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APPROXIMATIONS OF PARABOLIC VARIATIONAL INEQUALITIES

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Dedicated to Professor Miloš Zlámal on the occasion of his sixtieth birthday

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1. FORMULATION OF PROBLEMS

We shall approximate in various ways the following problem: Find a function $u: I \to V$ with the following properties: 1) $I = [0, T]$, $0 < T < \infty$ and V is a Hilbert space satisfying $H_0^1(\Omega) \subset V \subset H^1(\Omega)$; 2) $u(t) \in K$ for almost all $t \in I$, K being a closed convex subset of V; 3) $u \in AC(I; L_2(\Omega)) \cap L_\infty(I; V)$, $\dot{u} \in L_2(I; L_2(\Omega))$ and relations (1), (2) are satisfied:

$$(1)\qquad (\dot{u}(t), v - u(t)) + a(u(t), v - u(t)) \geqq (f(t), v - u(t)) \quad \forall v \in K$$

$$\forall t \in I \setminus E_v\,,$$

$$(2)\qquad u(0) = u_0 \in K\,,$$

where $\dot{u} = \partial u/\partial t$, E_v is a set with the property mes $E_v = 0$, the symbol $(\cdot, \cdot)$ denotes the scalar product in $L_2(\Omega)$ and Ω is a bounded domain in the x_1, x_2-plane with a sufficiently smooth boundary Γ. The norm in V is induced by the norm $\|\cdot\|_1$ in $H^1(\Omega)$; $\|v\|_1^2 = \|v\|_0^2 + |v|_1^2 = (v, v) + (\text{grad } v, \text{ grad } v)$. (The norm in the space $H^k(\Omega)$ will be denoted by $\|\cdot\|_k$, the seminorm by $|\cdot|_k$.) Further, we assume that

$$(3)\qquad f \in C^{0,1}(I; L_2(\Omega))\,.$$

All notation concerning the function spaces is the same as in [10].

The space V will have the form

$$(4)\qquad V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}\,,$$

where Γ_1 is part of the boundary Γ such that mes $\Gamma_1 > 0$. Finally, we assume that the form $a(v, w)$: $H^1(\Omega) \times H^1(\Omega) \to R$ has a potential $J(v)$ (this means that there exists a functional $J(v)$: $H^1(\Omega) \to R$ which is G-differentiable at arbitrary $v \in H^1(\Omega)$

and satisfies

(5) $$a(v, w) = J'(v, w) \quad \forall v, w \in H^1(\Omega)$$

(for more detail see [2, Chapter 2])) and that $J(v)$ is twice G-differentiable at arbitrary $v \in H^1(\Omega)$ and has the following properties:

(6) $$J(0) = 0\,, \quad J'(0, w) = 0 \quad \forall w \in H^1(\Omega)\,,$$

(7) $$|J''(v, w, z)| \leqq \beta_2 |w|_1 \, |z|_1 \quad \forall v, w, z \in H^1(\Omega)\,,$$

(8) $$|J''(v, w, w)| \geqq \beta_1 |w|_1^2 \quad \forall v, w \in H^1(\Omega)\,,$$

where β_1, β_2 are positive constants not depending on v, w, z.

The assumptions (5)–(8) have the following consequences: 1) the form $a(v, w)$ is linear in w; 2) the form $a(v, w)$ is hemicontinuous, i.e. $\lambda \to a(v + \lambda z, w)$ is a continuous function on R for all $v, w, z \in H^1(\Omega)$; 3) if we use Taylor's theorem in the form

$$J'(\omega + \psi, \varphi) = J'(\omega, \varphi) + J''(\omega + \vartheta\psi, \varphi, \psi)$$

where $0 < \vartheta < 1$ and ω, φ, ψ are arbitrary functions from $H^1(\Omega)$ (see [2]), we find that the form $a(v, w)$ is bounded, coercive, strongly monotone and lipschitz with respect to v:

(9) $$|a(v, w)| \leqq \beta_2 |v|_1 \, |w|_1 \quad \forall v, w \in H^1(\Omega)\,,$$

(10) $$a(v, v) \geqq \beta_1 |v|_1^2 \quad \forall v \in H^1(\Omega)\,,$$

(11) $$a(v, v - w) - a(w, v - w) \geqq \beta_1 |v - w|_1^2 \quad \forall v, w \in H^1(\Omega)\,,$$

(12) $$|a(v, w) - a(z, w)| \leqq \beta_2 |v - z|_1 \, |w|_1 \quad \forall v, w, z \in H^1(\Omega)\,;$$

4) if we use Taylor's theorem in the forms (see [2])

$$J(v) = J(0) + J'(0, v) + \tfrac{1}{2} J''(\vartheta v, v, v)\,,$$

$$J(w) = J(v) + J'(v, w - v) + \tfrac{1}{2} J''(v + \vartheta(w - v), w - v, w - v)\,,$$

where $0 < \vartheta < 1$, we find

(13) $$\tfrac{1}{2}\beta_1 |v|_1^2 \leqq J(v) \leqq \tfrac{1}{2}\beta_2 |v|_1^2 \quad \forall v \in H^1(\Omega)\,,$$

(14) $$a(v, w - v) + J(v) - J(w) \geqq -\tfrac{1}{2}\beta_2 |v - w|_1^2 \quad \forall v, w \in H^1(\Omega)\,.$$

Example 1. We give an example of $a(v, w)$ and $J(v)$ satisfying (5)–(8). Let $m(s) \in C^1([0, \infty))$ be a function with the property

(15) $$\beta_1 \leqq \frac{\mathrm{d}}{\mathrm{d}s} [s\, m(s)] \leqq \beta_2 \quad \forall s \in [0, \infty)\,,$$

where β_1, β_2 are positive constants. Let us define a function

$$F(y) = \int_0^y s\, m(s)\, \mathrm{d}s\,.$$

Using the function $F(y)$ let us define a functional

$$J(v) = \int_\Omega F(|\text{grad}\, v|)\, dx\,,$$

where

$$|\text{grad}\, v| = \sqrt{[(\partial v/\partial x_1)^2 + (\partial v/\partial x_2)^2]}\,.$$

According to [2, Chapter 2] we have

$$J'(v, w) = \left[\frac{d}{d\vartheta}\int_\Omega F(|\text{grad}\,(v + \vartheta w)|)\, dx\right]_{\vartheta=0}.$$

After a simple calculation we find that $J(v)$ is the potential corresponding to the form

$$a(v, w) = \int_\Omega m(|\text{grad}\, v|)\, \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\, dx\,. \tag{16}$$

In (16) and in what follows the summation convention over a repeated subscript is adopted. Further,

$$J''(v, w, z) = \left[\frac{d}{d\vartheta} J'(v + \vartheta z, w)\right]_{\vartheta=0} = \tag{17}$$

$$= \int_\Omega \left\{m'(\eta)\, \eta^{-1} \frac{\partial v}{\partial x_i}\frac{\partial z}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_j} + m(\eta)\frac{\partial w}{\partial x_i}\frac{\partial z}{\partial x_i}\right\} dx\,,$$

where $m'(s) = dm(s)/ds$ and $\eta = |\text{grad}\, v|$. In [16], estimates (7), (8) are derived for $J''(v, w, z)$ defined by (17).

Example 2. We give an example of a problem the variational formulation of which is of the form (1), (2). Let us consider the following initial-boundary value problem:

$$\dot{u} - \frac{\partial}{\partial x_i}\left(m(\xi)\frac{\partial u}{\partial x_i}\right) = f \quad \text{in} \quad \Omega \times (0, T]\,, \tag{18}$$

$$u = 0 \quad \text{on} \quad \Gamma_1\,, \tag{19}$$

$$u \geqq 0\,, \quad m(\xi)\frac{\partial u}{\partial \nu} \geqq 0\,, \quad m(\xi)\, u\, \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_2\,, \tag{20}$$

$$u(x, 0) = u_0(x)\,, \quad x \in \Omega\,, \tag{21}$$

where $\xi = |\text{grad}\, u|$, f satisfies (3), Γ_1 is the same as in (4), $\Gamma_2 = \Gamma - \Gamma_1$, $m(s)$ is the function from Example 1, $u_0 \in H^1(\Omega)$ and $\partial/\partial\nu$ is the normal derivative. Let us define

$$K = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1, v \geqq 0 \text{ on } \Gamma_2\}\,. \tag{22}$$

Then K is a closed convex subset of V. Let us multiply (18) by an arbitrary function $v \in K$ and integrate over the domain Ω. Using integration by parts and Green's theorem we obtain

$$(23)\qquad (\dot{u}, v) + a(u, v) - \int_{\Omega} m(\xi) \frac{\partial u}{\partial \nu} v \, \mathrm{d}s = (f, v) \quad \forall v \in K\,,$$

where $a(u, v)$ is given by (16). According to (19) and (20_1), we have $u \in K$. Thus relations (20_3) and (23) imply

$$(24)\qquad (\dot{u}, u) + a(u, u) = (f, u)\,.$$

Relations (20_2) and (22) give

$$(25)\qquad \int_{\Gamma_2} m(\xi) \frac{\partial u}{\partial \nu} v \, \mathrm{d}s \geqq 0 \quad \forall v \in K\,.$$

According to (23) and (25) we have

$$(26)\qquad (\dot{u}, v) + a(u, v) \geqq (f, v) \quad \forall v \in K\,.$$

Substracting (24) from (26) we obtain (1). Thus problem (1), (2), where $a(u, v)$ is given by (16) and K by (22), is a variational formulation of problem (18)–(21) (under the condition that the right-hand side of (21) belongs to K).

We give four approximate formulations of problem (1), (2). Let us start with the discretization in time. We introduce $\Delta t = T/r$, r being a natural number, and consider the partition of $I = [0, T]$ with the nodes

$$t_i = i\Delta t \quad (i = 0, \ldots, r)\,.$$

We set

$$f^i = f(t_i)$$

and define Problem 1: Find $U^i \in K$ $(i = 1, \ldots, r)$ such that

$$(27)\qquad \frac{1}{\Delta t}(U^i - U^{i-1}, v - U^i) + a(U^i, v - U^i) \geqq (f^i, v - U^i) \quad \forall v \in K\,,$$

$$(28)\qquad U^0 = u_0 \in K\,.$$

We have obtained (27) from (1) by means of the implicit Euler's method.

To solve Problem 1 means to solve an elliptic variational inequality with a nonlinear elliptic form $a(u, v)$. Thus we shall define another discrete problem where this nonlinearity is removed. To this end let us write inequality (1) in the form

$$(29)\qquad (\dot{u}(t), v - u(t)) + \Theta(u(t), v - u(t))_1 \geqq$$
$$\geqq \omega(u(t), v - u(t)) + (f(t), v - u(t)) \quad \forall v \in K \quad \forall t \in I \setminus E_v$$

where Θ is a constant satisfying

$$(30)\qquad \Theta > \beta_2/2$$

and the forms $(\cdot, \cdot)_1$, $\omega(\cdot, \cdot)$ are defined by

$$(v, w)_1 = (\text{grad } v, \text{grad } w)\,, \tag{31}$$

$$\omega(v, w) = \Theta(v, w)_1 - a(v, w)\,. \tag{32}$$

Discretizing the left-hand side of (29) by the implicit Euler's method and the right-hand side by the explicit Euler's method we come to Problem 2: Let U^0 be given by (28). Find $U^i \in K$ $(i = 1, \ldots, r)$ such that

$$\frac{1}{\Delta t}(U^i - U^{i-1}, v - U^i) + \Theta(U^i, v - U^i)_1 \geqq \tag{33}$$

$$\geqq \omega(U^{i-1}, v - U^i) + (f^{i-1}, v - U^i) \quad \forall v \in K\,.$$

Let us point out that the test functions $v - U^i$ are the same on both sides of (33).

The remaining approximate problems will be obtained by discretizing Problems 1 and 2 in space. We shall use the finite element method: Let us triangulate the domain Ω, i.e., let us divide it into a finite number of triangles in such a way that two arbitrary triangles are either disjoint, or have a common vertex, or a common side. (If a part of the boundary Γ is curved then each corresponding boundary triangle has one curved side which is part of the boundary Γ. Thus we only consider ideal curved triangles which were defined in [14].) Every interior triangle, i.e. a triangle having at most one point common with the boundary Γ, has straight sides only.

With every triangulation $\mathscr{T}$ we associate two parameters h and ϑ defined by

$$h = \max_{T \in \mathscr{T}} h_T\,, \quad \vartheta = \min_{T \in \mathscr{T}} \vartheta_T$$

where h_T and ϑ_T are the length of the greatest side and the magnitude of the smallest angle, respectively, of the triangle with straight sides which has the same vertices as the triangle T. We restrict ourselves to triangulations satiysfying

$$\vartheta \geqq \vartheta_0\,, \quad \vartheta_0 = \text{const} > 0\,.$$

A triangulation with this property will be denoted by $\mathscr{T}_h$.

On every triangulation $\mathscr{T}_h$ we define a finite dimensional space Z_h with the following properties:

a) $Z_h \subset C(\bar{\Omega})$;

b) every function $v \in Z_h$ is uniquely determined by its values prescribed at the vertices P_j of the triangles of $\mathscr{T}_h$ (the vertices P_j will be called the nodal points of $\mathscr{T}_h$);

c) the restriction of $v \in Z_h$ to an arbitrary interior triangle is a linear function.

In the case of a polygonal boundary Γ the space Z_h is formed by piecewise linear functions. The definition of the restriction of $v \in Z_h$ to an ideal curved triangle can be found in [14].

If a convex set $K \subset V$ is defined by a condition (C prescribed on a subset Λ of $\bar{\Omega}$) then we define the finite element approximation K_h of K in the following way: K_h consists of those functions of Z_h which satisfy condition (C) at the nodal points $P_j \in \Lambda$.

In this paper we restrict our considerations to convex sets K which satisfy

$$K_h \subset K \quad \forall h \,, \tag{34}$$

$$\overline{C^\infty(\bar{\Omega}) \cap K} = K \,, \tag{35}$$

where the closure is taken in the norm $\|\cdot\|_1$.

The convex set K defined by (22) has both the properties (34) and (35). Property (34) is evident because in the case of (22) we have

$$K_h = \{v \in Z_h \colon v = 0 \text{ on } \Gamma_1, v \geqq 0 \text{ on } \Gamma_2\}$$

and $Z_h \subset H^1(\Omega)$. The proof of property (35) is a special case of the proof of [5, Lemma A1].

Using (35) and the interpolation properties of functions from Z_h we can prove in the same way as in [13] (see also [3, pp. 134–135] or [6, p. 31]) that

$$\lim_{h \to 0} \inf_{v_h \in K_h} \|w - v_h\|_1 = 0 \quad \forall w \in K \,. \tag{36}$$

It follows from (2) and (36) that there exists $u_{0h} \in K_h$ such that

$$\lim_{h \to 0} \|u_0 - u_{0h}\|_1 = 0 \,. \tag{37}$$

Discretizing Problem 1 in space we obtain Problem 3: Let K_h be given. Let

$$U^0 = u_{0h} \in K_h \,. \tag{38}$$

Find $U^i \in K_h$ $(i = 1, \ldots, r)$ such that

$$\frac{1}{\Delta t}(U^i - U^{i-1}, v - U^i) + a(U^i, v - U^i) \geqq (f^i, v - U^i) \quad \forall v \in K_h \,. \tag{39}$$

Finally, discretizing Problem 2 in space we come to Problem 4: Let K_h be given. Let U^0 be defined by (38). Find $U^i \in K_h$ $(i = 1, \ldots, r)$ such that

$$\frac{1}{\Delta t}(U^i - U^{i-1}, v - U^i) + \Theta(U^i, v - U^i)_1 \geqq \tag{40}$$

$$\geqq \omega(U^{i-1}, v - U^i) + (f^{i-1}, v - U^i) \quad \forall v \in K_h \,.$$

It can be proved in the same way as in [1] that each of Problems 1 and 3 has only one solution. Now we prove existence and uniqueness of the solutions of Problems 2 and 4. Let us write inequality (40) in the form

$$b(U^i, v - U^i) \geqq l(v - U^i) \quad \forall v \in K_h \,, \tag{41}$$

where

$$b(v, w) = \frac{1}{\Delta t}(v, w) + \Theta(v, w)_1,$$

$$l(v) = \frac{1}{\Delta t}(U^{i-1}, v) + \omega(U^{i-1}, v) + (f^{i-1}, v).$$

The bilinear form $b(v, w)$ is symmetric, bounded on $V \times V$ and V-eliptic,

$$\Theta C_0 \|v\|_1^2 \leqq b(v, v) \quad \forall v \in V,$$

where $C_0 > 0$ is the constant from Friedrichs' inequality; the linear form $l(v)$ is bounded in the space V and K_h is a closed convex subset of V. Thus, according to [3, Theorems 1.1.1 and 1.1.2], inequality (41) has a unique solution $U^i \in K_h$. The proof in the case of Problem 2 is the same (we only replace K_h by K).

In the case of Problems 1 and 2 let us define an abstract function $U_r(t)$ by the relation

$$(42) \quad U_r(t) = U^{i-1} + \frac{t - t_{i-1}}{\Delta t}(U^i - U^{i-1}), \quad t \in [t_{i-1}, t_i], \quad i = 1, \dots, r.$$

The function $U_r(t)$ is a continuous extension of the solution $U^0, U^1, \dots, U^r$ of Problem 1 (or Problem 2) to the whole interval I.

Bock and Kačur [1] proved that problem (1), (2) has a unique solution and that the sequence $\{U_r(t)\}$ generated by the solutions of Problems 1 converges for $r \to \infty$ to the exact solution of (1), (2) in the space $C(I; L_2(\Omega))$. They did not use assumptions (5)–(8) concerning the form $a(v, w)$; they only assumed (9), (10) and the monotonicity of $a(v, w)$ on K:

$$a(v, v - w) - a(w, v - w) \geqq 0 \quad \forall v, w \in K.$$

However, their assumptions concerning the initial value u_0 are stronger.

The first aim of this paper is to prove, under the assumptions (5)–(8) concerning the form $a(v, w)$, that the sequence $\{U_r(t)\}$ generated by the solutions of Problems 2 converges for $r \to \infty$ to the exact solution $u(t)$ of problem (1), (2) in the space $C(I; L_2(\Omega))$. This will be done in Theorem 1.

In the case of Problems 3 and 4 let us define an abstract function $U^\delta(t)$ by the relation

$$(43) \quad U^\delta(t) = U^{i-1} + \frac{t - t_{i-1}}{\Delta t}(U^i - U^{i-1}), \quad t \in [t_{i-1}, t_i], \quad i = 1, \dots, r,$$

where $\delta = (h, \Delta t)$. In what follows we shall consider a sequence $\{U^{\delta_n}\}$ of functions belonging to the set $\{U^\delta\}$ and its subsequences. The sequence $\{U^{\delta_n}\}$ will be chosen in such a way that

$$\delta_n = (h_n, (\Delta t)_n) \to 0 \quad \text{if} \quad n \to \infty.$$

It is essential that h_n and $(\Delta t)_n$ are mutually independent. For greater simplicity we shall use the symbol $U_n(t)$ instead of the symbol $U^{\delta_n}(t)$. The symbols $\{U_k\}$, $\{U_j\}$ will denote subsequences of $\{U_n\}$. Instead of the symbols of the type $\lim_{\delta_n \to 0} \|U^{\delta_n}\|$ the symbol $\lim_{n\to\infty} \|U_n\|$ will be used.

Instead of the symbol K_{h_n}, which denotes the approximation of K corresponding to the triangulation $\mathscr{T}_{h_n}$, we shall use the symbol K_n.

In accordance with [1], the functions $U_r(t)$ defined by (42) will be called Rothe's functions. In order to stress the discretization in space, the functions $U_n(t)$ defined by (43) will be called the finite element Rothe's functions.

In Section 2 the convergence of the sequence of finite element Rothe's functions, which are generated by the solutions of Problems 4, in the space $C(I; L_2(\Omega))$ is proved. In Section 3, the convergence in the space $L_2(I; V)$ is studied. In Section 4, under additional regularity assumptions, some simple error estimates for the solution of Problem 4 are presented.

2. CONVERGENCE IN THE SPACE $C(I; L_2(\Omega))$

In this section we study in detail the convergence of the finite element Rothe's functions (43) corresponding to the solutions of Problems 4. The method of the proof is based on the compactness and monotonicity method. In Lemmas 1–3 the existence of a limit function $u(t)$ is established and in the course of proof of Theorem 2 it is shown that $u(t)$ is the only solution of problem (1), (2).

Similar results can be obtained for Rothe's functions (42) corresponding to the solutions of Problems 2 (see Theorem 1).

Lemma 1. *The solution U^i $(i = 1, \ldots, r)$ of the implicit-explicit scheme* (38), (40) *satisfies the following relations*:

$$\|U^m\|_1 \leqq C \qquad (1 \leqq m \leqq r) \quad \forall r\,,$$

$$\sum_{i=1}^{m} \|\Delta U^i\|_0^2 \leqq C\Delta t \quad (1 \leqq m \leqq r) \quad \forall r\,,$$

$$\sum_{i=1}^{m} \|\Delta U^i\|_1^2 \leqq C \qquad (1 \leqq m \leqq r) \quad \forall r$$

where

$$\Delta U^i = U^i - U^{i-1}\,.$$

In Lemma 1 and in what follows the symbol C denotes a positive constant independent of h, Δt and n and not necessarily the same at any two places.

Proof. We follow the ideas used in the proof of relation [16, (3.16)]. To the both sides of (14) let us add the expression $\Theta|w - v|_1^2$. Denoting $\varkappa = \Theta - \beta_2/2 > 0$ (see (30)) we obtain

(44) $$\Theta|w - v|_1^2 + a(v, w - v) \geqq \varkappa|w - v|_1^2 + J(w) - J(v) \quad \forall v, w \in H^1(\Omega).$$

Let us set $v = U^{i-1} \in K_n$ in (40) and multiply the obtained relation by $-\Delta t$. We get, according to (32),

(45) $$\|\Delta U^i\|_0^2 + \Delta t\, \Theta|\Delta U^i|_1^2 + \Delta t\, a(U^{i-1}, \Delta U^i) \leqq \Delta t(f^{i-1}, \Delta U^i).$$

Relation (44) with $w = U^i$, $v = U^{i-1}$ and relation (45) imply

$$\|\Delta U^i\|_0^2 + \varkappa\, \Delta t|\Delta U^i|_1^2 + \Delta t[J(U^i) - J(U^{i-1})] \leqq \Delta t(f^{i-1}, \Delta U^i).$$

Let us sum up this relation from $i = 1$ to $i = m$ and use (13) and Friedrichs' inequality $|\Delta U^i|_1^2 \geqq C_0\| \Delta U^i\|_1^2$. After summing by parts on the right-hand side we obtain

$$\sum_{i=1}^{m} \|\Delta U^i\|_0^2 + \varkappa C_0\, \Delta t \sum_{i=1}^{m} \|\Delta U^i\|_1^2 + \tfrac{1}{2}C_0\beta_1\, \Delta t\|U^m\|_1^2 \leqq$$
$$\leqq \tfrac{1}{2}\beta_2\, \Delta t\|U^0\|_1^2 + \Delta t\|f^0\|_0\, \|U^0\|_0 +$$
$$+ \Delta t\|f^{m-1}\|_0\, \|U^m\|_1 + \Delta t \sum_{i=1}^{m} \|\Delta f^i\|_0\, \|U^i\|_1.$$

Using assumption (3) and the inequality

(46) $$|ab| \leqq a^2/(2\gamma) + \gamma b^2/2$$

with various values of γ we obtain from the above relation

(47) $$\sum_{i=1}^{m} \|\Delta U^i\|_0^2 + c_1\, \Delta t \sum_{i=1}^{m} \|\Delta U^i\|_1^2 + c_1\, \Delta t\|U^m\|_1^2 \leqq$$
$$\leqq c_2\, \Delta t + c_2(\Delta t)^2 \sum_{i=1}^{m-1} \|U^i\|_1^2 \quad (1 \leqq m \leqq r)$$

where c_1, c_2 are positive constants not depending on m, r, n and h. Relation (47) implies

$$\|U^m\|_1^2 \leqq C + C\, \Delta t \sum_{i=1}^{m-1} \|U^i\|_1^2.$$

Using the discrete Gronwall's inequality we obtain the first relation of Lemma 1. As $\Delta t = T/r$ this relation implies

(48) $$\sum_{i=1}^{m} \|U^m\|_1^2 \leqq C/\Delta t.$$

Inserting (48) into the right-hand side of (47) we obtain the last two relations of Lemma 1.

Corollary 1. *The finite element Rothe's functions* (43) *satisfy*

(49) $$\|U_n(t)\|_1 \leqq C \quad \forall t \in I \quad \forall n,$$

$$\int_0^T \|\dot{U}_n(t)\|_0^2 \, dt \leqq C \quad \forall n . \tag{50}$$

Proof. Relation (50) is the second relation of Lemma 1 written in another form. Relation (49) follows immediately from the first relation of Lemma 1, using the fact that $0 \leqq t - t_{i-1} < \Delta t$ for $t \in [t_{i-1}, t_i)$.

In our considerations we shall need the following step-functions:

$$\bar{u}_n(t) = U^{i-1}, \quad t \in [t_{i-1}, t_i) \quad (i = 1, \ldots, r), \quad \bar{u}_n(T) = U^{r-1} \tag{51}$$

$$\bar{U}_n(t) = U^i, \quad t \in (t_{i-1}, t_i] \quad (i = 1, \ldots, r), \quad \bar{U}_n(0) = U^1 \tag{52}$$

$$\bar{f}_n(t) = f^{i-1}, \quad t \in [t_{i-1}, t_i) \quad (i = 1, \ldots, r), \quad \bar{f}_n(T) = f^{r-1} . \tag{53}$$

Corollary 2. *The step-functions $\bar{u}_n(t)$, $\bar{U}_n(t)$ and the finite element Rothe's functions $U_n(t)$ satisfy the relations:*

$$\|\bar{u}_n(t)\|_1 \leqq C, \quad \|\bar{U}_n(t)\|_1 \leqq C \quad \forall n \quad \forall t \in I, \tag{54}$$

$$\int_0^T \|\bar{U}_n(t) - U_n(t)\|_1^2 \, dt \leqq C(\Delta t)_n, \tag{55}$$

$$\int_0^T \|\bar{u}_n(t) - U_n(t)\|_1^2 \, dt \leqq C(\Delta t)_n . \tag{56}$$

Relations (54)–(56) are immediate consequences of the first and third relations of Lemma 1 and definitions of functions $\bar{u}_n(t)$, $\bar{U}_n(t)$, $U_n(t)$.

Lemma 2. *There exist a function $u \in C(I; L_2(\Omega)) \cap L_\infty(I; V)$ and a subsequence $\{U_k\}$ of $\{U_n\}$ such that*

$$U_k \to u \quad in \quad C(I; L_2(\Omega)), \tag{57}$$

$$U_k(t) \rightharpoonup u(t) \quad weakly\ in\ V \quad \forall t \in I, \tag{58}$$

$$U_k \rightharpoonup u \quad weakly\ in\ L_2(I; V), \tag{59}$$

$$\bar{u}_k \rightharpoonup u \quad weakly\ in\ L_2(I; V). \tag{60}$$

Proof. According to (50), we have

$$\|U_n(t'') - U_n(t')\|_0 = \left\| \int_{t'}^{t''} \dot{U}_n(t) \, dt \right\|_0 \leqq C|t'' - t'|^{1/2} \quad \forall t', t'' \in I = [0, T] .$$

Thus the functions $U_n(t)$ $(n = 1, 2, \ldots)$ are equicontinuous on I in the norm $\|\cdot\|_0$. Relations (49) and Rellich's theorem (see, e.g., [12, p. 17]) imply that the sequence $\{U_n(t)\}$ is relatively compact in $L_2(\Omega)$ for every $t \in I$. According to the generalization of the Arzela-Ascoli theorem (see, e. g., [10, p. 42]), there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that relation (57) holds, where $u \in C(I; L_2(\Omega))$.

Let $t \in I$ be arbitrary but fixed. Relation (49) and the compactness theorem (see, e.g., [2 ,Chapter 1, Theorem 4.2]) imply that we can extract a subsequence $\{U_j(t)\}$ of $\{U_k(t)\}$ converging weakly in V to an element $w \in V$. As (z, v) ($z \in L_2(\Omega)$ fixed, $v \in V$) is a linear bounded functional on V we see that also $U_j(t) \rightharpoonup w$ in $L_2(\Omega)$. Relation (57) implies that $w = u(t)$.

If $\{U_j(t)\} \neq \{U_k(t)\}$ then there exists an infinite subsequence $\{U_m(t)\}$ of $\{U_k(t)\}$ with the property that no subsequence of $\{U_m(t)\}$ converges to $u(t)$ weakly in V. Using (49) and repeating the preceding considerations we find that a subsequence $\{U_i(t)\}$ of $\{U_m(t)\}$ converges to $u(t)$ weakly in V. This is a contradiction. Thus $\{U_j(t)\} = \{U_k(t)\}$ and (58) is proved.

As the norm $\|\cdot\|_1$ is weakly lower semicontinuous on V relations (49) and (58) imply

$$\|u(t)\|_1 \leqq \liminf_{k\to\infty} \|U_k(t)\|_1 \leqq C \quad \forall t \in I \,.$$

Thus $u \in L_\infty(I; V)$.

Relation (49) and the compactness theorem imply that a subsequence $\{U_j\}$ of $\{U_k\}$ converges weakly in $L_2(I; V)$ to an element $w \in L_2(I; V)$. As the form

$$\int_0^T (z(t), v(t))\, \mathrm{d}t \; (z \in L_2(I; L_2(\Omega)) \text{ fixed}, v \in L_2(I; V))$$

is a linear bounded functional on $L_2(I; V)$ we see, according to [10, p. 125], that also $U_j \rightharpoonup w$ in $L_2(I; L_2(\Omega))$. Relation (57) implies that $w = u$. The rest of the proof of (59) is the same as in the preceding case.

Relation (60) is a consequence of (56) and (59). Lemma 2 is proved.

Lemma 3. *The limit function $u(t)$ satisfies (2), $u \in AC(I; L_2(\Omega))$, the strong derivative $\dot{u}(t)$ exists almost everywhere on I, $\dot{u} \in L_2(I; L_2(\Omega))$ and*

$$\dot{U}_k \rightharpoonup \dot{u} \quad \text{weakly in } L_2(I; L_2(\Omega))\,, \tag{61}$$

where $\{U_k\}$ is the same subsequence of $\{U_n\}$ as in Lemma 2.

Proof. For every $t \in I$ and for every k we have

$$(U_k(t), v) - (u_{0k}, v) = \int_0^t (\dot{U}_k(\tau), v)\, \mathrm{d}\tau \quad \forall v \in V\,, \tag{62}$$

using the notation u_{0k} for u_{0h_k}. According to (50) and the compactness theorem, we can extract a subsequence $\{U_m\}$ of $\{U_k\}$ such that

$$\dot{U}_m \rightharpoonup g \quad \text{weakly in } L_2(I, L_2(\Omega))\,. \tag{63}$$

Using (37), (57) and (63) we obtain from (62)

$$(u(t), v) - (u_0, v) = \int_0^t (g(\tau), v)\, \mathrm{d}\tau \quad \forall v \in V\,.$$

As V is dense in $L_2(\Omega)$ and as (see [4, p. 126])

$$\int_0^t (g(\tau), v)\, d\tau = \left(\int_0^t g(\tau)\, d\tau, v\right)$$

we get from the last relation

$$u(t) = u_0 + \int_0^t g(\tau)\, d\tau .$$

Thus $u \in AC(I, L_2(\Omega))$, $u(t)$ satisfies the initial condition (2) and we have

$$\dot{u}(t) = g(t) \quad \text{a.e. in } I .$$

The proof of the relation $\{U_m\} = \{U_k\}$ is similar to the corresponding proof in Lemma 2. Lemma 3 is proved.

Theorem 1. *Let K be a closed convex subset of the space V, which is given by* (4), *let f be a function satisfying* (3) *and let the form $a(v, w)$: $H^1(\Omega) \times H^1(\Omega) \to R$ have a potential $J(v)$: $H^1(\Omega) \to R$, which is twice G-differentiable at arbitrary $v \in H^1(\Omega)$ and satisfies conditions* (6)–(8). *Then there exists a unique function $u(t)$ with the properties*

$$u(t) \in K \quad \forall t \in I , \quad u \in AC(I; L_2(\Omega)) \cap L_\infty(I; V), \quad \dot{u} \in L_2(I; L_2(\Omega))$$

and satisfying relations (1), (2). *Further, every infinite sequence $\{U_r\}$ of Rothe's functions* (42), *where $U^1, \dots, U^r$ is the unique solution of the implicit-explicit scheme* (28), (33), *converges to the solution $u(t)$ of problem* (1), (2) *in the space $C(I, L_2(\Omega))$:*

$$\lim_{r \to \infty} \|u - U_r\|_{C(I; L_2(\Omega))} = 0 .$$

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied and let the convex set K satisfy conditions* (34), (35). *Then every infinite sequence $\{U_n\}$ of finite element Rothe's functions* (43), *where $U^1, \dots, U^r$ is the unique solution of the implicit-explicit scheme* (38), (40) *and where $\delta_n \to 0$, converges to the solution $u(t)$ of problem* (1), (2) *in the space $C(I; L_2(\Omega))$:*

$$\lim_{n \to \infty} \|u - U_n\|_{C(I; L_2(\Omega))} = 0 .$$

Proof. First we shall prove Theorem 2. Then we shall mention the differences in the proof of Theorem 1.

We have to prove that the limit function $u(t)$ appearing in Lemmas 2 and 3 has the following properties: 1) $u(t) \in K$ $\forall t \in I$; 2) $u(t)$ is the only solution of problem (1), (2); 3) the whole sequence $\{U_n\}$ converges to u in $C(I; L_2(\Omega))$.

It follows from the assumptions of Theorem 2 that every function $U_n(t)$ is defined on a closed convex set K_n and the sequence $\{K_n\}$ has the following property: To every $w \in K$ we can find a sequence $\{w_n\}$, where $w_n \in K_n$, such that

(64) $$\lim_{n \to \infty} \|w_n - w\|_1 = 0 .$$

Moreover, according to interpolation properties of Z_n, for every $w \in H^2(\Omega) \cap K$ we have

(65) $$\|w - \Pi_n w\|_1 \leqq Ch_n \|w\|_2 ,$$

where $\Pi_n w \in K_n$ is the interpolate of w in Z_n.

Lemma 4. *Let $w^* \in C^{0,1}(I; H^2(\Omega))$. Let $t_0 = 0 < t_1 < t_2 < \ldots < t_{r_n - 1} < t_{r_n} = T$ be the nodal points of the n-th partition of I into r_n subintervals of the length $(\Delta t)_n = T/r_n$. Let*

$$\bar{w}_n^*(t) = \Pi_n(w^*(t_i)) , \quad t_{i-1} < t \leqq t_i \quad (i = 1, \ldots, r_n) .$$

Then

$$\lim_{n \to \infty} \|\bar{w}_n^* - w^*\|_{L_2(I;V)} = 0 .$$

Proof. We have, according to (65) and Lipschitz continuity,

$$\int_0^T \|\bar{w}_n^*(t) - w^*(t)\|_1^2 \, dt \leqq 2 \sum_{i=1}^{r_n} \int_{t_{i-1}}^{t_i} \{\|\bar{w}_n^*(t_i) - w^*(t_i)\|_1^2 +$$
$$+ \|w^*(t_i) - w^*(t)\|_1^2\} \, dt \leqq 2T\{C \max_{t \in I} \|w^*(t)\|_2^2 \, h_n^2 + M(w^*) \, (\Delta t)_n^2\}$$

where $M(w^*)$ is a positive constant depending on w^* only. Lemma 4 is proved.

Lemma 5. *The set $\mathscr{S}_1 = \{w \in C^{0,1}(I; H^2(\Omega)) \colon w(t) \in K \; \forall t \in I\}$ is dense in the set $\mathscr{S}_2 = \{v \in L_2(I; V) \colon v(t) \in K \; \forall t \in I\}$.*

Proof. Let us choose $v \in \mathscr{S}_2$ and $\varepsilon > 0$ arbitrarily. Let $\bar{v} \in \mathscr{S}_2$ be a step function

$$\bar{v}(t) = z^1 \quad \text{on} \quad [0, t_1] , \quad \bar{v}(t) = z^i , \quad t \in (t_{i-1}, t_i] \quad (i = 2, \ldots, n)$$

such that $\|\bar{v} - v\|_{L_2(I;V)} < \varepsilon/2$ (we can achieve it using (87) with $\Delta t = T/n$ sufficiently small). Let $M = \max \|z^i - z^j\|_1$ $(i, j = 1, \ldots, n)$. Let us choose δ satisfying $0 < \delta < \varepsilon^2/(4M^2 n)$ and let us define

$$\tau_i^1 = t_i - \delta/2 , \quad \tau_i^2 = t_i + \delta/2 \quad (i = 1, \ldots, n - 1) ,$$

$$\tilde{w}(t) = z^{i-1} + \frac{z^i - z^{i-1}}{\delta} (t - \tau_i^1) \quad \text{on} \quad [\tau_i^1, \tau_i^2] \quad (i = 1, \ldots, n - 1) ,$$

$$\tilde{w}(t) = \bar{v}(t) \quad \text{on} \quad I \setminus \bigcup_{i=1}^{n-1} (\tau_i^1, \tau_i^2) .$$

We have $\|\tilde{w} - \bar{v}\|_{L_2(I;V)} < \varepsilon/2$, $\|\tilde{w} - v\|_{L_2(I;V)} < \varepsilon$. Thus the set $\mathscr{S}_3 = \{w \in C(I; V) \colon w(t) \in K \; \forall t \in I\}$ is dense in the set $\mathscr{S}_2$.

Let us choose $v \in \mathscr{S}_3$ and $\varepsilon > 0$ arbitrarily. Let $\delta > 0$ be such that $\|v(t') - v(t'')\|_1 < \varepsilon/5$ for all $t', t'' \in I$ satisfying the inequality $|t' - t''| < \delta$.

Let $t_0 = 0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = T$ be points for which $t_i - t_{i-1} < \delta$ $(i = 1, \ldots, n)$. Owing to (35) we can find $w_i \in H^2(\Omega) \cap K$ such that $\|w_i - v(t_i)\|_1 < \varepsilon/5$ $(i = 0, 1, \ldots, n)$. Let us define the function

$$w_\varepsilon(t) = w_{i-1} + \frac{w_i - w_{i-1}}{t_i - t_{i-1}}(t - t_{i-1}) \quad \text{on} \quad [t_{i-1}, t_i] \quad (i = 1, \ldots, n).$$

Then we have $\|w_\varepsilon(t) - v(t)\|_1 < \varepsilon \ \forall t \in I$. As $w_\varepsilon(t) \in \mathscr{S}_1$ we see that the set $\mathscr{S}_1$ is dense in the set $\mathscr{S}_3$.

Let us choose $v \in \mathscr{S}_2$ and $\varepsilon > 0$ arbitrarily. Using the preceding results we can easily find $w \in \mathscr{S}_1$ such $\|w - v\|_{L_2(I;V)} < \varepsilon$. Lemma 5 is proved.

The proof of Theorem 2 is divided into four parts $A)-D)$:

$A)$ The closed convex set K us weakly closed. Thus, according to relation (58), we have $u(t) \in K \ \forall t \in I$.

$B)$ We shall prove that the limit function $u(t)$ is a solution of problem (1), (2). Let us write relation (40) by means of functions $U_k(t)$, $\bar{u}_k(t)$, $\bar{U}_k(t)$ and $\bar{f}_k(t)$ in the form

$$(66) \qquad (\dot{U}_k(t), v - \bar{U}_k(t)) + \Theta(\bar{U}_k(t) - \bar{u}_k(t), v - \bar{U}_k(t))_1 - \\ - (\bar{f}_k(t), v - \bar{U}_k(t)) \geqq a(\bar{u}_k(t), \bar{U}_k(t) - v) \quad \forall v \in K_k \quad \forall t \in I \setminus E \quad \forall k,$$

where mes $E = 0$. Let us choose a function $w^* \in \mathscr{S}_1$, let us set $v = \bar{w}_k^*(t)$ for a given $t \in I \setminus E$, where $\bar{w}_k^*(t)$ is the step function from Lemma 4, let us add the term $a(\bar{u}_k(t), w^*(t) - u(t))$ to the both sides of (66) and let us integrate (66) in (t', t''), where $t' < t''$ are arbitrary in I. We obtain

$$\int_{t'}^{t''} \{(\dot{U}_k(t), \bar{w}_k^*(t) - \bar{U}_k(t)) + \Theta(\bar{U}_k(t) - \bar{u}_k(t), \bar{w}_k^*(t) - \bar{U}_k(t))_1 - \\ - (\bar{f}_k(t), \bar{w}_k^*(t) - \bar{U}_k(t)) + a(\bar{u}_k(t), w^*(t) - u(t))\} \, dt \geqq \\ \geqq \int_{t'}^{t''} \{a(\bar{u}_k(t), w^*(t) - \bar{w}_k^*(t)) + a(\bar{u}_k(t), \bar{U}_k(t) - u(t))\} \, dt .$$

As

$$a(\bar{u}_k(t), \bar{U}_k(t)) = a(\bar{u}_k(t), \bar{u}_k(t)) + a(\bar{u}_k(t), \bar{U}_k(t) - \bar{u}_k(t)),$$
$$(\dot{U}_k(t), \bar{w}_k^*(t) - \bar{U}_k(t)) = (\dot{U}_k(t), w^*(t) - u(t)) + \\ + (\dot{U}_k(t), \bar{w}_k^*(t) - w^*(t) + u(t) - U_k(t) + U_k(t) - \bar{U}_k(t)),$$
$$(\bar{f}_k(t), \bar{w}_k^*(t) - \bar{U}_k(t)) = (\bar{f}_k(t) - f(t), \bar{w}_k^*(t) - \bar{U}_k(t)) + \\ + (f(t), w^*(t) - u(t) + \bar{w}_k^*(t) - w^*(t) + u(t) - U_k(t) + U_k(t) - \bar{U}_k(t))$$

we obtain after passing to the limit for $k \to \infty$ by means of Lemmas 2–4, Corollary 2 and assumption (3):

$$\limsup_{k \to \infty} \int_{t'}^{t''} a(\bar{u}_k(t), \bar{u}_k(t) - u(t)) \, dt \leqq \int_{t'}^{t''} (\dot{u}(t) - f(t), w^*(t) - u(t)) \, dt +$$

$$+ \limsup_{k\to\infty} \int_{t'}^{t''} a(\bar{u}_k(t), w^*(t) - u(t))\, \mathrm{d}t .$$

The right-hand side is bounded by $C\|w^* - u\|_{L_2(I;V)}$. As w^* is an arbitrary function from $\mathscr{S}_1$ the expression $\|u - w^*\|_{L_2(I;V)}$ can be arbitrarily small, according to Lemma 5. Thus

$$\limsup_{k\to\infty} \int_{t'}^{t''} a(\bar{u}_k(t), \bar{u}_k(t) - u(t))\, \mathrm{d}t \leqq 0 . \tag{67}$$

The form $b(v, w) = \int_{t'}^{t''} a(v(t), w(t))\, \mathrm{d}t$ defined for all $v, w \in L_2(t', t''; V)$ is bounded, hemicontinuous and monotone, according to (5)–(8). Thus, according to [11, Chapter 2, Section 2.4], the form $b(v, w)$ is pseudomonotone and relations (60), (67) imply

$$\liminf_{k\to\infty} b(\bar{u}_k, \bar{u}_k - v) \geqq b(u, u - v) \quad \forall v \in L_2(t', t''; V) .$$

Then we also have

$$\liminf_{k\to\infty} \int_{t'}^{t''} a(\bar{u}_k(t), \bar{u}_k(t) - v)\, \mathrm{d}t \geqq \int_{t'}^{t''} a(u(t), u(t) - v)\, \mathrm{d}t \tag{68}$$

$$\forall v \in K , \quad \forall t', t'' \in I , \quad t' < t'' .$$

Let us choose $v \in K$ arbitrarily. According to (64), we can find a sequence $\{v_k\}$ such that $v_k \in K_k$ and $v_k \to v$ in $H^1(\Omega)$. As we have

$$a(\bar{u}_k(t), \bar{u}_k(t) - v_k) = a(\bar{u}_k(t), \bar{u}_k(t) - v) + a(\bar{u}_k(t), v - v_k)$$

we obtain from (68):

$$\liminf_{k\to\infty} \int_{t'}^{t''} a(\bar{u}_k(t), \bar{u}_k(t) - v_k)\, \mathrm{d}t \geqq \int_{t'}^{t''} a(u(t), u(t) - v)\, \mathrm{d}t . \tag{69}$$

Setting $v = v_k$ in (66), adding $a(\bar{u}_k(t), \bar{u}_k(t) - \bar{U}_k(t))$ to the both sides of (66), integrating the result in (t', t''), where $t' < t''$, and letting $k \to \infty$ we obtain, according to relation (69), Corollary 2 and Lemmas 2, 3:

$$\int_{t'}^{t''} \{(\dot{u}(t), v - u(t)) + a(u(t), v - v(t)) - (f(t), v - u(t))\}\, \mathrm{d}t \geqq 0 \tag{70}$$

$$\forall v \in K \quad \forall t', t'' \in I , \quad t' < t'' .$$

As t', t'' are arbitrary we see that relation (1) is satisfied by the limit function $u(t)$ and its derivative $\dot{u}(t)$.

C) Now we prove the uniqueness of the solution. Let us choose $w(t) \in \mathscr{S}_2$ (see Lemma 5). We can find a sequence $\{\bar{w}_n(t)\} \subset \mathscr{S}_2$ of step functions such that

$$\bar{w}_n \to w \quad \text{in} \quad L_2(I; V) .$$

Using relation (70) (which can be obtained by integrating (1)) we can write

$$\int_0^t \{(\dot{u}(\tau), \bar{w}_n(\tau) - u(\tau)) + a(u(\tau), \bar{w}_n(\tau) - u(\tau)) - \\ - (f(\tau), \bar{w}_n(\tau) - u(\tau))\} \, d\tau \geqq 0 \, .$$

Passing to the limit for $n \to \infty$ we find

$$\int_0^t \{(\dot{u}(\tau), w(\tau) - u(\tau)) + a(u(\tau), w(\tau) - u(\tau)) - \\ - (f(\tau), w(\tau) - u(\tau))\} \, d\tau \geqq 0 \quad \forall w \in \mathscr{S}_2 \, .$$

Let u_1, u_2 be two solutions of problem (1), (2). Setting $u = u_1$, $w = u_2$ and $u = u_2$, $w = u_1$ in the last inequality and taking into account the monotonicity of $a(v, w)$ we obtain after adding up:

$$\int_0^t (\dot{u}_1(\tau) - \dot{u}_2(\tau), u_1(\tau) - u_2(\tau)) \, d\tau = \tfrac{1}{2}\|u_1(t) - u_2(t)\|_0^2 \leqq 0 \, ,$$

because $u_1(0) = u_2(0) = u_0$. Thus $u_1(t) = u_2(t)$ a.e. on $(0, T)$.

D) Let us assume that there exists an infinite subsequence $\{U_s\}$ of $\{U_n\}$ with the property that no subsequence of $\{U_s\}$ converges to u in $C(I; L_2(\Omega))$. Using (49) and repeating all preceding considerations we find that a subsequence $\{U_i\}$ of $\{U_s\}$ converges to a function φ in $C(I; L_2(\Omega))$ and that φ is a solution of problem (1), (2). At the same time $\varphi \neq u$ in $C(I; L_2(\Omega))$. This is a contradiction with the uniqueness of problem (1), (2). Thus $\{U_k\} = \{U_n\}$. Theorem 2 is proved.

The proof of Theorem 1 is similar to the proof of Theorem 2 but simpler. Lemmas 1–3 hold without any change for sequences of Rothe's functions U_r and corresponding step functions $\bar{u}_r$, $\bar{U}_r$. The situation is simpler because now $K_k = K$ and we can choose $v = u(t)$ (t fixed, $t \in I \setminus E$) as a test function in (66). Integrating (66) in (t', t'') and letting $k \to \infty$ we easily find (67) and then (68). Choosing $v \in K$ arbitrarily, integrating (66) in (t', t'') and letting $k \to \infty$ we find, according to (68), Corollary 2 and Lemmas 2, 3, that relation (1) is satisfied by the limit function $u(t)$. Parts A, C, D the proof remain the same. Theorem 1 is proved.

Remark. If K contains zero and satisfies (34), (35) and if $u_0 \equiv 0$ then in the case of Problem 3 we can easily prove convergence of finite element Rothe's functions to u in $C(I; L_2(\Omega))$ under the assumption that the form $a(v, w)$ is bounded, coercive and monotone.

3. CONVERGENCE IN THE SPACE $L_2(I; V)$

In the case of Problem 4, the convergence is proved under restrictive conditions: in Theorem 3 we assume a special form of the convex set K, in Theorem 5 the parameter h_n depends on the parameter $(\Delta t)_n$.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied and let K be a closed convex cone with its vertex at zero. Then, in addition to the results introduced in Theorem 2, we have*

$$\lim_{n\to\infty} \|U_n - u\|_{L_2(I;V)} = 0 .$$

Proof. The proof is a generalization of the proof of [16, Theorem 3.1]. Therefore, the reasoning, which is the same as in [16], is only sketched.

According to (11) and Friedrichs' inequality we have

$$\int_0^T a(u, u - U_n)\,\mathrm{d}t - \int_0^T a(U_n, u - U_n)\,\mathrm{d}t \geqq C\|u - U_n\|^2_{L_2(I;V)} . \tag{71}$$

Let us set $F_n(t) = a(u(t), u(t) - U_n(t))$. Then, according to (9) and Lemmas 1 and 2,

$$|F_n(t)| \leqq \beta_2 C(C + 1) \quad \forall n \quad \forall t \in I ,$$

$$\lim_{n\to\infty} F_n(t) = 0 \quad \forall t \in I .$$

All assumptions of the Lebesgue dominated convergence theorem [10, p. 60] are satisfied; hence

$$\lim_{n\to\infty} \int_0^T a(u, u - U_n)\,\mathrm{d}t = 0 . \tag{72}$$

It remains to prove that the second integral on the left-hand side of (71) tends to zero. First we prove relation (76). To this end let us set $v = 0$ and $v = 2U^i$ in (40). Summing up the resulting equation from $i = 1$ to $i = r$ we obtain

$$\sum_{i=1}^{r} (\Delta U^i, U^i) + \Theta\Delta t \sum_{i=1}^{r} (\Delta U^i, U^i)_1 + \Delta t \sum_{i=1}^{r} a(U^{i-1}, U^i) = \tag{73}$$

$$= \Delta t \sum_{i=1}^{r} (f^{i-1}, U^i) .$$

Let us pass to the limit for $n \to \infty$ in (73). Using (9), (12), Lemmas 1 and 2 and the same argument as in the text between (3.26) and (3.27) of [16] we obtain

$$\lim_{n\to\infty} \int_0^T a(U_n, U_n)\,\mathrm{d}t = \int_0^T (f, u)\,\mathrm{d}t + \tfrac{1}{2}\|u(0)\|_0^2 - \tfrac{1}{2}\|u(T)\|_0^2 . \tag{74}$$

Let $t \in I \setminus E$ be arbitrary. Let us set $v = 0$ and $v = 2\,u(t)$ in (1). We get

$$(\dot{u}(t) - f(t), u(t)) + a(u(t), u(t)) = 0 \quad \forall t \in I \setminus E . \tag{75}$$

Integrating this relation in $(0, T)$ and comparing the result with (74) we find

$$\lim_{n\to\infty} \int_0^T a(U_n, U_n)\,\mathrm{d}t = \int_0^T a(u, u)\,\mathrm{d}t . \tag{76}$$

Now we prove that

$$\lim_{k\to\infty} \int_0^T a(U_k, u - U_k)\,\mathrm{d}t = 0\,, \tag{77}$$

where $\{U_k\}$ is a subsequence of $\{U_n\}$. Let us set

$$\langle \varphi_n(t), v\rangle = a(U_n(t), v) \quad \forall v \in V. \tag{78}$$

Taking into account (9) and (49) we see that

$$\varphi_n \in L_\infty(I, V^*)\,, \quad \|\varphi_n\|_{L_\infty(I;V^*)} \leqq C\,.$$

Therefore, there exist an element $\varphi \in L_\infty(I; V^*)$ and a subsequence $\{\varphi_k\}$ of $\{\varphi_n\}$ such that

$$\varphi_k \rightharpoonup \varphi \quad \text{weakly* in } L_\infty(I; V^*)\,. \tag{79}$$

Let us set $v = U^i + z$ in (40), where $z \in K_k \equiv K_{h_k}$. Using (32) we obtain

$$\frac{1}{\Delta t}(\Delta U^i, z) + \Theta(\Delta U^i, z)_1 + a(U^{i-1}, z) \geqq (f^{i-1}, z) \quad \forall z \in K_k\,. \tag{80}$$

Let us choose $v \in K$ arbitrarily and let $\{v_k\}$, $v_k \in K_k$, be such a sequence that

$$\lim_{k\to\infty} \|v_k - v\|_1 = 0\,.$$

Let us consider a function $\psi(t) \in C^\infty(I)$ with the property $\psi(t) \geqq 0$ and let us set $z = v_k\,\psi(t_i)$ in (80). After summing (80) from $i = 1$ to $i = r$ and after multiplying (80) by Δt let us pass to the limit for $\delta_k \to 0$ in the resulting relation. Using (79), Lemmas 1–3 and an argument similar to that used in the text between (3.24) and (3.25) in [16] we find

$$\int_0^T \{(\dot u(t) - f(t), v) + \langle \varphi(t), v\rangle\}\,\psi(t)\,\mathrm{d}t \geqq 0 \quad \forall v \in K \quad \forall \psi \in C^\infty(I)\,,$$
$$\psi(t) \geqq 0\,.$$

Let M be an arbitrary measurable subset of I and $\chi_M(t)$ its characteristic function. We can replace $\psi(t)$ by $\chi_M(t)$ in the last relation [because the proof of density of $C^\infty(I)$ in $L_2(I)$ implies that we can find a sequence $\{\psi_j\}$ such that

$$\lim_{j\to\infty} \|\psi_j - \chi_M\|_{L_2(I)} = 0 \quad \psi_j(t) \geqq 0$$

and because $\dot u - f \in L_2(I; L_2(\Omega))$, $\varphi \in L_\infty(I, V^*)$]. Thus we have

$$(\dot u(t) - f(t), v) + \langle \varphi(t), v\rangle \geqq 0 \quad \forall v \in K \quad \forall t \in I \setminus E\,. \tag{81}$$

For every element $y \in L_2(\Omega)$ there exists a unique element $g_y \in V^*$ for which

$$(y, v) = \langle g_y, v\rangle \quad \forall v \in V$$

(see [4, Chapter 1]). Let us set

$$\sigma(t) = g_{\dot{u}}(t) - g_f(t) + \varphi(t).$$

Then we can write, according to (81),

$$\langle \sigma(t), v \rangle \geqq 0 \quad \forall v \in K \quad \forall t \in I \setminus E \tag{82}$$

and we have

$$(\dot{u}(t) - f(t), v) + \langle \varphi(t) - \sigma(t), v \rangle = 0 \quad \forall v \in K \quad \forall t \in I \setminus E. \tag{83}$$

Let us choose $t \in I \setminus E$ arbitrarily and set $v = u(t)$ in (83). Comparing the result with (75) we find

$$\langle \varphi(t), u(t) \rangle = a(u(t), u(t)) + \langle \sigma(t), u(t) \rangle \quad \forall t \in I \setminus E. \tag{84}$$

Relations (78), (79) and (84) imply

$$\lim_{k \to \infty} \int_0^T a(U_k, u) \, \mathrm{d}t = \int_0^T a(u, u) \, \mathrm{d}t + \int_0^T \langle \sigma, u \rangle \, \mathrm{d}t.$$

This result and relation (76) give

$$\lim_{k \to \infty} \int_0^T a(U_k, u - U_k) \, \mathrm{d}t = \int_0^T \langle \sigma, u \rangle \, \mathrm{d}t.$$

As $a(U_n, u - U_n) \leqq a(u, u - U_n)$ we obtain from (72) that the right-hand side of the last relation is less than or equal to zero. This result together with relation (82) gives (77).

From the assertion of Theorem 2 it is easy to see that whole sequence $\{U_n\}$ converges to u in $L_2(I; V)$. Theorem 3 is proved.

In the case of Problem 2 we can prove the general result:

Theorem 4. *Let the assumptions of Theorem* 1 *be satisfied. Then*

$$\lim_{n \to \infty} \|U_n - u\|_{L_2(I;V)} = 0$$

where $\{U_n\}$ is a sequence of Rothe's functions.

Proof. Using the functions (42), (51)–(53) we obtain from relation (33) (we write n instead of r)

$$\int_0^T \{(\dot{U}_n, v_n - \bar{U}_n) + \Theta(\bar{U}_n - \bar{u}_n, v_n - \bar{U}_n)_1 - (\bar{f}_n, v_n - \bar{U}_n)\} \, \mathrm{d}t \geqq \tag{85}$$
$$\geqq \int_0^T a(\bar{u}_n, \bar{U}_n - v_n) \, \mathrm{d}t$$

where $v_n = v_n(t)$ is an arbitrary step function of the form

$$v_n(t) = g^i, \quad t \in (t_{i-1}, t_i], \quad g^i \in K \quad (i = 1, \ldots, n). \tag{86}$$

We have $u(t) \in L_2(I; V)$ and $u(t) \in K\ \forall t \in I$. Let us set

$$z^i = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} u(\tau)\, d\tau\,. \tag{87}$$

As $z^i \in K$ we can define a step function $z_n(t)$ of the type (86) by setting $g^i = z^i$. We have (see [9])

$$z_n \to u \quad \text{in} \quad L_2(I; V)\,. \tag{88}$$

Let us set $v_n(t) = z_n(t)$ in (85) and let us pass to the limit for $n \to \infty$. Owing to (88) and Lemmas 2, 3 we find that the left-hand side of (85) tends to zero. Thus

$$\limsup_{n\to\infty} \int_0^T a(\bar{u}_n(t), \bar{U}_n(t) - z_n(t))\, dt \leqq 0\,.$$

Using again (88) we see that the last relation implies

$$\limsup_{n\to\infty} \int_0^T a(\bar{u}_n(t), \bar{u}_n(t) - u(t))\, dt \leqq 0\,.$$

Using Corollary 2 of Lemma 1 and the same argument as in deriving relation (72) we find

$$\lim_{n\to\infty} \int_0^T a(u(t), \bar{u}_n(t) - u(t))\, dt = 0\,.$$

The last two relations together with the strong monotonicity (11) imply

$$\limsup_{n\to\infty} \|\bar{u}_n - u\|^2_{L_2(I;V)} = 0\,.$$

Using this result and Corollary 2 we obtain the assertion of Theorem 4.

If we consider an arbitrary convex set K in the case of Problem 4 we can prove only the following result:

Theorem 5. *Let the assumptions of Theorem 2 be satisfied. Then for every sequence $\{(\Delta t)_n\}$, where $(\Delta t)_n \to 0$, we can find a sequence of finite element convex sets K_n such that we have*

$$\lim_{n\to\infty} \|U_n - u\|_{L_2(I;V)} = 0\,,$$

where $\{U_n\}$ is the corresponding sequence of finite element Rothe's functions.

Proof. For every $(\Delta t)_n$ let us construct $z^1, \ldots, z^n$ $(n = T/(\Delta t)_n)$ by means of (87). Let us choose the corresponding triangulation $\mathcal{T}_n$ (and thus the finite element space Z_n – or simply the parameter h_n) in such a way that

$$\|v^i - z^i\|_1 \leqq C^*((\Delta t)_n)^\varepsilon \quad (i = 1, \ldots, n) \tag{89}$$

where $v^i \in K_n$ is the finite element approximation of z^i, ε is an arbitrary positive fixed number independent of n and C^* is an arbitrary positive constant independent of n.

Using the functions (43), (51)–(53) we obtain from (40) relation (85) where now

(90) $$v_n(t) = g^i\,, \quad t \in (t_{i-1}, t_i]\,, \quad g^i \in K_n \quad (i = 1, \ldots, n)\,.$$

Let us set $g^i = v^i$ in (90). Then we easily find by means of (88) and (89) that

(91) $$v_n \to u \quad \text{in} \quad L_2(I; V)\,.$$

Passing to the limit for $n \to \infty$ in (85) we find owing to (91) and Lemmas 2, 3:

$$\limsup_{n\to\infty} \int_0^T a(\bar{u}_n(t), \bar{U}_n(t) - v_n(t))\,\mathrm{d}t \leqq 0\,.$$

The rest of the proof is the same as in the case of Theorem 4. Theorem 5 is proved.

4. SOME ERROR ESTIMATES

For greater simplicity we restrict ourselves to the case $V = H_0^1(\Omega)$. The convex set K will be defined by

(92) $$K = \{v \in H_0^1(\Omega): v \geqq 0 \text{ a.e. in } \Omega\}\,.$$

In this case Green's theorem gives (under the condition that u is sufficiently smooth)

(93) $$a(u, v) = -(Au, v) \quad \forall v \in H_0^1(\Omega)\,,$$

where A is the operator generating the form $a(\cdot, \cdot)$; in the case of Example 1 we have

$$Au = \frac{\partial}{\partial x_i}\left(m(|\text{grad } u|)\,\frac{\partial u}{\partial x_i}\right).$$

We shall consider only domains Ω with polygonal boundaries and the approximate solutions defined by Problem 4.

Theorem 6. *Let the assumptions of Theorem* 1 *be satisfied and let K be of the form* (92). *Let the solution u of problem* (1), (2) *be such that*

$$u \in C(I; H^2(\Omega))\,, \quad \dot{u} \in L_2(I; H^1(\Omega))\,,$$
$$\ddot{u} \in L_2(I; V^*)\,, \quad Au \in C(I; L_2(\Omega))\,.$$

Then we have

$$\max_{i=1,\ldots,r} \|u^i - U^i\|_0 + \{\Delta t \sum_{i=1}^{r} \|u^i - U^i\|_1^2\}^{1/2} \leqq C(h + \Delta t + \|u_0 - U^0\|_1)\,,$$

where $U^1, \ldots, U^r$ is the solution of Problem 4 *and C is a constant independent of h and Δt.*

Proof. According to [8], for every $v \in H_0^1(\Omega)$ there exists an interpolate $I_h v \in Z_h$

such that

$$I_h v \in K_h = Z_h \cap K \quad \text{if} \quad v \in K\,,$$

(94) $$\|v - I_h v\|_j \leqq C\, h^{k-j} \|v\|_k \quad (j = 0, 1; k = 1, 2)\,.$$

According to [16], for all v, w, z from $H^1(\Omega)$ we have

(95) $$|\omega(v, w) - \omega(z, w)| \leqq \tau |v - z|_1 \, |w|_1$$

where the form $\omega(\cdot, \cdot)$ is given by (32) and τ is a constant independent of v, w. z and such that

$$0 < \tau < \Theta\,.$$

Let us set $t = t_i$, $v = U^i$ in (29) and $v = I_h u^i$ in (40), where $u^i = u(t_i)$. Multiplying (29) by $-\Delta t$ and (40) by -1 and adding up the resulting inequalities we obtain

(96) $$\Delta t(\dot{u}^i, e^i) + (\Delta U^i, U^i - I_h u^i) + \Theta \Delta t(u^i, e^i)_1 + \Theta \Delta t(U^i, U^i - I_h u^i)_1 \leqq$$
$$\leqq \Delta t\, \omega(U^{i-1}, U^i - I_h u^i) + \Delta t\, \omega(u^i, e^i) + \Delta t(f^i, e^i) + \Delta t(f^{i-1}, U^i - I_h u^i)\,,$$

where

(97) $$e^i = u^i - U^i\,.$$

Let us set

(98) $$\eta^i = u^i - I_h u^i\,.$$

Then

$$U^i - I_h u^i = \eta^i - e^i\,.$$

Let us write ΔU^i in the form $\Delta U^i = \Delta u^i - \Delta e^i$ and let us add $\Delta t[\omega(u^{i-1}, e^i + \eta^i) - \omega(u^{i-1}, e^i + \eta^i)]$ to the right-hand side of (96). Using (32) we obtain after rearranging the terms in (96):

$$B_1^i + \Delta t(B_2^i - |B_3^i|) \leqq \sum_{k=4}^{6} B_k^i + \Delta t \sum_{k=7}^{13} B_k^i\,,$$

where

$$\begin{aligned}
B_1^i &= (\Delta e^i, e^i)\,, & B_2^i &= \Theta(e^i, e^i)_1\,,\\
B_3^i &= \omega(u^{i-1}, e^i) - \omega(U^{i-1}, e^i)\,, & B_4^i &= (\Delta e^i, \eta^i)\,,\\
B_5^i &= (\Delta u^i - \Delta t \dot{u}^i, e^i)\,, & B_6^i &= -(\Delta u^i, \eta^i)\,,\\
B_7^i &= (\Delta f^i, e^i)\,, \quad B_8^i = (f^{i-1}, \eta^i)\,, & B_9^i &= \Theta(e^i, \eta^i)_1\,,\\
B_{10}^i &= \omega(u^i, e^i) - \omega(u^{i-1}, e^i)\,, & B_{11}^i &= \Theta(\Delta u^i, \eta^i)_1\,,\\
B_{12}^i &= \omega(U^{i-1}, \eta^i) - \omega(u^{i-1}, \eta^i)\,, & B_{13}^i &= -a(u^{i-1}, \eta^i)\,.
\end{aligned}$$

Let $1 \leqq m \leqq r$ and let us denote

$$S_k = \sum_{i=1}^{m} B_k^i \quad (k = 4, \ldots, 13)\,.$$

Then we can write

$$\sum_{i=1}^{m} B_1^i + \Delta t \sum_{i=1}^{m} (B_2^i - |B_3^i|) \leqq \sum_{k=4}^{7} S_k + \Delta t \sum_{k=7}^{13} S_k .$$

We have

$$\sum_{i=1}^{m} B_1^i \geqq \tfrac{1}{2}\|e^m\|_0^2 - \tfrac{1}{2}\|e^0\|_0^2 .$$

Using (95) and Friedrichs' inequality $|v|_1^2 \geqq C_0\|v\|_1^2$ we obtain

$$\Delta t \sum_{i=1}^{m} (B_2^i - |B_3^i|) \geqq M\, \Delta t \sum_{i=1}^{m} \|e^i\|_1^2 - \tfrac{1}{2}\tau\, \Delta t |e^0|_1^2 ,$$

where $M = (\Theta - \tau)\, C_0 > 0$. The last three inequalities imply, for $\Delta t \leqq 2/\tau$,

$$\|e^m\|_0^2 + 2M\Delta t \sum_{i=1}^{m} \|e^i\|_1^2 \leqq \|e^0\|_1^2 + 2\sum_{k=4}^{6} S_k + 2\Delta t \sum_{k=7}^{13} S_k . \tag{99}$$

Now we prove that the following estimates hold:

$$2S_4 \leqq \tfrac{1}{2}\|e^m\|_0^2 + \|e^0\|_0^2 + \tfrac{1}{8}M\Delta t \sum_{i=1}^{m} \|e^i\|_1^2 + \\ + Ch^4\|u\|_{C(I;H^2(\Omega))}^2 + Ch^2\|\dot{u}\|_{L_2(I;H^1(\Omega))}^2 , \tag{100}$$

$$2\, S_5 \leqq \tfrac{1}{8}M\Delta t \sum_{i=1}^{m} \|e^i\|_1^2 + C\Delta t^2 \|\ddot{u}\|_{L_2(I;V^*)}^2 , \tag{101}$$

$$2S_6 \leqq Ch^2(\|\dot{u}\|_{L_2(I;H^1(\Omega))}^2 + \|u\|_{C(I;H^2(\Omega))}^2) , \tag{102}$$

$$2\, \Delta t\, S_7 \leqq \tfrac{1}{8}M\, \Delta t \sum_{i=1}^{m} \|e^i\|_1^2 + C\Delta t^2 , \tag{103}$$

$$2\, \Delta t\, S_8 \leqq Ch^2\|u\|_{C(I;H^2(\Omega))} , \tag{104}$$

$$2\, \Delta t\, S_9 \leqq \tfrac{1}{8}M\Delta t \sum_{i=1}^{m} \|e^i\|_1^2 + Ch^2\|u\|_{C(I;H^2(\Omega))}^2 , \tag{105}$$

$$2\, \Delta t\, S_{10} \leqq \tfrac{1}{8}M\Delta t \sum_{i=1}^{m} \|e^i\|_1^2 + C\Delta t^2 \|\dot{u}\|_{L_2(I;H^1(\Omega))}^2 , \tag{106}$$

$$2\, \Delta t\, S_{11} \leqq C\Delta t^2 \|\dot{u}\|_{L_2(I;H^1(\Omega))}^2 + Ch^2\|u\|_{C(I;H^2(\Omega))}^2 , \tag{107}$$

$$2\, \Delta t\, S_{12} \leqq \tfrac{1}{8}M\Delta t \|e^0\|_1^2 + \tfrac{1}{8}M\Delta t \sum_{i=1}^{m} \|e^i\|_1^2 + \\ + Ch^2\|u\|_{C(I;H^2(\Omega))}^2 , \tag{108}$$

$$2\, \Delta t\, S_{13} \leqq Ch^2\|u\|_{C(I;H^2(\Omega))}\, \|Au\|_{C(I;L_2(\Omega))} . \tag{109}$$

First we prove (100). We have

$$S_4 = (e^m, \eta^m) - (e^0, \eta^1) - \sum_{i=1}^{m-1} (e^i, \Delta\eta^{i+1}) \leqq \tag{110}$$

$$\leqq \|e^m\|_0 \|\eta^m\|_0 + \|e^0\|_1 \|\eta^1\|_0 + \sum_{i=1}^{m-1} \|e^i\|_0 \|\Delta\eta^{i+1}\|_0 .$$

Relations (94), (98) and Taylor's theorem give

$$\|\eta^i\|_0 \leqq Ch^2 \|u^i\|_2 \leqq Ch^2 \|u\|_{C(I;H^2(\Omega))} , \tag{111}$$

$$\|\Delta\eta^{i+1}\|_0 \leqq Ch \|\Delta u^{i+1}\|_1 \leqq Ch \left\| \int_{t_i}^{t_{i+1}} \dot{u} \, dt \right\|_1 \leqq Ch \left\{ \Delta t \int_{t_i}^{t_{i+1}} \|\dot{u}\|_1^2 \, dt \right\}^{1/2} . \tag{112}$$

Inserting (111) and (112) into (110) and using inequality (46) several times with various values of γ we obtain (100).

Now we prove (101). We have, according to the formula for integration by parts [4, p. 148],

$$\int_{t_{i-1}}^{t_i} (t_{i-1} - t) \langle \ddot{u}(t), z \rangle \, dt = t_{i-1} \langle \Delta \dot{u}^i, z \rangle - \int_{t_{i-1}}^{t_i} \langle \ddot{u}(t), tz \rangle \, dt , \tag{113}$$

where $z \in K_h \subset K \subset H_0^1(\Omega)$. Integrating again by parts we find

$$\begin{aligned} - \int_{t_{i-1}}^{t_i} \langle \ddot{u}(t), tz \rangle \, dt &= \int_{t_{i-1}}^{t_i} \langle z, \dot{u}(t) \rangle \, dt - (\dot{u}^i, t_i z) + (\dot{u}^{i-1}, t_{i-1} z) = \\ &= \langle \Delta u^i, z \rangle - t_i \langle \dot{u}^i, z \rangle + t_{i-1} \langle \dot{u}^{i-1}, z \rangle , \end{aligned} \tag{114}$$

because $\langle v, w \rangle = (v, w)$ if $v \in L_2(\Omega)$ and $w \in V = H_0^1(\Omega)$. Inserting (114) into (113) we obtain

$$\int_{t_{i-1}}^{t_i} (t_{i-1} - t) \langle \ddot{u}(t), z \rangle \, dt = (\Delta u^i - \Delta t \dot{u}^i, z) . \tag{115}$$

Using (115) with $z = e^i$ we get

$$S_5 \leqq \Delta t \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} |\langle \ddot{u}(t), e^i \rangle| \, dt \leqq \Delta t \sum_{i=1}^{m} \left\{ \Delta t \int_{t_{i-1}}^{t_i} \|\ddot{u}\|_* \, dt \right\}^{1/2} \|e^i\|_1 . \tag{116}$$

Relation (101) follows from (116) by means of inequality (46).

Relations (102)–(109) can be proved similarly by means of (93), (95), (111), (112) and (46). Inserting (100)–(109) into (99) and using the standard argument we easily obtain the assertion of Theorem 6.

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Souhrn

APROXIMACE PARABOLICKÝCH VARIAČNÍCH NEROVNIC

Alexander Ženíšek

V článku jsou studovány různé aproximace parabolické variační nerovnice (1), kde $a(u, v)$ je nelineární eliptická forma mající potenciál $J(v)$, který je dvakrát G-diferencovatelný pro libovolné $v \in H^1(\Omega)$. Tato vlastnost formy $a(v, w)$ umožňuje dokázat konvergenci přibližných řešení definovaných linearizovanými schématy (33) a (40). Schéma (40) je plně diskretizováno – v prostoru metodou konečných prvků a v čase Eulerovou diferenční metodou (levá strana implicitní, pravá strana explicitní formulí). Silná konvergence jak v prostoru $C(I; L_2(\Omega))$, tak v prostoru $L_2(I; H^1(\Omega))$ je dokázána bez jakýchkoliv předpokladů o hladkosti přesného řešení. V závěru článku jsou provedeny odhady chyby za dodatečných předpokladů o hladkosti řešení.

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