqq \varepsilon$ a.e. in $\langle a, b\rangle$.

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Souhrn

KONTAKTNÍ PROBLÉM PRUŽNÝCH TĚLES. ČÁST II.: APROXIMACE METODOU KONEČNYCH PRVKŮ

JAROSLAV HASLINGER, IVAN HLAVÁČEK

Práce se zabývá aproximací kontaktního problému dvou rovinných pružných těles metodou konečných prvků. Navazuje bezprostředně na předchozí výsledky autorů, obsažené v [1]. Přípustná konvexní množina posunutí pro klasický variační princip se aproximuje po částech lineárními vektorovými funkcemi na trojúhelnících. Studuje se rychlost konvergence a konvergence přibližných řešení k řešení přesnému v závislosti na normě dělení. Uvažují se přitom jednak úlohy, v nichž se rozsah kontaktu během deformace nemění, jednak úlohy s proměnným rozsahem kontaktu.

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