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INVARIANT RESISTIVE NETWORKS IN EUCLIDEAN SPACES AND THEIR RELATION TO GEOMETRY

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1. INTRODUCTION

We shall be interested in properties of electrical resistive networks in a Euclidean n-space which are "invariant" in the following sense: For an arbitrary homogeneous electrical field in this n-space, the energy absorbed by the network does not depend on the (geometric) location of the network in the n-space (we allow Euclidean motions of the entire network only).

For instance, a network consisting of two mutually perpendicular line segments of the same lengths made of a homogeneous resistive wire is easily seen to be invariant in the Euclidean plane.

For simplicity, we shall deal with such electrical networks only which consist of a finite number of separately homogeneous line segments, i.e. each segment is an ideal one-dimensional resistive wire, the resisteance of its any connected portion being proportional to the length of this portion. In our considerations, we shall also allow "connections" of infinite resistance. Since a shift (translation) of such a homogeneous line segment in the considered space does not change absorption of energy in any homogeneous electrical field, we can, in fact, restrict ourselves to investigations of a finite set of vectors in the space such that with each vector a positive (but possibly infinite) resistance or, equivalently, a nonnegative conductivity as its reciprocal, is associated.

To describe the situation mathematically, assume that in a Euclidean n-space $\boldsymbol{E}_n$ with the inner product (x, y), a finite number of vectors $u_1, u_2, \ldots, u_N$ with conductivities $c_1, c_2, \ldots, c_N$ are given. If a homogeneous field in $\boldsymbol{E}_n$ is determined by the vector u orthogonal to the hyperplanes of constant potentials, the potentials at the end-points of u differing by one, the energy absorbed by one vector u_i is $c_i(u, u_i)^2/(u, u)^2$ since the potential at the end-points of u_i is $(u, u_i)/(u, u)$ and the current is $c_i(u, u_i)$: : (u, u). Thus we have

(1.1) **Theorem.** *A set of vectors* $u_1, \ldots, u_N$ *in* $\mathbf{E}_N$ *with the corresponding conductivities* $c_1, \ldots, c_N$ *(not all equal to zero) is electrically invariant iff there exists a constant* c *such that*

$$\sum_{i=1}^{N} c_i(x, u_i)^2 = c(\mathbf{x}, x) \tag{1}$$

for any vector $\mathbf{x} \in \mathbf{E}_n$. *If so, the absorption is* $c/(u, u)$ *if the field corresponds to the (non-zero) vector* u.

We shall say that the system $u_1, \ldots, u_N$ of vectors in $\mathbf{E}_n$ admits electrical invariance if there exist conductivities $c_1, \ldots, c_N$ (not all equal to zero) assigned to these vectors such that (1) is satisfied with $c > 0$.

In the sequel, we intend to characterize completely the systems of vectors which admit electrical invariance, present some classes of such systems and use the results to obtain a few geometrical theorems.

2. RESULTS

Let us first prove an algebraic lemma:

(2.1) **Lemma.** *Let* $u_1, \ldots, u_N$ *be vectors in* $\mathbf{E}_n$. *Then the following conditions are equivalent*:

(i)

$$\sum_{i=1}^{N} (x, u_i)^2 = (x, x) \quad \textit{for any} \quad x \in \mathbf{E}_n \,. \tag{2}$$

(ii) *if* $U_1, \ldots, U_N$ *are column vectors of coordinates of* $u_1, \ldots, u_N$ *with respect to a certain orthonormal basis of* $\mathbf{E}_n$, *the* $n \times N$ *matrix* $U = (U_1, \ldots, U_N)$ *satisfies*

$$UU^{\mathsf{T}} = I \,, \tag{3}$$

the identity matrix of order n;

(iii) *there exists a Euclidean N-space* $\mathbf{E}_N$ *containing* $\mathbf{E}_n$ *and orthonormal vectors* $v_1, \ldots, v_N$ *in* $\mathbf{E}_N$ *such that the orthogonal projection* P *of* $\mathbf{E}_N$ *onto* $\mathbf{E}_n$ *satisfies*

$$Pv_i = u_i \,, \quad i = 1, \ldots, N \,. \tag{4}$$

Proof. We shall prove implications (i) ⇒ (ii), (ii) ⇒ (iii) and (iii) ⇒ (i).

(i) ⇒ (ii). Let us denote by $\mathbf{E}'_n$ the arithmetic Euclidean space of real column vectors with the inner product $(X, Y) = \sum_{i=1}^{n} X_i Y_i$, where X_i, Y_i are the coordinates of X, Y, respectively.

Clearly, the coordinates of vectors in $\mathbf{E}_n$ with respect to an orthonormal basis belong to $\mathbf{E}'_n$ (and, in fact, both models are isomorphic). If we denote by X^{T} etc. the transpose of X, (X, Y) can also be written as $Y^{\mathsf{T}}X$. Accordingly, (2) can be expressed as

$$X^{\mathsf{T}}UU^{\mathsf{T}}X = X^{\mathsf{T}}X \quad \text{for all} \quad X \in \mathbf{E}'_n,$$

i.e.

$$X^{\mathsf{T}}(UU^{\mathsf{T}} - I)\,X = 0 \quad \text{for all} \quad X \in \mathbf{E}'_n$$

which implies (3).

(ii) $\Rightarrow$ (iii). It follows from $UU^{\mathsf{T}} = I$ that the n columns of U^{T} form an orthonormal set of vectors in $\mathbf{E}'_N$. By a well known theorem [4], $N \geqq n$ and for $N > n$, U^{T} can be completed by $N - n$ columns to an orthonormal basis in $\mathbf{E}'_N$. In other words, either $N = n$, U is an orthogonal matrix and (iii) is true, or $N > n$ and there exists an $n \times (N - n)$ matrix W such that the matrix

$$V = \begin{pmatrix} U \\ W \end{pmatrix}$$

is an orthogonal matrix. If we denote by $V_1, \ldots, V_N$ the columns of the matrix V and by P the (orthogonal) projection of $\mathbf{E}'_N$ onto $\mathbf{E}'_n$ which assigns to any vector X in $\mathbf{E}'_N$ the vector PX in $\mathbf{E}'_n$ having the same first n coordinates as X, we obtain $PV_i = U_i$, $i = 1, \ldots, N$. For the original space $\mathbf{E}_n$ this means exactly (4).

(iii) $\Rightarrow$ (i). Since $v_1, \ldots, v_N$ form an orthonormal basis in $\mathbf{E}_N$, we have, by Pythagorean theorem,

$$\sum_{i=1}^{N} (x, v_i)^2 = (x, x) \quad \text{for any} \quad x \in \mathbf{E}_N\,.$$

Thus also

$$\sum_{i=1}^{N} (Px, v_i)^2 = (Px, Px) \quad \text{for any} \quad x \in \mathbf{E}_N\,.$$

However, $(Px, v_i) = (P^2x, v_i) = (Px, Pv_i)$. Thus if (4) is satisfied,

$$\sum_{i=1}^{N} (Px, u_i)^2 = (Px, Px) \quad \text{for all} \quad x \in \mathbf{E}_N$$

and (2) follows.

We are able now to prove the main theorem.

(2.2) **Theorem.** *A finite system of N vectors $u_1, \ldots, u_N$ in a Euclidean n-space $\mathbf{E}_n$ admits electrical invariance iff a non-void subsystem $u_{i_1}, u_{i_2}, \ldots, u_{i_m}$ of $m \geqq n$ vectors of it is formed by orthogonal projection of m mutually orthogonal non-zero vectors $v_1, \ldots, v_m$ in a Euclidean m-space $\mathbf{E}_m$ containing $\mathbf{E}_n$. The corresponding conductivities $c_1, c_2, \ldots, c_N$ with respect to which $u_1, \ldots, u_N$ become electrically invariant can be chosen as follows: $c_{i_1} = 1/r_{i_1}, \ldots, c_{i_m} = 1/r_{i_m}$ where $r_{i_j}, j = 1, \ldots, m$, are squares of the lengths of the vectors $v_1, \ldots, v_m$ in $\mathbf{E}_m$, $c_i = 0$ in all other cases.*

Proof. Let first (1) be satisfied. Denote by M the set of those m indices $i_1, \ldots, i_m$ for which $c_i \neq 0$. Then (1) can be written as

$$\sum_{i \in M} (x, (c_i/c)^{1/2} u_i)^2 = (x, x) .$$

By Lemma (2.1), the (non-zero) vectors $(c_{i_k}/c)^{1/2} u_{i_k}$ are orthogonal projections of orthogonal vectors v_k, $k = 1, \ldots, m$, in $\mathbf{E}_m$ containing $\mathbf{E}_n$. Therefore, the vectors u_{i_k} are orthonormal projections of mutually orthogonal vectors $(c/c_{i_k})^{1/2} v_i$ (whose squares of lengths are proportional to $1/c_{i_k} = r_{i_k}$).

Conversely, if the vectors $u_1, \ldots, u_N$ have the property mentioned in the theorem then the vectors $w_i = v_i/|v_i|$, $|v_i|$ being the length of v_i, form an orthonormal basis in $\mathbf{E}_m$ containing $\mathbf{E}_n$. Therefore,

$$\sum_{i=1}^{m} (x, w_i)^2 = (x, x) \quad \text{for all} \quad x \in \mathbf{E}_m$$

which means, by Thm. (1.1),

$$\sum_{i=1}^{N} c_i(x, u_i)^2 = (x, x) \quad \text{for all} \quad x \in \mathbf{E}_n$$

for $c_{i_k} = 1/r_{i_k} = 1/|v_k|^2$, $k = 1, \ldots, m$, and $c_i = 0$ for all remaining indices.

(2.3) **Corollary.** *A system of vectors in $\mathbf{E}_n$ admitting electrical invariance (in $\mathbf{E}_n$) has at least n vectors. In this case of n vectors $u_1, \ldots, u_n$, the system admits electrical invariance in $\mathbf{E}_n$ iff these vectors are all non-zero and mutually orthogonal. The corresponding resistances $r_1, \ldots, r_n$ are necessarily finite and proportional to the squares of the lengths:*

$$r_i = k|u_i|^2 , \quad i = 1, \ldots, n .$$

(2.4) **Corollary.** *If a system of vectors in $\mathbf{E}_n$ admits electrical invariance in $\mathbf{E}_n$ then every larger system of vectors in $\mathbf{E}_n$ also has this property.*

In the next theorem, we prove a necessary condition which has to be satisfied by any system of vectors admitting electrical invariance. In its formulation, the term quadrant-space in $\mathbf{E}_n$ means the intersection of any two closed half-spaces in $\mathbf{E}_n$ whose boundary hyperplanes are orthogonal. The intersection of the two boundary hyperplanes will be called the basic space of the quadrant-space.

(2.5) **Theorem.** If $u_1, \ldots, u_N$ *are vectors in $\mathbf{E}_n$ which admit electrical invariance then any quadrant-space* S *of $\mathbf{E}_n$ contains a half-line which is parallel or anti-parallel to one of the vectors $u_1, \ldots, u_N$ but is not parallel to the basic space of* S.

Proof. Let $u_1, \ldots, u_N$ admit electrical invariance with respect to conductivities

$c_1, \ldots, c_N$. Let v, w be non-zero vectors for which the intersection of the half-spaces

$$(x, v) \geqq 0\,, \quad (x, w) \geqq 0$$

forms the quadrant-space S, $(v, w) = 0$, $(v, v) = (w, w) = 1$. Assume that S does not contain any half-line parallel or anti-parallel to $u_1, \ldots, u_N$. Then

$$(5) \qquad (u_k, v)\,(u_k, w) < 0$$

for all vectors u_k not parallel to the basic space of S while equality holds for the vectors u_k parallel to the basic space of S. Besides, (5) is satisfied for at least one k for which $c_k \neq 0$ since by Thm. (2.2), not all these vectors of the system can be contained in an $(n\text{-}2)$-dimensional subspace of $\mathbf{E}_n$.

Denote $u = (1/\sqrt{2})\,(v + w)$, $u' = (1/\sqrt{2})\,(v - w)$. If for a vector u_k, (5) is satisfied, then

$$(u_k, u)^2 = \tfrac{1}{2}((u_k, v) + (u_k, w))^2 =$$
$$= \tfrac{1}{2}((u_k, v)^2 + (u_k, w)^2 + 2(u_k, v)\,(u_k, w)) < \tfrac{1}{2}((u_k, v)^2 + (u_k, w)^2 -$$
$$- 2(u_k, v)\,(u_k, w)) = (u_k, u')^2\,.$$

If u_k belongs to the basic space of S, $(u_k, u)^2 = (u_k, u')^2$. Therefore, by (1),

$$c(u, u) = \sum_{k=1}^{N} c_k(u, u_k)^2 < \sum_{k=1}^{N} c_k(u', u_k)^2 = c(u', u')\,,$$

a contradiction since $(u, u) = (u', u') = 1$.

Remark. It is an open question whether this condition is also sufficient. Let us formulate a conjecture.

Conjecture. *Let a finite system V of (non-zero) vectors in $\mathbf{E}_n$ satisfy the condition: For any quadrant-space S in $\mathbf{E}_n$, there is a half-line in S which is parallel or anti-parallel to some vector in V but is not parallel to the basis of S. Then V admits electrical invariance in $\mathbf{E}_n$.*

3. PARTICULAR CASES

We shall begin with a suitable analytical approach to the simplex geometry. As is well known, an n-simplex in a Euclidean point space $\mathbf{E}_n^p$, being a generalization of a triangle in $\mathbf{E}_2^p$ and of a tetrahedron in $\mathbf{E}_3^p$, is determined by $n + 1$ linearly independent points of $\mathbf{E}_n^p$, called vertices. The simplex itself is usually considered as the convex hull of its vertices. The line segments joining two different vertices are called edges. If $A_1, \ldots, A_{n+1}$ are the vertices of an n-simplex Σ then any n points of them determine an $(n - 1)$-simplex called an $(n - 1)$-face of Σ. The $(n - 1)$-face which is determined by all vertices $A_1, \ldots, A_{n+1}$ except A_i will be called the $(n-1)$-face opposite to A_i.

Denote by u_{ik}, $i, k = 1, \ldots, n+1$, the vectors

$$(6) \qquad u_{ik} = A_i - A_k\,.$$

Thus $u_{ki} = -u_{ik}$, $u_{ii} = 0$. The vectors $u_{1,n+1}, u_{2,n+1}, \ldots, u_{n,n+1}$ being linearly independent, there exist vectors $p_1, \ldots, p_n$ which form with the former ones a biorthogonal system:

$$(7) \qquad (u_{i,n+1}, p_j) = \delta_{ij}\,, \quad i, j, 1, \ldots, n$$

(δ_{ij} is the Kronecker symbol).

If we define p_{n+1} as

$$(8) \qquad p_{n+1} = -\sum_{i=1}^{n} p_i\,,$$

the $n+1$ vectors $p_1, \ldots, p_{n+1}$ satisfy

$$(9) \qquad (u_{ik}, p_j) = \delta_{ij} - \delta_{jk}\,, \quad i, j, k = 1, \ldots, n+1$$

and are thus perpendicular to the $(n-1)$-faces (p_j to that opposite to A_j).

(3.1) **Theorem.** *Let Σ be an n-simplex with vertices $A_1, \ldots, A_{n+1}$ in $\mathbf{E}_n^p$. If u_{ik} and p_j are vectors defined by* (6), (7) *and* (8), *the vectors $w_i = -p_i/(p_i, p_i)$ are vectors of altitudes of the simplex Σ, i.e. $w_i = P_i - A_i$ where P_i is the foot of the perpendicular from A_i to the opposite $(n-1)$-face. Thus w_i are also vectors of outer normals to the $(n-1)$-faces of Σ. The length of p_i is equal to the reciprocal of the length of the altitude from A_i.*

Proof. Since $w_i = P_i - A_i$ is parallel to p_i, $w_i = k_i p_i$. Now, $P_i = \sum_{j=1}^{n+1}{}^{(i)} \alpha_j A_j$, $\sum_{j=1}^{n+1}{}^{(i)} \alpha_j = 1$ and $\overset{(i)}{\alpha_i} = 0$. Thus, $(w_i =) \sum_{j=1}^{n+1}{}^{(i)} \alpha_j (A_j - A_i) = k_i p_i$. By inner multiplication by p_i, one gets using (9) $k_i = -((p_i, p_i))^{-1}$. The rest is obvious.

This theorem enables us to find relation between the interior angles φ_{ik} ($i \neq k$, $i, k = 1, \ldots, n+1$) of faces opposite to A_i and A_k and the vectors p_i:

(3.2) **Theorem.** *In the notation of Thm.* (3.1),

$$(10) \qquad \cos \varphi_{ik} = -(p_i, p_k)/|p_i|\,|p_k|\,.$$

Proof. Since $-p_i$ are vectors of outer normals, the angle between p_i and p_k (for $i \neq k$) equals $\pi - \varphi_{ik}$.

An n-simplex Σ is called orthocentric if all the altitudes (as lines) meet in one point, called orthocenter of Σ. It can be shown [1] that the orthocenter is an interior point of Σ iff all its interior angles are acute.

We are able now to consider the case of a system S of $n + 1$ vectors in $\mathbf{E}_n$ which admits electrical invariance in $\mathbf{E}_n$. We have to distinguish two cases: Either there exists a subsystem $S' \subset S$ with $k + 1$ vectors $(k < n)$ which are all contained in a Euclidean k-space $\mathbf{E}_k$, or not. If the former case occurs, we shall say that S is reducible; in the latter case, S will be called irreducible. By the definition of a system admitting electrical invariance in $\mathbf{E}_n$, it follows in the first case (if we use rotations in $\mathbf{E}_n$ around the "axis" $\mathbf{E}_k$) that S' again admits electrical invariance in $\mathbf{E}_k$.

By Corollary (2.3), S' is electrically equivalent to k mutually orthogonal vectors which together with the remaining $n - k$ vectors in S form n vectors admitting electrical invariance in $\mathbf{E}_n$. By Corollary (2.3), the remaining $n - k$ vectors in S are mutually orthogonal and also orthogonal to $\mathbf{E}_k$. Thus we have:

(3.3) **Theorem.** *A reducible system of $n + 1$ vectors in $\mathbf{E}_n$ which admits electrical invariance always contains a unique irreducible subsystem of $k + 1$ vectors in $\mathbf{E}_k (k < n)$ which admits electrical invariance. The remaining $n - k$ vectors are mutually orthogonal and also orthogonal to $\mathbf{E}_k$.*

In the following characterization we can restrict ourselves, in view of (2.3) and (2.4), to irreducible systems of $n + 1$ vectors no n of which are mutually orthogonal.

(3.4) **Theorem.** *Let $S = \{u_1, \ldots, u_{n+1}\}$ be an irreducible system of $n + 1$ vectors in $\mathbf{E}_n$, no n of which being mutually orthogonal. Then the following assertions are equivalent:*

(i) *S admits electrical invariance;*

(ii) *there exists an acute-angled orthocentric n-simplex Σ in the point-space $\mathbf{E}_n^p$ to $\mathbf{E}_n$ with vertices $A_1, \ldots, A_{n+1}$ and orthocenter A_0 such that u_i is a non-zero multiple of $A_i - A_0$ for $i = 1, \ldots, n + 1$;*

(iii) *there is a single linearly independent relation $\sum_{i=1}^{n+1} \alpha_i u_i = 0$, the coefficints α_i are all different from zero and there exist positive numbers $k_1, \ldots, k_{n+1}$ such that for all $i, j = 1, \ldots, n + 1$, $i \neq j$,*

$$(u_i, u_j) = -\alpha_i \alpha_j k_i k_j \,. \tag{11}$$

The resistances $r_1, \ldots, r_{n+1}$ corresponding to the vectors $u_1, \ldots, u_{n+1}$ (as segments) are then proportional to $k_1, \ldots, k_{n+1}$:

$$r_i = \sigma k_i \,, \quad i = 1, \ldots, n + 1 \,. \tag{12}$$

Proof. We shall prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let S satisfy (i). By Thm. (2.2), there exists a Euclidean $(n+1)$-space $\mathbf{E}_{n+1}$ containing $\mathbf{E}_n$ and $n+1$ orthogonal vectors $v_1, \ldots, v_{n+1}$ in $\mathbf{E}_{n+1}$ such that for the orthogonal projection P of $\mathbf{E}_{n+1}$ onto $\mathbf{E}_n$, $Pv_i = u_i$, $i = 1, \ldots, n+1$. Since S is irreducible, the (unique up to a non-zero factor) vector u for which $Pu = 0$ is not orthogonal to any of the vectors $v_1, \ldots, v_n$. In the point spaces $\mathbf{E}^p_{n+1}$ and $\mathbf{E}^p_n$ containing $\mathbf{E}_{n+1}$ and $\mathbf{E}_n$ with a common origin A_0, let A_i, $i = 1, \ldots, n+1$ be points determined by

$$A_i - A_0 = u_i/(u, v_i)\,, \quad i = 1, \ldots, n+1\,.$$

This can also be written as

$$A_i - A_0 = v_i/(u, v_i) - u/(u, u)\,. \tag{13}$$

We shall show that $A_1, \ldots, A_{n+1}$ are vertices of an orthocentric n-simplex Σ with the orthocenter A_0 which is its interior point. Let i, j, k be mutually different indices from $\{1, 2, \ldots, n+1\}$. By (13),

$$\begin{aligned}(A_i - A_0, A_j - A_k) &= (A_i - A_0, A_j - A_0 - (A_k - A_0)) = \\ &= (v_i/(u, v_i) - u/(u, u)\,,\ \ v_j/(u, v_j) - v_k/(u, v_k)) = 0\end{aligned}$$

since v_i is orthogonal to v_j and v_k. Thus $A_i - A_0$ is orthogonal to the $(n-1)$-face of Σ opposite to A_i, $i = 1, \ldots, n+1$, which means that Σ is orthocentric with the orthocenter A_0. Since

$$u = \sum_{i=1}^{n+1} \frac{(u, v_i)}{(v_i, v_i)} v_i\,, \quad \text{we have by (13)}$$

$$u = \sum_{i=1}^{n+1} \frac{(u, v_i)^2}{(v_i, v_i)} (A_i - A_0) + \sum_{i=1}^{n+1} \frac{(u, v_i)^2}{(v_i, v_i)}\, u/(u, u)\,.$$

Since u is orthogonal to all $A_i - A_0$, it follows that

$$\sum_{i=1}^{n+1} \frac{(u, v_i)^2}{(v_i, v_i)} (A_i - A_0) = 0\,,$$

$$\sum_{i=1}^{n+1} \frac{(u, v_i)^2}{(v_i, v_i)} = (u, u)\,.$$

Thus we have

$$A_0 = \sum_{i=1}^{n+1} \xi_i A_i$$

with

$$\xi_i = \frac{(u, v_i)^2}{(u, u)(v_i, v_i)} > 0\,, \quad i = 1, \ldots, n+1$$

and

$$\sum_{i=1}^{n+1} \xi_i = 1 .$$

Thus A_0 is an interior point of Σ.

(ii) $\Rightarrow$ (iii). Let

(14) $$A_i - A_0 = \hat{\alpha}_i u_i\,, \quad \hat{\alpha}_i \neq 0\,, \quad i = 1, \ldots, n+1\,,$$

where $A_1, \ldots, A_{n+1}$ are vertices of an orthocentric n-simplex Σ whose orthocenter A_0 is an interior point of Σ. Since $A_i - A_0$ is orthogonal to the $(n-1)$-face opposite to A_i,

$$(A_i - A_0, A_j - A_k) = 0$$

whenever $j \neq i \neq k$. This means by (14) that

$$(\hat{\alpha}_i u_i, \hat{\alpha}_j u_j - \hat{\alpha}_k u_k) = 0$$

or,

$$\hat{\alpha}_i \hat{\alpha}_j (u_i, u_j) = \hat{\alpha}_i \hat{\alpha}_k (u_i, u_k) \quad \text{for all} \quad i, j, k\,,$$

$j \neq i \neq k$. It easily follows that

(15) $$\hat{\alpha}_i \hat{\alpha}_j (u_i, u_j) = K$$

for all $i, j = 1, \ldots, n+1$, $i \neq j$. Let us show that $K < 0$. There exist positive numbers $\beta_1, \ldots, \beta_{n+1}$ such that

$$A_0 = \sum_{i=1}^{n+1} \beta_i A_i\,, \quad \sum_{i=1}^{n+1} \beta_i = 1\,.$$

Consequently, by (14),

$$\sum_{i=1}^{n+1} \hat{\alpha}_i \beta_i u_i = 0\,.$$

Thus, we have by (15),

$$0 = (\hat{\alpha}_1 u_1, \sum_{i=1}^{n+1} \hat{\alpha}_i \beta_i u_i) = \hat{\alpha}_1^2 \beta_1 (u_1, u_1) + (\sum_{i=2}^{n+1} \beta_i) K$$

and $K < 0$.

Hence for $i, j = 1, \ldots, n+1, i \neq j$, if $\alpha_i = \hat{\alpha}_i \beta_i$,

$$(u_i, u_j) = \frac{K}{\hat{\alpha}_i \hat{\alpha}_j} = -\alpha_i \alpha_j k_i k_j$$

for $k_i = \beta_i \left|K\right|^{1/2} / \hat{\alpha}_i^2$, $i = 1, \ldots, n+1$.

(iii) $\Rightarrow$ (i). Let (iii) hold. We shall show that

$$\sum_{i=1}^{n+1} k_i^{-1}(x, u_i)^2 = \left(\sum_{i=1}^{n+1} \alpha_i^2 k_i\right)(x, x). \tag{16}$$

It suffices to show that

$$\sum_{i=1}^{n+1} k_i^{-1}(x, u_i)(y, u_i) = \left(\sum_{i=1}^{n+1} \alpha_i^2 k_i\right)(x, y) \tag{17}$$

for $y = u_p$, $p = 1, \dots, n + 1$.

We have

$$\sum_{i=1}^{n+1} k_i^{-1}(x, u_i)(u_p, u_i) = k_p^{-1}(u_p, u_p)(x, u_p) - \alpha_p k_p \sum_{i \neq p}(x, \alpha_i u_i) =$$
$$= \left(\alpha_p^2 k_p + k_p^{-1}(u_p, u_p)\right)(x, u_p).$$

However,

$$0 = \left(u_p, \sum_{i=1}^{n+1} \alpha_i u_i\right) = \alpha_p(u_p, u_p) - \alpha_p k_p \sum_{i \neq p} \alpha_i^2 k_i,$$

which implies

$$(u_p, u_p) = k_p\left(\sum_{i=1}^{n+1} \alpha_i^2 k_i - \alpha_p^2 k_p\right).$$

Therefore,

$$\alpha_p^2 k_p + k_p^{-1}(u_p, u_p) = \sum_{i=1}^{n+1} \alpha_i^2 k_i$$

and (17) is proved for $y = u_p$.

The rest follows immediately from (iii).

Another particular case which is closely connected to the previous one is that of a closed polygon in $\mathbf{E}_n$ with $n + 1$ sides. We can restrict ourselves to the case that the cyclically ordered vertices $B_1, B_2, \dots, B_{n+1}$ of this $(n + 1)$-gon are linearly independent. The vertices B_i, B_{i+1}, $i = 1, \dots, n$ as well as B_{n+1}, B_1 are called neighboring, and also the $(n - 1)$-faces of the corresponding n-simplex opposite to B_i, B_{i+1} as well as opposite to B_{n+1}, B_1 are called neighboring.

In [1], such an $(n + 1)$-gon was called cyclic if any two non-neighboring $(n - 1)$-faces are orthogonal. It was shown that of the remaining $n + 1$ interior angles (between the neighboring $(n - 1)$-faces), at least n are acute; one can be obtuse or right. Cyclic $(n + 1)$-gons have the property that, given the lengths $|B_i B_{i+1}| = l_i$, $i = 1, \dots, n + 1$ $(B_{n+2} = B_1)$, satisfying

$$2 \max_{k=1,\dots,n+1} l_k < \sum_{k=1}^{n+1} l_k),$$

there exists up to congruence a unique $(n + 1)$-gon in $\mathbf{E}_n$, the corresponding n-simplex of which has the maximum n-dimensional volume. This $(n+1)$-gon is always cyclic and has all interior angles between neighboring $(n-1)$-faces acute, has one this angle right, or has one this angle obtuse according to whether

$$2 \max_k l_k^2 < \sum_{k=1}^{n+1} l_k^2,\ 2 \max_k l_k^2 = \sum_k l_k^2 \text{ or } 2 \max_k l_k^2 > \sum_k l_k^2 .$$

We shall need only the first two cases. The cyclic $(n+1)$-gon with one right angle has, in fact, n mutually orthogonal sides, for instance $B_1B_2, B_2B_3, \ldots, B_nB_{n+1}$ (and another side $B_{n+1}B_1$). As for the cyclic $(n+1)$-gon $B_1, B_2, \ldots, B_{n+1}$ with acute angles between neighboring $(n-1)$-faces, it was proved in [1] that it is characterized by the fact there exist positive numbers $\pi_1, \ldots, \pi_{n+1}$ such that the vectors $a_i = B_{i+1} - B_i$, $i = 1, \ldots, n+1$, $B_{n+2} = B_1$, satisfy for $i \neq j$

$$(a_i, a_j) = -\pi_i\pi_j/\sigma\,, \quad i, j = 1, \ldots, n+1\,, \tag{18}$$

where

$$\sigma = \sum_{i=1}^{n+1} \pi_i\,.$$

It is easily checked by virtue of (18) that the vectors

$$p_i = \pi_{i-1}^{-1} a_{i-1} - \pi_i^{-1} a_i\,, \quad i = 1, \ldots, n+1\ (a_0 = a_{n+1}\,,\ \pi_0 = \pi_{n+1}) \tag{19}$$

are the vectors from (7), (8) and (9) corresponding to the n-simplex with vertices $B_1, \ldots, B_{n+1}$. Since, as is also easily computed,

$$(p_i, p_{i+1}) = -\pi_i^{-1}\,,$$

we have by (10) and Thm. (3.1)

$$\pi_i = \frac{|w_i|\,|w_{i+1}|}{\cos \varphi_{i,i+1}}\,, \tag{20}$$

where $|w_i|$ is the length of the altitude from B_i and $\varphi_{i,i+1}$ the interior angle between the faces opposite to B_i, B_{i+1}.

We are now able to prove:

(3.5) **Theorem.** *A closed $(n+1)$-gon $B_1B_2, \ldots, B_{n+1}$ in $\mathbf{E}_n$ admits electrical invariance in $\mathbf{E}_n$ iff it is cyclic without obtuse interior angles. If one interior angle between neighboring $(n-1)$-faces is right, the resistances corresponding to orthogonal sides are proportional to the squares of their lengths while the remaining resistance is infinite. If all interior angles between neighboring $(n-1)$-faces are acute, the resistance r_i of B_iB_{i+1} is*

$$r_i = \sigma|w_i|\,|w_{i+1}|/\cos \varphi_{i,i+1}\,,$$

where σ is an arbitrary positive constant, $|w_i|$ the length of the altitude from B_i in the corresponding n-simplex and $\varphi_{i,i+1}$ the interior angle between the $(n-1)$-faces opposite to B_i and B_{i+1}.

Proof. Let an $(n+1)$-gon $B_1, \ldots, B_{n+1}$ in $\mathbf{E}_n$ admit electrical invariance in $\mathbf{E}_n$. The vectors $a_i = B_{i+1} - B_i$, $i = 1, \ldots, n+1$ $(B_{n+2} = B_1)$ satisfy

$$\sum_{i=1}^{n+1} a_i = 0 \tag{21}$$

and, since they cannot be contained in a smaller space than $\mathbf{E}_n$, any n of these vectors are linearly independent. Let us distinguish two cases:

A) n of these vectors are mutually orthogonal. Then we clearly have the case of a cyclic $(n+1)$-gon with one angle between the neighboring $(n-1)$-faces right. The corresponding resistances are then by Thm. (2.3) as asserted.

B) No n of these vectors are mutually orthogonal; since none of these vectors a_i can be orthogonal to the remaining ones, these vectors form an irreducible system in the sense of Thm. (3.4). By this theorem, since in (iii) the coefficients α_i can be taken as ones, there exist positive numbers $k_1, \ldots, k_{n+1}$ satisfying for all $i, j = 1, \ldots$ $\ldots, n+1$, $i \neq j$

$$(a_i, a_j) = -k_i k_j .$$

If we set

$$\pi_i = k_i \sum_{j=1}^{n+1} k_j , \tag{22}$$

$$\sigma = \sum_{i=1}^{n+1} \pi_i ,$$

(18) will be satisfied so that the $(n+1)$-gon is cyclic with acute interior angles between neighboring $(n-1)$-faces as asserted. From (12), (22) and (20), we obtain the formula for resistances as asserted.

The converse is also true as is easily checked by using (2.3) in the case that n of the vectors a_i are mutually orthogonal and Thm. (3.4) in the other case.

For $n = 2$, we obtain the following corollary:

(3.6) **Corollary.** *A triangle (i.e. a closed 3-gon in $\mathbf{E}_2$) admits electrical invariance in $\mathbf{E}_2$ iff it has no obtuse angle. In this case, the corresponding resistances r_i $(i = 1, 2, 3)$ for the triangle $A_1A_2A_3$ are proportional to $l_i/\cos \alpha_i$ (or, to $\operatorname{tg} \alpha_i$) where l_i is the length of the side opposite to A_i and α_i is the angle at A_i, $i = 1, 2, 3$. (If one of the angles is right, the opposite side has infinite resistance.)*

Remark. It seems interesting that in the cases of n vectors in $\mathbf{E}_n$, of $n + 1$ vectors in $\mathbf{E}_n$ as well as of the $(n + 1)$-gon in $\mathbf{E}_n$, these vectors admit electrical invariance in the case only that the geometric configuration of the segments fulfils the following condition: Among all possible configurations with given lengths of the segments, these have the maximum possible n-dimensional volume of the convex hull of the segments.

Corollary (3.6) suggests importance of n-simplices without obtuse angles in this problem. This will also be confirmed in the following theorem:

(3.7) **Theorem.** *Let $A_1, \ldots, A_{n+1}$ be vertices of an n-simplex Σ in $\mathbf{E}_n$. The set of edges A_iA_k $(i \neq k,\ i, k = 1, \ldots, n + 1)$ as segments admits electrical invariance in $\mathbf{E}_n$ iff Σ has no obtuse interior angle between $(n - 1)$-faces. In this case, the resistances $r_{ik}(=r_{ki},\ i \neq k)$ corresponding to the edges A_iA_k can be chosen as*

$$r_{ik} = |w_i|\,|w_k|/\cos \varphi_{ik}\,, \tag{23}$$

where $|w_i|$ is the length of the altitude of Σ from A_i and φ_{ik} is the interior angle between the $(n - 1)$-faces opposite to A_i and A_k. (If $\varphi_{ik} = \pi/2$, r_{ik} will be infinite.)

Proof. Assume that the vectors $u_{ik} = A_i - A_k$, $i < k$, $i, k = 1, \ldots, n + 1$ (or, in fact, the corresponding edges as segments) admit electrical invariance. Suppose Σ has an obtuse interior angle between the faces, say, ω_1 and ω_2 opposite to A_1 and A_2. One can then choose a quadrant-space S in $\mathbf{E}_n$ whose basic space is the intersection of ω_1 and ω_2, which is contained in the intersection of half-spaces $(\omega_1A_1) \cap (\omega_2A_2)$ but contains neither A_1 nor A_2. It is geometrically clear that S contains no half-line parallel or anti-parallel to u_{ik} except those parallel to its basis. By Thm. (2.5), we obtain a contradiction.

To prove the converse, assume none of the interior angles of Σ is obtuse. By Thm. (1.1), it suffices to prove that

$$\sum_{\substack{i,k=1\\ i<k}}^{n} r_{ik}^{-1}(x, u_{ik})^2 = (x, x) \tag{24}$$

for r_{ik} defined by (23).

It follows immediately from (23), Thm. (3.1) and (10) that

$$r_{ik}^{-1} = -(p_i, p_k)$$

where p_i are vectors satisfying (7), (8) and (9). Thus (24) is equivalent to

$$-\tfrac{1}{2}\sum_{i,k=1}^{n} (p_i, p_k)\,(x, u_{ik})\,(y, u_{ik}) = (x, y) \tag{25}$$

where, of course, $u_{ki} = -u_{ik}$ and $u_{ii} = 0$, $i, k = 1, \ldots, n + 1$, and it suffices to prove this for $x = p_r$, $y = p_s$, $r, s = 1, \ldots, n + 1$.

By (9) and (8),

$$-\tfrac{1}{2}\sum_{i,k=1}^{n+1}(p_i, p_k)(p_r, u_{ik})(p_s, u_{ik}) =$$

$$= -\tfrac{1}{2}\sum_{i,k=1}^{n+1}(p_i, p_k)(\delta_{ir} - \delta_{kr})(\delta_{is} - \delta_{ks}).$$

If $r = s$, the right-hand side is

$$-\tfrac{1}{2}\sum_{i,k=1}^{n+1}(p_i, p_k)(\delta_{ir} + \delta_{kr} - 2\delta_{ir}\delta_{kr}) =$$

$$= -(p_r, \sum_{k=1}^{n+1} p_k) + (p_r, p_r) = (p_r, p_r).$$

If $r \neq s$, we obtain, since $\delta_{ir}\delta_{is} = 0$,

$$-\tfrac{1}{2}\sum_{i,k=1}^{n+1}(p_i, p_k)(-\delta_{ir}\delta_{ks} - \delta_{kr}\delta_{is}) = (p_r, p_s).$$

The proof is complete.

We are now able to prove a geometric theorem.

(3.8) **Theorem.** *A necessary and sufficient condition that an n-simplex Σ in $\mathbf{E}_n$ has no interior angle obtuse is: for any quadrant space S in $\mathbf{E}_n$, there is an edge of Σ which is not parallel with the basis of S but which is parallel with a half-line in S.*

Proof. Let Σ have an obtuse interior angle. Then as in the proof of Thm. (3.7) there is a quadrant space S which contains no line parallel to an edge of Σ except those parallel to its basis. Conversely, if Σ has no interior angle obtuse, the set of edges admits electrical invariance by Thm. (3.7). By Thm. (2.5), the set of edges has the property mentioned above.

Let us recall now that in [3], and previously without proof in [2], it was shown that to any connected electrical resistive network (i.e., a network containing resistors only) with $n + 1$ nodes $N_1, \ldots, N_{n+1}$ and conductivity c_{ik} in the branch between N_i and N_k ($i \neq k$, $k = 1, \ldots, n + 1$), there exists in $\mathbf{E}_n$ an n-simplex Σ with vertices $A_1, \ldots, A_{n+1}$ such that

$$c_{ik} = -(p_i, p_k), \quad i, k = 1, \ldots, n + 1, \quad i \neq k, \tag{26}$$

p_i being the vectors defined by (7) and (8). This simplex Σ has thus no obtuse interior angle. For any $i, k = 1, \ldots, n + 1$, $i \neq k$, the total resistance R_{ik} of the network between the nodes N_i and N_k is equal to the square of the length of the edge A_iA_k.

We shall prove the following theorem:

(3.9) **Theorem.** *Any connected electrical resistive network with $n + 1$ nodes can be realized by a network in $\mathbf{E}_n$ which is electrically invariant.*

Proof. By the recalled result, there exists an n-simplex Σ in $\mathbf{E}_n$ satisfying (2.6). Comparing this with the formula in Thm. (3.7), it follows that edges of Σ admit electrical invariance exactly with given resistances.

In the conclusion, we shall investigate this simplicial network Σ in more detail. First, let us prove:

(3.10) **Lemma.** *Let A_k be a fixed vertex of Σ, let w be a non-zero vector parallel to the face opposite to A_k. Let F_w be a homogeneous field in $\mathbf{E}_n$ corresponding to the vector w. Then the absorption of energy of the system S_k of segments A_kA_j, $j = 1, \ldots$ $\ldots, n + 1$, $j \neq k$, between any two parallel hyperplanes $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ orthogonal to w is proportional to the distance of $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, whenever the (open) layer between $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ has a non-void intersection with S_k but does not contain any vertex A_j of Σ unless the vector A_jA_k is orthogonal to w.*

Proof. Denote by M_0, M_1, M_2 the sets of indices $j = 1, \ldots, n + 1$, $j \neq k$ for which $(u_{jk}, w) = 0$, $(u_{jk}, w) > 0$, $(u_{j,k}, w) < 0$, respectively. Let $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ be hyperplanes satisfying the conditions above and let $\boldsymbol{H}$ be the hyperplane orthogonal to w which contains A_k. Then either $\boldsymbol{H}$ is contained in the (open) layer between $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$, or it is not. In the former case, the absorption A_{12} of S_k between $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ is equal to the sum of absorptions A_{01} between $\boldsymbol{H}$ and $\boldsymbol{H}_1$ and A_{02} between $\boldsymbol{H}$ and $\boldsymbol{H}_2$. Since one of the hyperplanes $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ intersects all segments A_iA_k for $i \in M_1$, the other all segments A_iA_k for $i \in M_2$. we can assume that $\boldsymbol{H}_1$ is the first and $\boldsymbol{H}_2$ the other. If d_i is the distance between $\boldsymbol{H}$ and $\boldsymbol{H}_i$, $i = 1, 2$, we have

$$A_{0i} = -\sum_{j \in M_i} (p_j, p_k) \frac{|(u_{ik}, w)|}{d_i|w|} \cdot \frac{d_i^2}{|w|^2} =$$

$$= \frac{d_i}{|w|^3} \Big(-\sum_{j \in M_i} (p_j, p_k) \, |(u_{jk}, w)|\Big)$$

since the conductivity of the part of A_jA_k between $\boldsymbol{H}_i$ and $\boldsymbol{H}$ is $-(p_j, p_k) \, |(u_{jk}, w)|/(d_i|w|)$ while the potential between $\boldsymbol{H}_i$ and $\boldsymbol{H}$ is $d_i/|w|$.

Let us show that

$$-\sum_{j \in M_1} (p_j, p_k) \, |(u_{jk}, w)| = -\sum_{j \in M_2} (p_j, p_k) \, |(u_{jk}, w)| \, . \tag{27}$$

For any vector $x \in \mathbf{E}_n$,

$$x = \sum_{j=1}^{n+1} (p_j, x) \, u_{jk}$$

since this is true, by (9), for the basis vector u_{mk}, $m = 1, \ldots, n+1$, $m \neq k$. Consequently,

$$\sum_{j=1}^{n+1} (p_j, p_k)(u_{jk}, w) = \Big(\sum_{j=1}^{n+1} (p_j, p_k)\, u_{jk}, w\Big) = (p_k, w) = 0$$

since w is in the face opposite to A_k and thus orthogonal to p_k. Therefore,

$$\sum_{j \in M_1} (p_j, p_k)(u_{jk}, w) + \sum_{j \in M_2} (p_j, p_k)(u_{jk}, w) = 0$$

(since $(u_{jk}, w) = 0$ for $j \in M_0$), which implies (27). Consequently,

$$A_{01} = Cd_1\,, \quad A_{02} = Cd_2$$

and thus A_{12} is a C-multiple of the distance between $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. It is easy to check that the same is true if $\boldsymbol{H}$ is not contained in the open layer between $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$.

We are now able to prove:

(3.11) **Theorem.** *The resistive simplicial network Σ in $\mathbf{E}_n$ described above has the following property: Let S_1, S_2 be any non-void disjoint subsets of the set of vertices of Σ, let $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ be (uniquely determined) parallel hyperplanes* in $\mathbf{E}_n$, *$\boldsymbol{H}_i$ containing S_i, $i = 1, 2$, both orthogonal to all the $(n-1)$-faces opposite to the vertices A_j which belong neither to S_1 nor to S_2. If w is a non-zero vector orthogonal to $\boldsymbol{H}_1$ and F_w a homogeneous electrical field corresponding to w then the absorption of energy of the part of Σ between $\boldsymbol{H}_1$ and a variable hyperplane $\boldsymbol{H}$ parallel to $\boldsymbol{H}_1$ and lying in the (closed) layer between $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ is a linear function of the distance of $\boldsymbol{H}$ from $\boldsymbol{H}_1$.*

Proof. Let $\boldsymbol{H}' \neq \boldsymbol{H}_1$ be the nearest hyperplane to $\boldsymbol{H}_1$ satisfying the above conditions which contains at least one vertex of Σ. The mentioned function Φ is clearly linear for $\boldsymbol{H}$ between $\boldsymbol{H}_1$ and $\boldsymbol{H}'$. If $\boldsymbol{H}' = \boldsymbol{H}_2$, we are finished. If not, let us show that Φ is linear also in the neighborhood of $\boldsymbol{H}'$. Let $\boldsymbol{H}'$ contain exactly the vertices $A_{k_1}, \ldots, A_{k_s}$. By Lemma (3.10), the contributions of the sets $S_{k_1}, \ldots, S_{k_s}$ to Φ are all linear in the neighborhood of $\boldsymbol{H}'$. The segments which belong to two of these sets do not intervene since they are parallel to $\boldsymbol{H}_1$, and eventual segments joining other pairs of vertices contribute linearly to Φ. Therefore, Φ is linear up to the second nearest hyperplane to $\boldsymbol{H}_1$ containing at least one vertex of Σ and the same argument shows that Φ is linear in the whole layer between $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$.

This theorem enables us to state a theorem on a general connected resistive network. It proves and strengthens the result from [3] already mentioned above.

(3.12) **Theorem.**. *Let $\mathcal{N}$ be a resistive network with $n+1$ nodes, let S_1, S_2 be disjoint subsets of the set of nodes. Let Σ be the corresponding simplicial network, $\bar{S}_1$, $\bar{S}_2$ the sets of vertices corresponding to S_1 and S_2. Then the total resistance $R(S_1, S_2)$ of $\mathcal{N}$ between S_1 and S_2* (each *S_i is considered as joined by shortcuts*) *is equal to the square $d^2(\bar{S}_1, \bar{S}_2)$ of the distance of the hyperplanes $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ defined in Thm. (3.11). The potential P at a node N_k, if $\bar{S}_1$, $\bar{S}_2$ have potentials P_1, P_2, is given by*

$$P = \lambda P_1 + (1 - \lambda) P_2 , \tag{28}$$

where λ is the number satisfying

$$A_k = \lambda B_k + (1 - \lambda) C_k , \tag{29}$$

B_k, C_k being the feet of perpendiculars from A_k to $\boldsymbol{H}_1$, $\boldsymbol{H}_2$.

Proof. By Thm. (3.9), there exists a simplicial network Σ in $\boldsymbol{E}_n$ which is electrically invariant and has the same resistances between its nodes as $\mathcal{N}$. If $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ are hyperplanes satisfying the conditions of Thm. (3.11) with respect to $\bar{S}_1$, $\bar{S}_2$, choose a homogenous field F in $\boldsymbol{E}_n$ for which the potential on $\boldsymbol{H}_i$ is constant and equal to P_i, $i = 1, 2$, $P_1 \neq P_2$. By (24), the absorption of energy of Σ with respect to F is then

$$\frac{1}{(w, w)^2} \sum_{1 \leq i \leq k \leq n+1} -(p_i, p_k)(u_{ik}, w)^2$$

where $w = (d(\bar{S}_1, \bar{S}_2)/|P_1 - P_2|) w_0$, w_0 being a unit vector orthogonal to $\boldsymbol{H}_1$. This is, by (24), equal to

$$(w, w)^{-1} = (P_1 - P_2)^2/d^2(\bar{S}_1, \bar{S}_2) . \tag{30}$$

On the other hand, it follows from linearity in Thm. (3.11) that Σ behaves in F like a homogeneous resistive segment in $\boldsymbol{E}_n$ which is perpendicular to $\boldsymbol{H}_1$, has length $d(\bar{S}_1, \bar{S}_2)$, touches both $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ and whose resistance R yields the same absorption $(P_1 - P_2)^2/R$ as Σ. By (30), $(P_1 - P_2)^2/d^2(\bar{S}_1, \bar{S}_2) = (P_1 - P_2)^2/R$ so that

$$R = d^2(\bar{S}_1, \bar{S}_2) .$$

However, R is equal to the total resistance of $\mathcal{N}$ between S_1 and S_2 since the current between S_1 and S_2, with potentials P_1 and P_2, is the same as in the field F. Therefore, also the potentials in the nodes N_k are the same as the potentials of A_k in F which proves (28) with (29).

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Souhrn

INVARIANTNÍ ODPOROVÉ OBVODY V EUKLIDOVSKÝCH PROSTORECH A JEJICH SOUVISLOST S GEOMETRIÍ

MIROSLAV FIEDLER

Vyšetřují se geometrické vlastnosti konečných soustav složených z homogenních odporových elementů tvaru úsečky s vlastností, že absorpce energie soustavy v libovolném elektrickém poli se nezmění při jakékoliv ortogonální transformaci soustavy.

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