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LEAST SQUARE METHOD FOR SOLVING CONTACT PROBLEMS WITH FRICTION OBEYING THE COULOMB LAW

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INTRODUCTION

Let us assume a structure consisting of two or more deformable bodies in mutual contact, involving friction on common surfaces. It is well-known that problems of such a kind can be formulated in terms of variational inequalities (see [1], [5]). One of the most classical models of friction, namely that obeying the Coulomb law, has been recently analyzed mathematically ([2]). In [4] the relation between the continuous problem and its discrete version, obtained by applying finite elements, is studied. The question of the numerical realization has still remained open. The aim of the present paper is to propose one possible way, based on the least square method. The original variational inequality formulation is replaced in finite dimension by a family of nonlinear equations, using the technique of the simultaneous penalization and regularization. These equations can be viewed as the state equations for a cost functional J, the global minimum of which will be searched. The paper is organized as follows: in Section 1, the continuous model is presented. Section 2 analyzes the finite element discretization of the continuous model, based on a mixed variational formulation introduced in [4]. The least square method is described in Section 3 and its relation to the method presented in Section 2 is established. Some remarks, concerning the numerical realization, especially how to calculate the gradient of J, are included in Section 4.

1. SETTING OF THE PROBLEM

Let an elastic body be represented by a polygonal domain $\Omega \subset \mathbb{R}_2$, the boundary $\partial\Omega$ of which consists of 3 disjoint and non-empty parts Γ_u, Γ_P and Γ_K, i.e.:

$$\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_P \cup \bar{\Gamma}_K .$$

We suppose that Γ_K (a contact part) is represented by one straight line segment parallel to the x_2 − axis (see Fig. 1).

On each part of $\partial\Omega$, different boundary conditions will be assumed. On Γ_u, the body is supposed to be fixed, i.e.:

$$u_i = 0 \quad \text{on} \quad \Gamma_u\,, \quad i = 1, 2\,. \tag{1.1}$$

On Γ_P, surface tractions are prescribed:

$$\tau_{ij}(\boldsymbol{u})\, n_j = P_i \quad \text{on} \quad \Gamma_P\,, \quad i = 1, 2\,. \tag{1.2}$$

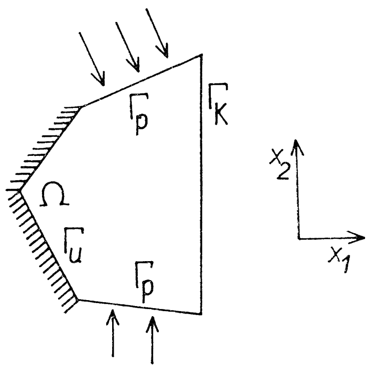

Fig. 1.

Finally, along Γ_K the body is unilaterally supported by a rigid foundation and the influence of friction is taken into account, i.e.

$$u_n \leqq 0\,, \quad T_n(\boldsymbol{u}) \leqq 0\,, \quad u_n\, T_n(\boldsymbol{u}) = 0 \quad \text{on} \quad \Gamma_K \tag{1.3}$$

(unilateral conditions),

$$\begin{cases} |T_t(\boldsymbol{u})| \leqq \mathscr{F}|T_n(\boldsymbol{u})| \\ \text{if} \quad |T_t(\boldsymbol{u})| < \mathscr{F}|T_n(\boldsymbol{u})| \quad \text{then} \quad u_t = 0 \\ \text{if} \quad |T_t(\boldsymbol{u})| = \mathscr{F}|T_n(\boldsymbol{u})| \quad \text{then there exists} \quad \lambda \geqq 0 \quad \text{such that} \end{cases} \tag{1.4}$$

$$u_t = -\lambda T_t(\boldsymbol{u})$$

(Coulomb law of friction)

on Γ_K.

Symbol $\tau(\boldsymbol{u}) = \{\tau_{ij}(\boldsymbol{u})\}_{i,j=1}^2$ denotes the stress tensor related to the linearized strain tensor $\varepsilon(\boldsymbol{u}) = \{\varepsilon_{ij}\}_{i,j=1}^2$ by means of the linear Hooke's law:

$$\tau_{ij}(\boldsymbol{u}) = c_{ijkl}\, \varepsilon_{kl}(\boldsymbol{u})\,, \quad \varepsilon_{kl}(\boldsymbol{u}) = 1/2(\partial u_k/\partial x_l + \partial u_l/\partial x_k)\,. \tag{1.5}$$

Elasticity coefficients c_{ijkl} are supposed to be bounded and measurable in Ω (i.e. $c_{ijkl} \in L^\infty(\Omega)$), satisfying the usual symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klij} \quad \text{a.e. in} \quad \Omega$$

and the ellipticity condition:

$$\exists \tilde{\alpha} > 0 \quad \text{such that} \quad c_{ijkl}\zeta_{ij}\,\zeta_{kl} \geqq \tilde{\alpha}\zeta_{ij}\zeta_{ij} \quad \forall \zeta_{ij} = \zeta_{ji} \in \mathbb{R}_1 \quad \text{a.e. in} \quad \Omega\,.$$

u_n, u_t are respectively, the normal and tangential components of the displacement field $\mathbf{u} = (u_1, u_2)$. Similarly, $T_n(\mathbf{u})$, $T_t(\mathbf{u})$ denote the normal and tangential components, respectively, of the stress vector $\mathbf{T}(\mathbf{u}) = (\tau_{1j}(\mathbf{u})\, n_j, \tau_{2j}(\mathbf{u})\, n_j)$. Finally, $\mathscr{F}$ is the coefficient of the Coulomb friction. By *a classical solution* of the Signorini problem with friction obeying the Coulomb law, we mean a displacement field $\mathbf{u}$ which is in the equilibrium state with a given body force $\mathbf{F} = (F_1, F_2)$, i.e. satisfies the equilibrium equations

$$(1.6) \qquad \partial\tau_{ij}/\partial x_j + F_i = 0 \quad \text{in} \quad \Omega\,, \quad i = 1, 2$$

and the boundary conditions (1.1)–(1.4). Justification and derivation of (1.3) and (1.4) can be found in [1].

In order to give the weak form of the problem in question, we shall assume a simpler model involving friction, the so called *model with a given friction*. The classical formulation of such a problem can be formally obtained by replacing the unknown value $|T_n(\mathbf{u})|$ by a known function (or more generally, functional) g. Let us introduce the following sets:

$$V = \{v \in H^1(\Omega) |\; v = 0 \text{ on } \Gamma_u\}\,,$$

$$\mathbf{V} = V \times V\,,$$

$$K = \{\mathbf{v} \in \mathbf{V} |\; v_n \leqq 0 \text{ on } \Gamma_K\}\,,$$

$$H^{1/2}(\Gamma_K) = \{\mu \in L^2(\Gamma_K) \mid \exists \mu \in V : \mu = v \text{ on } \Gamma_K\}\,,$$

$$H^{-1/2}(\Gamma_K) = (H^{1/2}(\Gamma_K))' \quad (\text{the dual space to } H^{1/2}(\Gamma_K))\,,$$

$$H_+^{-1/2}(\Gamma_K) = \{\mu^* \in H^{-1/2}(\Gamma_K) \mid \langle \mu^*, v\rangle \geqq 0 \;\forall v \in V, v \geqq 0 \text{ on } \Gamma_K\}\,.$$

The symbol $\langle\ ,\ \rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma_K)$ and $H^{1/2}(\Gamma_K)$.

Let $g \in H_+^{-1/2}(\Gamma_K)$ be given. By *a weak solution of the Signorini problem with a given friction* we mean a function $\mathbf{u} \equiv \mathbf{u}(g) \in K$ such that

$$(\mathscr{P}) \qquad a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \langle \mathscr{F} g, |v_t| - |u_t| \rangle \geqq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K\,,$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \tau_{ij}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})\, \mathrm{d}x\,,$$

$$L(\mathbf{v}) = \int_\Omega F_i v_i \,\mathrm{d}x + \int_{\Gamma_P} P_i v_i \,\mathrm{d}s\,, \quad \mathbf{F} \in (L^2(\Omega))^2, \mathbf{P} \in (L^2(\Gamma_P))^2\,.$$

Using classical results of the calculus of variations one can easily prove the existence and the uniqueness of $\mathbf{u} \in K$, solving $(\mathscr{P})$. Applying Green's formula to $(\mathscr{P})$ it is readily seen that $-T_n(\mathbf{u}(g)) \in H_+^{-1/2}(\Gamma_K)$. Hence a mapping $\Phi : H_+^{-1/2}(\Gamma_K) \to H_+^{-1/2}(\Gamma_K)$ can be defined by

$$(1.7) \qquad \Phi(g) = -T_n(\mathbf{u})\,.$$

By a *variational solution of the Signorini problem with Coulomb friction* we mean any function $\boldsymbol{u} \in K$ satisfying

$$\Phi(-T_n(\boldsymbol{u})) = -T_n(\boldsymbol{u}),$$

i.e. $-T_n(\boldsymbol{u})$ is a fixed point of the mapping Φ in $H_+^{-1/2}(\Gamma_K)$. The existence of such a fixed point has been studied in [2] in the case when Ω is an infinitely long strip and $\Gamma_P = \emptyset$ and in [3] for a bounded domain with a smooth boundary $\partial\Omega$.

2. FINITE ELEMENT DISCRETIZATION

An approximation of the Signorini problem with friction obeying the Coulomb law can be defined by means of finite elements. Let $\{\mathscr{T}_h\}$, $h \to 0+$ be *a regular family* of triangulations of $\bar{\Omega}$, which is consistent with the decomposition of $\partial\Omega$ into Γ_u, Γ_P and Γ_K. With any $\mathscr{T}_h$ the following finite dimensional spaces will be associated:

$$V_h = \{v_h \in C(\bar{\Omega}) \mid v_{h|T} \in P_1(T),\ v_h = 0 \text{ on } \Gamma_u\},$$

$$\boldsymbol{V}_h = V_h \times V_h,$$

i.e. V_h contains all piecewise linear functions over a given triangulation $\mathscr{T}_h$. Let $\{\mathscr{T}_H\}$, $H \to 0+$ be a partition of Γ_K, nodes of which will be denoted by $b_1, \ldots, b_{m(H)}$. In the sequel we shall consider families of $\{\mathscr{T}_H\}$ satisfying

$$\exists \tilde{\beta} > 0 : \frac{\min_i H_i}{H} \geqq \tilde{\beta},$$

where H_i = length of $\overline{b_i b_{i+1}}$, $H = \max_i H_i$. Let

$$L_H = \{\mu_H \in L^2(\Gamma_K) \mid \mu_{H|b_i b_{i+1}} \in P_0(b_i b_{i+1}),\ i = 1, \ldots, m(H)\},$$

$$\Lambda_H = \{\mu_H \in L_H \mid \mu_H \geqq 0 \text{ on } \Gamma_K\},$$

i.e. Λ_H contains all non-negative, piecewise-constant functions over $\mathscr{T}_H$. Analogously to the continuous case, we start with the approximation of the auxiliary problem $(\mathscr{P})$.

Let $g_H \in \Lambda_H$ be given. We look for a pair $\{\boldsymbol{u}_h, \lambda_H\} \in \boldsymbol{V}_h \times \Lambda_H$, satisfying

$$(\mathscr{P})_{hH} \begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + \langle \lambda_H, v_{hn} - u_{hn} \rangle + \langle \mathscr{F} g_H, |v_{ht}| - |u_{ht}| \rangle \geqq L(\boldsymbol{v}_h - \boldsymbol{u}_h) \\ \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \langle \mu_H - \lambda_H, u_{hn} \rangle \leqq 0 \quad \forall \mu_H \in \Lambda_H. \end{cases}$$

The symbol $\langle\ ,\ \rangle$ denotes the scalar product in $L^2(\Gamma_K)$.

Remark 2.1. $\lambda_H \in \Lambda_H$ satisfying $(\mathscr{P})_{hH}$ is the Lagrange multiplier associated with the unilateral boundary condition on Γ_K. $-\lambda_H$ plays the role of the approximate normal stress along Γ_K.

Next, we shall suppose that the following condition is satisfied:

$$(\mathbf{S}) \qquad \mu_H \in L_H \quad \langle \mu_H, z_h \rangle = 0 \quad \forall z_h \in V_h \Rightarrow \mu_H = 0 .$$

An equivalent form of $(\mathbf{S})$ is

$$\exists \beta > 0 \quad \forall \mu_H \in L_H : \sup_{z_h \in V_h} \frac{\langle \mu_H, z_h \rangle}{\|z_h\|_{1,\Omega}} \geqq \beta .$$

One can easily verify that under the condition $(\mathbf{S})$, there exists a unique solution $\{\mathbf{u}_h, \lambda_H\}$ of $(\mathscr{P})_{hH}$.

Interpretation of $(\mathscr{P})_{hH}$

Let

$$K_{hH} = \{\mathbf{v}_h \in \mathbf{V}_h \mid \langle \mu_H, v_{hn} \rangle \leqq 0 \ \forall \mu_H \in \Lambda_H\} .$$

K_{hH} contains all functions from $\mathbf{V}_h$, the mean value of the normal component v_{hn} of which is non-positive on any $\overline{b_i b_{i+1}}$. $i = 1, \ldots, m(H)$.

Substituting $\mu_H = 0, 2\lambda_H$ into the second relation of $(\mathscr{P})_{hH}$, we have

$$\langle \lambda_H, u_{hn} \rangle = 0 , \quad \langle \mu_H, u_{hn} \rangle \leqq 0 \quad \forall \mu_H \in \Lambda_H ,$$

i.e. $\mathbf{u}_h \in K_{hH}$ and

$$(2.1) \qquad a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + \langle \mathscr{F} g_H, |v_{ht}| - |u_{ht}| \rangle \geqq L(\mathbf{v}_h - \mathbf{u}_h) \quad \forall \mathbf{v}_h \in K_{hH} .$$

Let $\Phi_H : \Lambda_H \to \Lambda_H$ be a mapping defined as follows:

$$(\mathbf{P})_H \qquad \Phi_H(g_H) = \lambda_H .$$

Φ_H can be viewed as an approximation of the mapping Φ defined by (1.7). The main result of this section is

Theorem 2.1. *For any $\mathscr{F} \in C(\Gamma_K)$, $\mathscr{F} \geqq 0$ there exists at least one solution of $(\mathbf{P})_H$.*

Proof. i) Φ_H is a continuous mapping from Λ_H into itself (see [4], Th. 2.3).

ii) We shall show that

$$\Phi_H(B_r \cap \Lambda_H) \subset B_r \cap \Lambda_H$$

for any $r \geqq r_0$, where r_0 does not depend on $\mathscr{F}$. B_r denotes the ball with the center at the origin and the radius equal to r measured in a suitable topology (see (2.6) below) Substituting $\mathbf{v}_h = 0, 2\mathbf{u}_h$ into (2.1) we get

$$(2.2) \qquad a(\mathbf{u}_h, \mathbf{u}_h) + \langle \mathscr{F} g_H, |u_{ht}| \rangle = L(\mathbf{u}_h) ,$$

hence

$$(2.3) \qquad \|\mathbf{u}_h\|_{1,\Omega} \leqq 1/\alpha(\|\mathbf{F}\|_{0,\Omega} + \|\mathbf{P}\|_{0,\Gamma_P})$$

by virtue of Korn's inequality.

Let

$$\mathring{\mathbf{V}}_h = \{\mathbf{v}_h \in \mathbf{V}_h \mid \mathbf{v}_h = (v_{h1}, 0)\} . \tag{2.4}$$

As $\Gamma_K \parallel x_2$, we have $v_{hn} = v_{h1}$, $v_{ht} = 0$ if $\mathbf{v}_h \in \mathring{\mathbf{V}}_h$ and

$$a(\mathbf{u}_h, \mathbf{v}_h) + \langle \lambda_H, v_{h1} \rangle = L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathring{\mathbf{V}}_h .$$

Hence

$$\sup_{V_h} \frac{\langle \lambda_H, v_{h1} \rangle}{\|v_{h1}\|_{1,\Omega}} \leqq M\|\mathbf{u}_h\|_{1,\Omega} + (\|\mathbf{F}\|_{0,\Omega} + \|\mathbf{P}\|_{0,\Gamma_P}) . \tag{2.5}$$

Let us introduce the following notation:

$$\|\mu_H\|_{-1/2,h} = \sup_{z_h \in V_h} \frac{\langle \mu_H, z_h \rangle}{\|z_h\|_{1,\Omega}} , \quad \mu_H \in L_H . \tag{2.6}$$

If the condition (**S**) is satisfied, then (2.6) defines a norm on L_H. Moreover,

$$\exists \gamma > 0 \quad \forall \mu_H \in L_H : \|\mu_H\|_{-1/2,h} \geqq \gamma \|\mu_H\|_{-1/2} . \tag{2.7}$$

The constant γ in general depends on h, H. (2.5) and (2.6) result in

$$\|\lambda_H\|_{-1/2,h} \leqq (M/\alpha + 1)(\|\mathbf{F}\|_{0,\Omega} + \|\mathbf{P}\|_{0,\Gamma_P}) .$$

Let us set

$$r_0 = (M/\alpha + 1)(\|\mathbf{F}\|_{0,\Omega} + \|\mathbf{P}\|_{0,\Gamma_P}) .$$

Then $\Phi_H(B_r \cap \Lambda_H) \subset B_r \cap \Lambda_H$ for any $r \geqq r_0$. Using the Schauder fixed-point theorem we arrive at the assertion.

It can be shown that

$$\|\lambda_H - \bar{\lambda}_H\|_{-1/2} \equiv \|\Phi_H(g_H) - \Phi_H(\bar{g}_H)\|_{-1/2} \leqq q\|g_H - \bar{g}_H\|_{-1/2} ,$$

where $q = C(H)[\mathscr{F}]$, $[\mathscr{F}] = \max_{\Gamma_K} \mathscr{F}(x)$ and $C(H) \to +\infty$ if $H \to 0+$ (for the proof see [4]). If

$$[\mathscr{F}] < 1/C(H) , \tag{2.8}$$

then Φ_H is contractive and its unique fixed-point can be found by the method of successive approximations. Unfortunately, to keep $q \in (0, 1)$, $[\mathscr{F}]$ has to tend to zero whenever $H \to 0+$. This is the reason for which the method of successive approximations need not be successful, in general. Below we present an alternative approach, based on the smoothening of $(\mathscr{P})_{hH}$ combined with the least square method.

3. LEAST SQUARE METHOD FOR NUMERICAL SOLUTION OF $(\mathbf{P})_H$

Let $\beta : C^1 \to R_1$ be a function such that

- $\beta(x) \geqq 0 \ \forall x \in R_1$ and $\beta(x) = 0$ if and only if $x \leqq 0$
- β is monotone on R_1.

For any $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$ let

$$(\tilde{\beta}(\mathbf{u}_h), \mathbf{v}_h) = \sum_{i=1}^{M(H)} \beta(\bar{u}^i_{hn})\, \bar{v}^i_{hn} H_i = \sum_{i=1}^{M(H)} \beta(\bar{u}^i_{h1})\, \bar{v}^i_{h1} H_i$$

and

$$j_\varepsilon(g_H, \mathbf{v}_h) = \langle \mathscr{F} g_H, \sqrt{(v^2_{ht} + \varepsilon^2)} \rangle\,, \quad \varepsilon > 0\,.$$

Here $\bar{u}^i_{h1}$ denotes the mean value of u_{h1} on $\overline{b_i b_{i+1}}$:

$$\bar{u}^i_{h1} = 1/H_i \int_{b_i b_{i+1}} u_{h1}\, \mathrm{d}s\,.$$

Lemma 3.1. *The following identity holds:*

$$(\tilde{\beta}(\mathbf{u}_h), \mathbf{v}_h) = \langle \omega_H, v_{hn} \rangle\,,$$

where $\omega_H \in \Lambda_H$ is defined by

$$\omega_{H|b_i b_{i+1}} = \beta(\bar{u}^i_{h1})\, \chi_i\,,$$

with χ_i being the characteristic function of $\overline{b_i b_{i+1}}$.

Proof.

$$(\tilde{\beta}(\mathbf{u}_h), \mathbf{v}_h) = \sum_{i=1}^{M(H)} \beta(\bar{u}^i_{h1})\, \bar{v}^i_{h1} H_i = \sum_{i=1}^{M(H)} \beta(\bar{u}^i_{h1}) \int_{b_i b_{i+1}} v_{h1}\, \mathrm{d}s = \int_{\Gamma_K} \omega_H v_{h1}\, \mathrm{d}s\,.$$

Lemma 3.2. *The following equivalence holds:*

$$\mathbf{u}_h \in \mathbf{V}_h\,, \quad (\tilde{\beta}(\mathbf{u}_h), \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h \Leftrightarrow \mathbf{u}_h \in K_{hH}\,.$$

Proof. Let $\mathbf{u}_h \in \mathbf{V}_h$ be such that

$$(\tilde{\beta}(\mathbf{u}_h), \mathbf{v}) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h\,.$$

From this and Lemma 3.1 one has

$$\langle \omega_H, v_{hn} \rangle = \langle \omega_H, v_{h1} \rangle = 0 \quad \forall v_{h1} \in V_h\,,$$

so that $\omega_H = 0$ on Γ_K due to the condition $(\mathbf{S})$. Definitions of ω_H and K_{hH} yield the assertion of the lemma.

Let $\varepsilon > 0$ be a parameter tending to zero and let us consider the following penalized-regularized problem:

$$(\mathscr{P})_\varepsilon \quad \begin{cases} \text{find} \quad \mathbf{u}^\varepsilon_h \in \mathbf{V}_h \quad \text{such that} \\ a(\mathbf{u}^\varepsilon_h, \mathbf{v}) + 1/\varepsilon(\tilde{\beta}(\mathbf{u}^\varepsilon_h), \mathbf{v}_h) + j'_\varepsilon(g_H, \mathbf{u}^\varepsilon_h)\, \mathbf{v}_h = L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h\,, \end{cases}$$

with

$$j'_\varepsilon(g_H, \mathbf{u}^\varepsilon_h)\, \mathbf{v}_h = \int_{\Gamma_K} \mathscr{F} g_H \frac{u^\varepsilon_{ht} v_{ht}}{\sqrt{(u^{\varepsilon 2}_{ht} + \varepsilon^2)}}\, \mathrm{d}s\,.$$

It is readily seen that for any $\varepsilon > 0$ there exists a unique solution $\boldsymbol{u}_h^\varepsilon$ of $(\mathscr{P})_\varepsilon$. Now, we shall define a mapping $\Psi_H^\varepsilon : \Lambda_H \to \Lambda_H$ by means of

$$\Psi_H^\varepsilon(g_H)\big|_{b_i b_{i+1}} = 1/\varepsilon\, \beta(\bar{u}_{h1}^{\varepsilon i})\, \chi_i \equiv 1/\varepsilon \omega_H^\varepsilon\big|_{b_i b_{i+1}} . \tag{3.1}$$

Remark 3.1. The function $-\Psi_H^\varepsilon(g_H)$ will play again the role of the approximate normal stress along Γ_K. A function $\boldsymbol{u}_h^\varepsilon$ satisfying $(\mathscr{P})_\varepsilon$, can be obtained by solving a nonlinear system of algebraic equations.

Analogously to the approach used in the last section, we shall consider the problem of finding a fixed point of the mapping Ψ_H^ε in Λ_H, i.e.: find $\lambda_H^\varepsilon \in \Lambda_H$ such that

$$\Psi_H^\varepsilon(\lambda_H^\varepsilon) = \lambda_H^\varepsilon . \tag{$\mathbf{P}$}_\varepsilon$$

Next, we shall study

i) the existence of λ_H^ε;
ii) the relation between the solutions of $(\mathbf{P})_\varepsilon$ and $(\mathbf{P})$ if $\varepsilon \to 0+$.

Theorem 3.1. *For any $\mathscr{F} \in C(\Gamma_K)$, $\mathscr{F} \geqq 0$ and $\varepsilon > 0$ there exists at least one solution of $(\mathbf{P})_\varepsilon$.*

Proof is analogous to that of Theorem 2.1. From the definition of $j_\varepsilon(\boldsymbol{v}_h)$ and $(\tilde{\beta}(\boldsymbol{u}_h^\varepsilon), \boldsymbol{v}_h)$ it follows that

$$\alpha\|\boldsymbol{u}_h^\varepsilon\|_{1,\Omega}^2 \leqq a(\boldsymbol{u}_h^\varepsilon, \boldsymbol{u}_h^\varepsilon) \leqq a(\boldsymbol{u}_h^\varepsilon, \boldsymbol{u}_h^\varepsilon) + 1/\varepsilon(\tilde{\beta}(\boldsymbol{u}_h^\varepsilon), \boldsymbol{u}_h^\varepsilon) + j_\varepsilon'(g_H, \boldsymbol{u}_h^\varepsilon)\, \boldsymbol{u}_h^\varepsilon = L(\boldsymbol{u}_h^\varepsilon) ,$$

from which

$$\|\boldsymbol{u}_h^\varepsilon\|_{1,\Omega} \leqq 1/\alpha(\|\boldsymbol{F}\|_{0,\Omega} + \|\boldsymbol{P}\|_{0,\Gamma_P})$$

independently of $\varepsilon > 0$. Let us substitute a function $\boldsymbol{v}_h \in \mathring{\boldsymbol{V}}_h$ into $(\mathscr{P})_\varepsilon$. As $v_{ht} = 0$, we immediately get

$$a(\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}_h) + \langle 1/\varepsilon\omega_H^\varepsilon, v_{hn}\rangle = L(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathring{\boldsymbol{V}}_h ,$$

so that

$$\|1/\varepsilon\omega_H^\varepsilon\|_{-1/2,h} = \sup_{v_h} \frac{\langle 1/\varepsilon\omega_H^\varepsilon, v_{h1}\rangle}{\|v_{h1}\|_{1,\Omega}} \leqq M\|\boldsymbol{u}_h^\varepsilon\|_{1,\Omega} + \|\boldsymbol{F}\|_{0,\Omega} + \|\boldsymbol{P}\|_{0,\Gamma_P} \leqq r_0 . \tag{3.2}$$

Hence the mapping Ψ_H^ε maps a set $\Lambda_H \cap B_r$ into itself, where

$$B_r = \{\mu_H \in L_H \mid \|\mu_H\|_{-1/2,h} \leqq r,\ r \geqq r_0\} .$$

It remains to verify that Ψ_H^ε is continuous.

Let $g_H, \bar{g}_H \in \Lambda_H$ be given and let $\boldsymbol{u}_h^\varepsilon, \boldsymbol{z}_h^\varepsilon \in \boldsymbol{V}_h$ be the corresponding solutions of the penalized – regularized problems:

$$a(\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}_h) + 1/\varepsilon(\tilde{\beta}(\boldsymbol{u}_h^\varepsilon), \boldsymbol{v}_h) + j_\varepsilon'(g_H, \boldsymbol{u}_h^\varepsilon)\, \boldsymbol{v}_h = L(\boldsymbol{v}_h) ,$$

$$a(\boldsymbol{z}_h^\varepsilon, \boldsymbol{v}_h) + 1/\varepsilon(\tilde{\beta}(\boldsymbol{z}_h^\varepsilon), \boldsymbol{v}_h) + j_\varepsilon'(g_H, \boldsymbol{z}_h^\varepsilon)\, \boldsymbol{v}_h = L(\boldsymbol{v}_h) .$$

Substituting $\mathbf{z}_h^\varepsilon - \mathbf{u}_h^\varepsilon$, $\mathbf{u}_h^\varepsilon - \mathbf{z}_h^\varepsilon$ into the first and the second equation, respectively, and summing up these equations one has

$$(3.2) \qquad a(\mathbf{u}_h^\varepsilon - \mathbf{z}_h^\varepsilon, \mathbf{u}_h^\varepsilon - \mathbf{z}_h^\varepsilon) \leqq 1/\varepsilon(\tilde{\beta}(\mathbf{u}_h^\varepsilon) - \tilde{\beta}(\mathbf{z}_h^\varepsilon), \mathbf{z}_h^\varepsilon - \mathbf{u}_h^\varepsilon) + $$
$$+ \int_{\Gamma_K} \mathscr{F}(g_H - \bar{g}_H) \frac{z_{ht}^\varepsilon}{\sqrt{(z_{ht}^{\varepsilon 2} + \varepsilon^2)}} (z_{ht}^\varepsilon - u_{ht}^\varepsilon) \, ds +$$
$$+ \int_{\Gamma_K} \mathscr{F} g_H \left(\frac{u_{ht}^\varepsilon}{\sqrt{(u_{ht}^{\varepsilon 2} + \varepsilon^2)}} - \frac{z_{ht}^\varepsilon}{\sqrt{(z_{ht}^{\varepsilon 2} + \varepsilon^2)}} \right) (z_{ht}^\varepsilon - u_{ht}^\varepsilon) \, ds \leqq$$
$$\leqq \int_{\Gamma_K} \mathscr{F}(g_H - \bar{g}_H) \frac{z_{ht}^\varepsilon}{\sqrt{(z_{ht}^{\varepsilon 2} + \varepsilon^2)}} (z_{ht}^\varepsilon - u_{ht}^\varepsilon) \, ds \leqq c[\mathscr{F}] \, \|g_H - \bar{g}_H\|_{0,\Gamma_K} \, \|\mathbf{u}_h^\varepsilon - \mathbf{z}_h^\varepsilon\|_{1,\Omega} \, ,$$

the monotonicity of $\tilde{\beta}$ and $j'_\varepsilon(g_H, \mathbf{u}_h^\varepsilon)$ being taken into account. From (3.2) it follows that

$$(3.3) \qquad \|\mathbf{u}_h^\varepsilon - \mathbf{z}_h^\varepsilon\|_{1,\Omega} \leqq c[\mathscr{F}] \, \|g_H - \bar{g}_H\|_{0,\Gamma_K} \, .$$

Using the same approach as at the beginning of the proof, we get

$$(3.4) \qquad \|1/\varepsilon\omega_H^\varepsilon - 1/\varepsilon\bar{\omega}_H^\varepsilon\|_{-1/2,h} \leqq M \|\mathbf{u}_h^\varepsilon - \mathbf{z}_h^\varepsilon\|_{1,\Omega} \, ,$$

where

$$\omega_H^\varepsilon|_{b_i b_{i+1}} = \beta(\bar{u}_{h1}^{\varepsilon i}) \, \chi_i \, ,$$
$$\bar{\omega}_H^\varepsilon|_{b_i b_{i+1}} = \beta(\bar{z}_{h1}^{\varepsilon i}) \, \chi_i \, ,$$

and $\bar{u}_{h1}^{\varepsilon i}$, $\bar{z}_{h1}^{\varepsilon i}$ are the mean values of u_{h1}^ε, z_{h1}^ε on $b_i b_{i+1}$, respectively. Combining (3.3) with (3.4) we finally get

$$\|1/\varepsilon\omega_H^\varepsilon - 1/\varepsilon\bar{\omega}_H^\varepsilon\|_{-1/2,h} \leqq c[\mathscr{F}] \, M \|g_H - \bar{g}_H\|_{0,\Gamma_K}$$

which yields the continuity of the mapping Ψ_H^ε. The existence of a fixed point of Ψ_H^ε in $\Lambda_H \cap B_r$, $r \geqq r_0$, is then a direct consequence of the Schauder theorem. ■

A natural question arises if there is any relation between $(\mathbf{P})$ and $(\mathbf{P})_\varepsilon$. The answer is given by

Theorem 3.2. *Let* $\{\lambda_H^\varepsilon\}$, $\varepsilon \to 0+$ *be fixed points of the mappings* Ψ_H^ε *in* $\Lambda_H \cap B_r$, $r \geqq r_0$,

$$\lambda_H^\varepsilon|_{b_i b_{i+1}} = 1/\varepsilon \, \beta(\bar{u}_{h1}^{\varepsilon i}) \, \chi_i \, .$$

Then there exist subsequences $\{\mathbf{u}_h^{\varepsilon'}\} \subset \{\mathbf{u}_h^\varepsilon\}$, $\{\lambda_H^{\varepsilon'}\} \subset \{\lambda_H^\varepsilon\}$ *and elements* $\mathbf{u}_h^*$, λ_H^* *such that*

$$(3.5) \qquad \begin{aligned} \mathbf{u}_h^{\varepsilon'} &\to \mathbf{u}_h^* \, , \\ \lambda_H^{\varepsilon'} &\to \lambda_H^* \, , \quad \varepsilon' \to 0+ \, . \end{aligned}$$

At the same time λ_H^* *is a fixed point of* Φ_H *and* $\mathbf{u}_h^*$ *is a solution of* $(\mathscr{P})_{hH}$ *with* $g_H = \lambda_H^*$.

Proof. Let $\lambda_H^\varepsilon \in \Lambda_H \cup B_r$ be fixed points of Ψ_H^ε,

$$\lambda_H^\varepsilon\big|_{b_i b_{i+1}} = 1/\varepsilon\, \beta(\bar{u}_{h1}^{\varepsilon i})\, \chi_i\,.$$

Here $\boldsymbol{u}_h^\varepsilon \in \boldsymbol{V}_h$ denotes the solution of $(\mathscr{P})_\varepsilon$ with g_H equal to λ_H^ε. $\{\|\boldsymbol{u}_h^\varepsilon\|_{1,\Omega}\}$, $\{\|\lambda_H^\varepsilon\|_{-1/2,h}\}$ are bounded independently of ε as follows from Theorem 3.1. Therefore there exist subsequences $\{\boldsymbol{u}_h^{\varepsilon'}\} \subset \{\boldsymbol{u}_h^\varepsilon\}$, $\{\lambda_H^{\varepsilon'}\} \subset \{\lambda_H^\varepsilon\}$ and elements $\boldsymbol{u}_h^* \in \boldsymbol{V}_h$, $\lambda_H^* \in \Lambda_H$ such that (3.5) is satisfied. Let us write $\boldsymbol{v}_h - \boldsymbol{u}_h^{\varepsilon'}$ instead of $\boldsymbol{v}_h$ in $(\mathscr{P})_{\varepsilon'}$:

$$a(\boldsymbol{u}_h^{\varepsilon'}, \boldsymbol{v}_h - \boldsymbol{u}_h^{\varepsilon'}) + \langle \lambda_H^{\varepsilon'}, v_{hn} - u_{hn}^{\varepsilon'}\rangle + j'_{\varepsilon'}(\lambda_H^{\varepsilon'}, \boldsymbol{u}_h^{\varepsilon'})(\boldsymbol{v}_h - \boldsymbol{u}_h^{\varepsilon'}) = L(\boldsymbol{v}_h - \boldsymbol{u}_h^{\varepsilon'})$$
$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h\,.$$

Passing to the limit for $\varepsilon' \to 0+$ we have

$$a(\boldsymbol{u}_h^{\varepsilon'}, \boldsymbol{v}_h - \boldsymbol{u}_h^{\varepsilon'}) \to a(\boldsymbol{u}_h^*, \boldsymbol{v}_h - \boldsymbol{u}_h^*)\,,$$
$$\langle \lambda_H^{\varepsilon'}, v_{hn} - u_{hn}^{\varepsilon'}\rangle \to \langle \lambda_H^*, v_{hn} - u_{hn}^*\rangle\,,$$
$$j'_{\varepsilon'}(\lambda_H^{\varepsilon'}, \boldsymbol{u}_h^{\varepsilon'})(\boldsymbol{v}_h - \boldsymbol{u}_h^{\varepsilon'}) = \int_{\Gamma_K} \mathscr{F}\lambda_H^{\varepsilon'} \frac{u_{ht}^{\varepsilon'}}{\sqrt{((u_{ht}^{\varepsilon'})^2 + \varepsilon'^2)}}(v_{ht} - u_{ht})\,\mathrm{d}s \to$$
$$\to \int_{\Gamma_K} \mathscr{F}\lambda_H^* \operatorname{sign} u_{ht}^*(v_{ht} - u_{ht}^*)\,\mathrm{d}s = \langle \mathscr{F}\lambda_H^*, \operatorname{sign} u_{ht}^* v_{ht}\rangle -$$
$$- \langle \mathscr{F}\lambda_H^*, |u_{ht}^*|\rangle \leqq \langle \mathscr{F}\lambda_H^*, |v_{ht}| - |u_{ht}^*|\rangle\,.$$

These limits yield

$$a(\boldsymbol{u}_h^*, \boldsymbol{v}_h - \boldsymbol{u}_h^*) + \langle \lambda_H^*, v_{hn} - u_{hn}^*\rangle + \langle \mathscr{F}\lambda_H^*, |v_{ht}| - |u_{ht}^*|\rangle \geqq \geqq L(\boldsymbol{v}_h - \boldsymbol{u}_h^*) \quad \forall v_h \in \boldsymbol{V}_h\,. \tag{3.6}$$

Now we prove that

$$\langle \mu_H - \lambda_H^*, u_{hn}^*\rangle \leqq 0 \quad \forall \mu_H \in \Lambda_H\,.$$

First of all,

$$0 \leqq 1/\varepsilon' \sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^{\varepsilon' i}) \leqq c\,,$$

so that

$$\sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^{\varepsilon' i}) \to 0\,, \quad \varepsilon' \to 0+\,. \tag{3.7}$$

On the other hand,

$$\sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^{\varepsilon' i}) \to \sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^{*i})\,. \tag{3.8}$$

Comparing (3.7) with (3.8) we see that

$$\sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^{*i}) = 0\,,$$

i.e. $\boldsymbol{u}_h^* \in K_{hH}$.

Let $\mu_H \in \Lambda_H$ be arbitrary. Then

$$(3.9) \qquad \langle \mu_H, u^*_{hn} \rangle = \sum_{i=1}^{M(H)} \int_{b_i b_{i+1}} \mu_H u^*_{hn} \, ds = \sum_{i=1}^{M(H)} \mu_H \bar{u}^{*i}_{hn} H_i \leqq 0$$

as follows from the definition of Λ_H and the fact that $\bar{u}^{*i}_{hn} \leqq 0$. Finally, let us show that $\langle \lambda^*_H, u^*_{hn} \rangle = 0$. Indeed,

$$(3.10) \qquad \langle \lambda^*_H, u^*_{hn} \rangle = \lim_{\varepsilon' \to 0+} \langle \lambda^{\varepsilon'}_H, u^{\varepsilon'}_{hn} \rangle = \lim_{\varepsilon' \to 0+} \sum_{i=1}^{M(H)} 1/\varepsilon' \, \beta(\bar{u}^{\varepsilon' i}_{hn}) \, \bar{u}^{\varepsilon' i}_{hn} \geqq 0 \, .$$

At the same time $\langle \lambda^*_H, u^*_{hn} \rangle$ has to be non-positive as follows from (3.9). From (3.9) and (3.10) we finally get

$$(3.11) \qquad \langle \mu_H - \lambda^*_H, u^*_{hn} \rangle \leqq 0 \quad \forall \mu_H \in \Lambda_H \, .$$

(3.6) and (3.11) yield the assertion of the theorem.

4. NUMERICAL REALIZATION OF $(\mathbf{P})_\varepsilon$

Taking into account the results of the last section we see that the problem of finding a fixed point of the mapping Φ_H in Λ_H can be replaced by the same problem for a mapping Ψ^ε_H. Both problems are close in a certain sense (see Theorem 3.2). The least square method will be used for numerical realization of $(\mathbf{P})_\varepsilon$.

Let $J : \Lambda_H \to \mathbb{R}_1$ be the functional given by

$$(4.1) \qquad J(g_H) = \tfrac{1}{2} \| \Psi^\varepsilon_H(g_H) - g_H \|^2_{0,\Gamma_K} \, ,$$

where $\Psi^\varepsilon_H(g_H) \in \Lambda_H$ is defined by

$$\Psi^\varepsilon_H(g_H)|_{b_i b_{i+1}} = 1/\varepsilon \, \beta(\bar{u}^{\varepsilon i}_{hn}) \, \chi_i \, ,$$

$$\bar{u}^{\varepsilon i}_{hn} = 1/H_i \int_{b_i b_{i+1}} u^\varepsilon_{hn} \, ds$$

and $\mathbf{u}^\varepsilon_h \in \mathbf{V}_h$ is the solution of

$$(4.2) \qquad a(\mathbf{u}^\varepsilon_h, \mathbf{v}_h) + 1/\varepsilon(\tilde{\beta}(\mathbf{u}^\varepsilon_h), \mathbf{v}_h) + j'_{\varepsilon'}(g_H, \mathbf{u}^\varepsilon_h) \, \mathbf{v}_h = L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h \, .$$

The problem $(\mathbf{P})_\varepsilon$ can be now equivalently stated as the problem of finding global minimizers of J in Λ_H (at which J is equal to zero).

Remark 4.1. This formulation of $(\mathbf{P})_\varepsilon$ can be expressed in terms of the optimal control theory: J is *a cost functional*, $\mathbf{u}^\varepsilon_h$ is *the state variable* defined by *the state equation* (4.2) and $g_H \in \Lambda_H$ is *the control* of our problem.

For the numerical realization of the minimization of J over Λ_H, different optimization procedures may be used. Most of them require the knowledge of the gradient of J. This is why we sketch how to calculate it. To simplify notations, we omit the symbols ε, h, H and we shall write $\mathbf{u}$, g, ... instead of $\mathbf{u}^\varepsilon_h$, g^ε_H, ...

Let $\varphi \in L_H$ be given. Then

$$(4.3) \qquad J'(g)\,\varphi = (g - \Psi(g), \varphi)_{0,\Gamma_K} - (g - \Psi(g), \Psi'_g(g)\,\varphi)_{0,\Gamma_K} =$$
$$= (g - \Psi(g), \varphi)_{0,\Gamma_K} - (g - \Psi(g), \Psi'_u(\mathbf{u}(g))\,\mathbf{u}'_g\varphi)_{0,\Gamma_K} =$$
$$= (g - \Psi(g), \varphi)_{0,\Gamma_K} - (g - \Psi(g), \Psi'_u(\mathbf{u}(g))\,\boldsymbol{\omega})_{0,\Gamma_K},$$

where $\boldsymbol{\omega} \equiv \mathbf{u}'_g\varphi$. The symbols Ψ'_g, Ψ'_u etc. denote the differentiation of Ψ with respect to g, $\mathbf{u}$ etc. respectively.

Writing the state equation (4.2) for g and $g + \varphi$ we immediately get

$$(4.4) \qquad a(\boldsymbol{\omega}, \mathbf{v}) + 1/\varepsilon(\tilde{\beta}'_u(\mathbf{u})\,\boldsymbol{\omega}, \mathbf{v}) + j'_\varepsilon(\varphi, \mathbf{u})\,\mathbf{v} + \langle j'_{\varepsilon u}(g, \mathbf{u})\,\boldsymbol{\omega}, \mathbf{v}\rangle = 0 \quad \forall \mathbf{v} \in \mathbf{V}.$$

Here

$$\langle \tilde{\beta}'_{\varepsilon u}(\mathbf{u})\,\mathbf{s}, \mathbf{t}\rangle = \sum_{i=1}^{M(H)} \beta'(\bar{u}^i_n)\,\bar{s}^i_n \bar{t}^i_n \quad \forall \mathbf{s}, \mathbf{t} \in \mathbf{V}_h,$$

$$\langle j'_{\varepsilon u}(g, \mathbf{u})\,\boldsymbol{\omega}, \mathbf{v}\rangle = \int_{\Gamma_K} \mathscr{F} g \frac{\varepsilon^2 \omega_t}{(u_t^2 + \varepsilon^2)\sqrt{(u_t^2 + \varepsilon^2)}}\, v_t \,\mathrm{d}s.$$

Let $\boldsymbol{\varrho} \in \mathbf{V}_h$ be the solution of *the adjoint equation*

$$(4.5) \qquad a(\boldsymbol{\varrho}, \mathbf{v}) + 1/\varepsilon(\tilde{\beta}'_u(\mathbf{u})\,\boldsymbol{\varrho}, \mathbf{v}) + \langle j'_{\varepsilon u}(g, \mathbf{u})\,\boldsymbol{\varrho}, \mathbf{v}\rangle =$$
$$= -(\Psi(g) - g, \Psi'_u(\mathbf{u})\,\mathbf{v})_{0,\Gamma_K} \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

Inserting $\boldsymbol{\omega}$ into (4.5) instead of $\mathbf{v}$ and comparing it with (4.4) we see that

$$-(\Psi(g) - g, \Psi'_u(\mathbf{u})\,\boldsymbol{\omega})_{0,\Gamma_K} = a(\boldsymbol{\varrho}, \boldsymbol{\omega}) + 1/\varepsilon(\tilde{\beta}'_u(\mathbf{u})\,\boldsymbol{\varrho}, \boldsymbol{\omega}) +$$
$$+ \langle j'_{\varepsilon u}(g, \mathbf{u})\,\boldsymbol{\varrho}, \boldsymbol{\omega}\rangle = a(\boldsymbol{\omega}, \boldsymbol{\varrho}) + 1/\varepsilon(\tilde{\beta}'_u(\mathbf{u})\,\boldsymbol{\omega}, \boldsymbol{\varrho}) + \langle j'_{\varepsilon u}(g, \mathbf{u})\boldsymbol{\omega}, \boldsymbol{\varrho}\rangle =$$
$$= -j'_\varepsilon(\varphi, \mathbf{u})\,\boldsymbol{\varrho}.$$

Hence

$$J'(g)\,\varphi = (g - \Psi(g), \varphi)_{0,\Gamma_K} - j'_\varepsilon(\varphi, \mathbf{u})\,\boldsymbol{\varrho} \equiv$$
$$= \int_{\Gamma_K} (g - \Psi(g))\,\varphi \,\mathrm{d}s - \int_{\Gamma_K} \mathscr{F} \frac{\varphi u_{ht}}{\sqrt{(u_{ht}^2 + \varepsilon^2)}}\, \varrho_t \,\mathrm{d}s,$$

where $\boldsymbol{\varrho} \in \mathbf{V}$ is the unique solution of (4.5).

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Souhrn

METODA NEJMENŠÍCH ČTVERCŮ PRO ŘEŠENÍ KONTAKTNÍCH ÚLOH S COULOMBOVSKÝM TŘENÍM

JAROSLAV HASLINGER

Předložená práce se zabývá numerickou realizací kontaktních úloh s coulombovským třením. Původní úloha je formulována jako problém nalezení pevného bodu jistého operátoru, generovaného variační nerovnicí. Tato nerovnice je pomocí penalizační a regularizační metody transformována na systém variačních nelineárních rovnic, které generují jiné operátory, jež jsou však v jistém smyslu blízké k výše vzpomenutému. Problém nalezení pevných bodů těchto operátorů se řeší pomocí metody nejmenších čtverců, v níž příslušné rovnice vystupují coby stavové rovnice a odpovídající kvadratická odchylka hraje úlohu kriteriální funkce.

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