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PERIODIC MOVING AVERAGE PROCESS

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Periodic moving average processes are representatives of the class of periodic models suitable for the description of some seasonal time series and for the construction of multivariate moving average models. The attention being lately concentrated mainly on the periodic autoregressions, some methods of statistical analysis of the periodic moving average processes are suggested in the paper. These methods include the estimation procedure (based on Durbin's construction of the parameter estimators in the moving average processes and on Pagano's results for the periodic autoregressions) and the test of the periodic structure. The results are demonstrated by means of numerical simulations.

1. INTRODUCTION

Periodic time series models whose coefficients change periodically in time were originally suggested for modeling time series with seasonal character (see e.g. [3], [6]) but later their usefulness in the multivariate time series analysis has been shown as well (see [7], [8]). However, the attention was concentrated mainly on the case of the periodic autoregressions which was studied from various points of view (see [1], [6], [7], [8]), while the periodic moving average processes were neglected (e.g. Cleveland and Tiao [3] investigated them only from the identification point of view).

A periodic moving average process $\{X_t\}$ is given by the following general relation (see e.g. [9])

$$X_t = \varepsilon_t + \beta_1(t)\,\varepsilon_{t-1} + \ldots + \beta_{q_t}(t)\,\varepsilon_{t-q_t}\,, \tag{1.1}$$

where the coefficients $\beta_j(t)$ are periodic functions of time with a period d, i.e.

$$\beta_j(t) = \beta_j(t + d) \tag{1.2}$$

and

$$q_t = q_{t+d}\,. \tag{1.3}$$

The process $\{\varepsilon_t\}$ is a normal white noise with zero mean and a variance $\sigma^2 > 0$ (we shall not consider the more general case in which the variance σ^2 also changes periodically). It is obvious that all the above coefficients $\beta_j(t)$ are fully determined by the vectors

$$(1.4) \qquad \boldsymbol{\beta}(1) = (\beta_1(1), \ldots, \beta_{q_1}(1))', \ldots, \boldsymbol{\beta}(d) = (\beta_1(d), \ldots, \beta_{q_d}(d))' .$$

The following invertibility conditions are supposed to be fulfilled for all these vectors: all roots of the characteristic equations

$$(1.5) \qquad z^{q_i} + \beta_1(i)\, z^{q_i - 1} + \ldots + \beta_{q_i}(i) = 0\,, \quad i = 1, \ldots, d$$

are in the absolute value less than one. The process (1.1) is called a periodic moving average process with the period d and the orders $q_1, \ldots, q_d$.

Although the application of the periodic moving average processes for the parameter estimation in the multivariate moving average models will be the subject of another paper let us describe here its main idea to demonstrate the usefulness of these processes. With each d-dimensional moving average process $\{\mathbf{X}_t\}$ of an order q one can identify a model of the form

$$(1.6) \qquad \mathbf{X}_t = \boldsymbol{\beta}_0 \boldsymbol{\varepsilon}_t + \boldsymbol{\beta}_1 \boldsymbol{\varepsilon}_{t-1} + \ldots + \boldsymbol{\beta}_q \boldsymbol{\varepsilon}_{t-q}\,,$$

where $\boldsymbol{\beta}_j$ are $d \times d$ matrices of parameters such that $\boldsymbol{\beta}_0$ is a lower triangular matrix with positive numbers on the main diagonal and $\{\boldsymbol{\varepsilon}_t\}$ is a d-dimensional white noise with zero mean vector and a variance matrix equal to the identity matrix (if we consider the more usual form $\mathbf{X}_t = \boldsymbol{\eta}_t + \gamma_1 \boldsymbol{\eta}_{t-1} + \ldots + \gamma_q \boldsymbol{\eta}_{t-q}$ with a white noise $\{\boldsymbol{\eta}_t\}$ such that $\operatorname{var}(\boldsymbol{\eta}_t)$ is a general positive definite matrix we can set $\boldsymbol{\varepsilon}_t = \mathbf{T}^{-1}\boldsymbol{\eta}_t$, $\boldsymbol{\beta}_0 = \mathbf{T}$ and $\boldsymbol{\beta}_j = \gamma_j \mathbf{T}$, $j = 1, \ldots, q$, where the lower triangular matrix $\mathbf{T}$ is taken from the so called Cholesky decomposition $\operatorname{var}(\boldsymbol{\eta}_t) = \mathbf{T}\mathbf{T}'$). If we define a univariate process $\{X_t\}$ by means of the relation

$$(1.7) \qquad X_{j+d(t-1)} = \mathbf{X}_{jt}\,, \quad j = 1, \ldots, d\,,$$

where $\mathbf{X}_t = (\mathbf{X}_{1t}, \ldots, \mathbf{X}_{dt})'$, then one can see from (1.6) that $\{X_t\}$ is a periodic moving average process with the period d and the orders $1 + dq, 2 + dq, \ldots, d + dq$.

In this paper the estimation procedure for the model (1.1) and the test of the periodic structure are developed. The estimation procedure is based on the method suggested by Durbin [5] for the construction of asymptotically efficient parameter estimators in the univariate moving average models and on Pagano's results [8] for the periodic autoregressions. First the procedure is demonstrated on the simple case with $d = 2$ in Section 2 but then the general case is considered in Section 3. The test of periodicity is described in Section 4 and the results of some numerical simulations are given in Section 5.

2. CASE WITH PERIOD TWO

Let us consider the periodic moving average process (1.1) with the period $d = 2$ given by the relations denoted for simplicity as

$$(2.1) \qquad X_{2t} = \varepsilon_{2t} + \alpha_1 \varepsilon_{2t-1} + \ldots + \alpha_{q_1} \varepsilon_{2t-q_1},$$

$$X_{2t+1} = \varepsilon_{2t+1} + \beta_1 \varepsilon_{2t} + \ldots + \beta_{q_2} \varepsilon_{2t+1-q_2}.$$

Let us approximate (2.1) by the periodic autoregression (also with the period two) of the form

$$(2.2) \qquad X_{2t} + \gamma_1 X_{2t-1} + \ldots + \gamma_{k_1} X_{2t-k_1} = \varepsilon_{2t},$$

$$X_{2t+1} + \delta_1 X_{2t} + \ldots + \delta_{k_2} X_{2t+1-k_2} = \varepsilon_{2t+1},$$

where the numbers k_1 and k_2 are sufficiently large (e.g., in the simplest case with $q_1 = q_2 = 1$ we can write the explicit expression of (2.2), where $\gamma_1 = -\alpha_1, \gamma_2 = \alpha_1\beta_1$, $\gamma_3 = -\alpha_1^2\beta_1$, $\gamma_4 = \alpha_1^2\beta_1^2, \ldots, \delta_1 = -\beta_1$, $\delta_2 = \alpha_1\beta_1, \delta_3 = -\alpha_1\beta_1^2$, $\delta_4 = \alpha_1^2\beta_1^2, \ldots$). Such an approximation is in accordance with Durbin's approach [5] to the parameter estimation in the nonperiodic moving average models.

According to [8], for the parameters $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{k_1})'$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{k_2})'$ of (2.2) one can write two systems of the Yule-Walker equations

$$(2.3) \qquad \mathbf{R}_1 \boldsymbol{\gamma} = -\mathbf{g}, \quad \mathbf{R}_2 \boldsymbol{\delta} = -\mathbf{h},$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are $k_1 \times k_1$ and $k_2 \times k_2$ variance matrices of the form

$$(2.4) \qquad \mathbf{R}_1 = \operatorname{var}\{(X_{2t-1}, X_{2t-2}, \ldots, X_{2t-k_1})'\},$$

$$\mathbf{R}_2 = \operatorname{var}\{(X_{2t}, X_{2t-1}, \ldots, X_{2t+1-k_2})'\}$$

and the vectors $\mathbf{g} = (g_1, \ldots, g_{k_1})'$ and $\mathbf{h} = (h_1, \ldots, h_{k_2})'$ are defined by

$$(2.5) \qquad g_i = \operatorname{cov}(X_{2t}, X_{2t-i}), \quad i = 1, \ldots, k_1,$$

$$h_j = \operatorname{cov}(X_{2t+1}, X_{2t+1-j}), \quad j = 1, \ldots, k_2$$

(t in (2.4) and (2.5) can be arbitrary because (2.2) is the so called covariance stationary periodic autoregression in the sense of [8]).

The systems of equations (2.3) can be used for the construction of the estimators $\mathbf{c} = (c_1, \ldots, c_{k_1})'$ and $\mathbf{d} = (d_1, \ldots, d_{k_2})'$ of the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$; if the observations $X_1, \ldots, X_T$ are available (for simplicity, we assume T even, $T = 2N$) it suffices to replace all covariances of the type $\operatorname{cov}(X_u, X_v)$ in (2.4) and (2.5) by their estimates

$$(2.6) \qquad R_N(u, v) = \frac{1}{m_2 - m_1 + 1} \sum_{k=m_1}^{m_2} X_{u+2k} X_{v+2k},$$

where the limits m_1 and m_2 are chosen so that all terms in the preceding sum are known and their number is maximal. Pagano [8] showed that such estimators $\mathbf{c}$ and $\mathbf{d}$ have asymptotically normal distributions with mean vectors γ and $\boldsymbol{\delta}$, variance matrices $(\sigma^2/N)\,\mathbf{R}_1^{-1}$ and $(\sigma^2/N)\,\mathbf{R}_2^{-1}$, and are mutually uncorrelated.

Durbin's estimation procedure in the nonperiodic moving average models consists in maximizing the likelihood function derived from the asymptotic distribution of the estimated parameters in the autoregression approximation to the moving average model. If we use the above asymptotic normal distribution of $\mathbf{c}$ and $\mathbf{d}$ we can modify Durbin's approach for the periodic case by maximizing the function

$$Q = -\frac{N}{2\sigma^2}\{(\mathbf{c}-\gamma)'\,\mathbf{R}_1(\mathbf{c}-\gamma) + (\mathbf{d}-\boldsymbol{\delta})'\,\mathbf{R}_2(\mathbf{d}-\boldsymbol{\delta})\} \tag{2.7}$$

with respect to the original parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{q_1})'$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{q_2})'$. The function Q is obviously the dominating part of the asymptotic loglikelihood of $\mathbf{c}$ and $\mathbf{d}$ (at least for large T).

Using (2.3) we can rewrite (2.7) to the form

$$Q = -\frac{N}{2\sigma^2}(\mathbf{c}'\mathbf{R}_1\mathbf{c} + 2\mathbf{c}'\mathbf{g} - \gamma'\mathbf{g} + \mathbf{d}'\mathbf{R}_2\mathbf{d} + 2\mathbf{d}'\mathbf{h} - \boldsymbol{\delta}'\mathbf{h})\,. \tag{2.8}$$

Moreover, by virtue of (2.1) and (2.2) we have

$$\begin{aligned}\gamma'\mathbf{g} &= \gamma_1 \operatorname{cov}(X_{2t}, X_{2t-1}) + \ldots + \gamma_{k_1} \operatorname{cov}(X_{2t}, X_{2t-k_1}) = \\ &= \operatorname{cov}(X_{2t}, \gamma_1 X_{2t-1} + \ldots + \gamma_{k_1} X_{2t-k_1}) = \operatorname{cov}(X_{2t}, \varepsilon_{2t} - X_{2t}) = \\ &= -\sigma^2(\alpha_1^2 + \ldots + \alpha_{q_1}^2)\end{aligned} \tag{2.9}$$

and analogously

$$\boldsymbol{\delta}'\mathbf{h} = -\sigma^2(\beta_1^2 + \ldots + \beta_{q_2}^2)\,. \tag{2.10}$$

Therefore

$$Q = -\frac{N}{2\sigma^2}\left(\mathbf{c}'\mathbf{R}_1\mathbf{c} + 2\mathbf{c}'\mathbf{g} + \sigma_2 \sum_{i=1}^{q_1} \alpha_i^2 + \mathbf{d}'\mathbf{R}_2\mathbf{d} + 2\mathbf{d}'\mathbf{h} + \sigma^2 \sum_{j=1}^{q_2} \beta_j^2\right). \tag{2.11}$$

Finally, the estimators $\mathbf{a} = (a_1, \ldots, a_{q_1})'$ and $\mathbf{b} = (b_1, \ldots, b_{q_2})'$ of the parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{q_1})'$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{q_2})'$ are constructed by differentiating Q with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and equating the derivatives to zero (i.e. by solving the approximate maximum likelihood equations). For this purpose the elements of the matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ and of the vectors $\mathbf{g}$ and $\mathbf{h}$ must be expressed explicitly in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ using (2.1) (i.e.

$$\begin{aligned}\operatorname{cov}(X_{2t-1}, X_{2t-1}) &= \sigma^2(1 + \beta_1^2 + \ldots + \beta_{q_2}^2)\,, \\ \operatorname{cov}(X_{2t-1}, X_{2t-2}) &= \sigma^2(\beta_1 + \alpha_1\beta_2 + \ldots)\,, \\ \operatorname{cov}(X_{2t}, X_{2t}) &= \sigma^2(1 + \alpha_1^2 + \ldots + \alpha_{q_1}^2)\,, \\ \operatorname{cov}(X_{2t}, X_{2t-1}) &= \sigma^2(\alpha_1 + \alpha_2\beta_1 + \ldots) \quad \text{etc.,}\end{aligned}$$

where $\alpha_r = 0$ for $r > q_1$ and $\beta_s = 0$ for $s > q_2$ in the sums of the type $\beta_1 + \alpha_1\beta_2 + \ldots$). It is not difficult to derive by means of simple algebraic manipulations that such estimators $\boldsymbol{a}$ and $\boldsymbol{b}$ can be obtained as the solutions of two systems (2.12) and (2.13) of linear equations (let us define for simplicity $a_r = 0$ for $r > q_1$, $b_s = 0$ for $s > q_2$, $c_i = 0$ for $i > k_1$ and $d_j = 0$ for $j > k_2$)

$$(2.12)\quad \begin{aligned} &(1 + d_1^2 + c_2^2 + \ldots)\, a_1 + (d_1 + c_1c_2 + d_2d_3 + \ldots)\, b_2 + \\ &\quad + (c_2 + d_1d_3 + c_2c_4 + \ldots)\, a_3 + (d_3 + c_1c_4 + d_2d_5) \ldots)\, b_4 + \ldots = \\ &= -(c_1 + d_1d_2 + c_2c_3) + \ldots), \\ &(d_1 + c_1c_2 + d_2d_3 + \ldots)\, a_1 + (1 + c_1^2 + d_2^2 + \ldots)\, b_2 + \\ &\quad + (c_1 + d_1d_2 + c_2c_3 + \ldots)\, a_3 + (d_2 + c_1c_3 + d_2d_4 + \ldots)\, b_4 + \ldots = \\ &\quad = -(d_2 + c_1c_3 + d_2d_4 + \ldots), \\ &(c_2 + d_1d_3 + c_2c_4 + \ldots)\, a_1 + (c_1 + d_1d_2 + c_2c_3 + \ldots)\, b_2 + \\ &\quad + (1 + d_1^2 + c_2^2 + \ldots)\, a_3 + (d_1 + c_1c_2 + d_2d_3 + \ldots)\, b_4 \ldots = \\ &\quad = -(c_3 + d_1d_4 + c_2c_5 + \ldots), \\ &\qquad\qquad\qquad\qquad \vdots \end{aligned}$$

$$(2.13)\quad \begin{aligned} &(1 + c_1^2 + d_2^2 \ldots)\, b_1 + (c_1 + d_1d_2 + c_2c_3 + \ldots)\, a_2 + \\ &\quad + (d_2 + c_1c_3 + d_2d_4 + \ldots)\, b_3 + (c_3 + d_1d_4 + c_2c_5 + \ldots)\, a_4 + \ldots = \\ &\quad = -(d_1 + c_1c_2 + d_2d_3 + \ldots), \\ &(c_1 + d_1d_2 + c_2c_3 + \ldots)\, b_1 + (1 + d_1^2 + c_2^2 + \ldots)\, a_2 + \\ &\quad + (d_1 + c_1c_2 + d_2d_3 + \ldots)\, b_3 + (c_2 + d_1d_3 + c_2c_4 + \ldots)\, a_4 + \ldots = \\ &\quad = -(c_2 + d_1d_3 + c_2c_4 + \ldots), \\ &(d_2 + c_1c_3 + d_2d_4 + \ldots)\, b_1 + (d_1 + c_1c_2 + d_2d_3 + \ldots)\, a_2 + \\ &\quad + (1 + c_1^2 + d_2^2 + \ldots)\, b_3 + (c_1 + d_1d_2 + c_2c_3 + \ldots)\, a_4 + \ldots = \\ &\quad = -(d_3 + c_1c_4 + d_2d_5 + \ldots), \\ &\qquad\qquad\qquad\qquad \vdots \end{aligned}$$

(the first equation in the system (2.12) corresponds to $\partial Q/\partial\alpha_1 = 0$, the second equation to $\partial Q/\partial\beta_2 = 0$, etc., and the first equation in the system (2.13) corresponds to $\partial Q/\partial\beta_1 = 0$, the second equation to $\partial Q/\alpha_2 = 0$, etc.). The number of equations in (2.12) or (2.13) must be equal to the number of the unknown variables in these systems. The compact formulas for (2.12) and (2.13) are given in Section 3.

Moreover, the asymptotic covariance structure of the estimators $\boldsymbol{a}$ and $\boldsymbol{b}$ can be easily estimated. Let us define the following vectors (with the appropriate finite dimensions):

$$(2.14)\quad \xi_1 = (a_1, b_2, a_3, b_4, \ldots)', \quad \xi_2 = (b_1, a_2, b_3, a_4, \ldots)', \quad \xi = (\xi_1', \xi_2')'.$$

Since the vector ξ is constructed by means of the approximate maximum likelihood principle it can be considered to be asymptotically normal and unbiased and its asymptotic variance matrix can be expressed to the order $o(1/T)$ as $(\mathsf{E}\,\partial^2 Q/\partial\xi^2)^{-1}$ (see e.g. [4]). Using this formula we can estimate the variance matrix of ξ_1 as

(2.15) $$\boldsymbol{\Omega}_1 =$$

$$= \frac{1}{N}\begin{pmatrix} 1 + d_1^2 + c_2^2 + \ldots, & d_1 + c_1c_2 + d_2d_3 + \ldots, & c_2 + d_1d_3 + c_2c_4 + \ldots, & \ldots \\ d_1 + c_1c_2 + d_2d_3 + \ldots, & 1 + c_1^2 + d_2^2 + \ldots, & c_1 + d_1d_2 + c_2c_3 + \ldots, & \ldots \\ c_2 + d_1d_3 + c_2c_4 + \ldots, & c_1 + d_1d_2 + c_2c_3 + \ldots, & 1 + d_1^2 + c_2^2 + \ldots, & \ldots \\ & \vdots & & \end{pmatrix}^{-1}.$$

The estimator $\boldsymbol{\Omega}_2$ of the variance matrix of ξ_2 differs from (2.15) only by interchanging $\mathbf{c}$ and $\mathbf{d}$ and, finally, ξ_1 and ξ_2 can be taken asymptotically uncorrelated. These conclusions on the covariance structure will be used in Section 4 for the construction of the test of periodicity.

Finally, the variance σ^2 of the white noise can be estimated e.g. from the values $\hat{\varepsilon}_t$ calculated by means of (2.1) using the estimated coefficients and $\hat{\varepsilon}_0 = \hat{\varepsilon}_{-1} = \hat{\varepsilon}_{-2} = \ldots = 0$.

Remark. In the simplest case with $d = 2$ and $q_1 = q_2 = 1$ we obtain the following explicit results. The estimators a_1 and b_1 of the parameters α_1 and β_1 have the form

(2.16) $$a_1 = -\frac{c_1 + d_1d_2 + c_2c_3 + d_3d_4 + \ldots}{1 + d_1^2 + c_2^2 + d_3^2 + \ldots},$$

$$b_1 = -\frac{d_1 + c_1c_2 + d_2d_3 + c_3c_4 + \ldots}{1 + c_1^2 + d_2^2 + c_3^2 + \ldots}$$

(the construction of the estimators $\mathbf{c}$ and $\mathbf{d}$ remains unchanged). The estimators a_1 and b_1 are asymptotically uncorrelated with the asymptotic variances estimated by

(2.17) $$\frac{1}{N}\,\frac{1}{1 + d_1^2 + c_2^2 + d_3^2 + \ldots}, \quad \frac{1}{N}\,\frac{1}{1 + c_1^2 + d_2^2 + c_3^2 + \ldots}.$$

3. GENERAL CASE

In the general case we handle the periodic moving average process with a period d and orders $q_1, \ldots, q_d$ of the form

(3.1) $$\begin{aligned} X_{dt} &= \varepsilon_{dt} + \beta_1(1)\,\varepsilon_{dt-1} + \ldots + \beta_{q_1}(1)\,\varepsilon_{dt-q_1}, \\ X_{dt+1} &= \varepsilon_{dt+1} + \beta_1(2)\,\varepsilon_{dt} + \ldots + \beta_{q_2}(2)\,\varepsilon_{dt+1-q_2}, \\ &\vdots \\ X_{dt+d-1} &= \varepsilon_{dt+d-1} + \beta_1(d)\,\varepsilon_{dt+d-2} + \ldots + \beta_{q_d}(d)\,\varepsilon_{dt+d-1-q_d}. \end{aligned}$$

We can literally repeat the reasoning from Section 2 but now with the autoregression approximations

$$(3.2)\qquad \begin{aligned} &X_{dt} &&+ \alpha_1(1)\,X_{dt-1} &&+ \ldots + \alpha_{k_1}(1)\,X_{dt-k_1} &&= \varepsilon_{dt}\,,\\ &X_{dt+1} &&+ \alpha_1(2)\,X_{dt} &&+ \ldots + \alpha_{k_2}(2)\,X_{dt+1-k_2} &&= \varepsilon_{dt+1}\,,\\ &&&&&\vdots&&\\ &X_{dt+d-1} &&+ \alpha_1(d)\,X_{dt+d-2} &&+ \ldots + \alpha_{k_d}(d)\,X_{dt+d-1-k_d} &&= \varepsilon_{dt+d-1}\,. \end{aligned}$$

Let us denote by symbols $\boldsymbol{a}(1), \ldots, \boldsymbol{a}(d)$ the estimated vectors of their parameters constructed similarly as in Section 2 on the basis $X_1, \ldots, X_T$ with $T = dN$. Then the estimators $\boldsymbol{b}(1), \ldots, \boldsymbol{b}(d)$ of the parameters in (3.1) can be obtained as the solutions of d systems of linear equations. The i-th system $(i = 1, \ldots, d)$ which yields the values $b_1(i)$, $b_2(i+1)$, $b_3(i+2), \ldots$ as its solution has the form

$$(3.3)\qquad \sum_{k=1}^{j-1}\{\sum_{r=1} a_{r-1}(i+j+r-2)\,a_{j-k+r-1}(i+j+r-2)\}\,b_k(k+i-1) +$$
$$+ \sum_{k=j}\{\sum_{r=1} a_{r-1}(i+k+r-2)\,a_{k-j+r-1}(i+k+r-2)\}\,b_k(k+i-1) =$$
$$= -\sum_{r=1} a_{r-1}(i+j+r-2)\,a_{j+r-1}(i+j+r-2)\,,\quad j = 1, 2, \ldots,$$

where we take $a_r(i) = 0$ for $r > k_i$, $a_0(i) = 1$, $a_r(i) = a_r(i+d)$, $b_k(i) = 0$ for $k > q_i$, $b_k(i) = b_k(i+d)$. The number of equations in the i-th system (3.3) is equal to the number of its unknown variables $b_1(i)$, $b_2(i+1)$, $b_3(i+2), \ldots$. It can be easily verified that the systems (2.12) and (2.13) are special cases of (3.3) when $d = 2$.

Moreover, the vectors $\boldsymbol{b}^*(1) = (b_1(1), b_2(2), b_3(3), \ldots)'$, $\boldsymbol{b}^*(2) = (b_1(2), b_2(3), b_3(4), \ldots)'$, $\ldots$, $\boldsymbol{b}^*(d) = (b_1(d), b_2(1), b_3(2), \ldots)'$ of the solutions of the particular systems (3.3) are mutually uncorrelated and the asymptotic variance matrices of these vectors can be estimated by the inverse matrices to the matrices of the left-handside coefficients in the particular systems equations (3.3) divided by N (the arguments for these conclusions are the same as in Section 2).

4. TEST OF PERIODIC STRUCTURE

When testing the periodic structure of a moving average process we can make use of the asymptotic normality of the described estimators which was mentioned in the previous text including the estimated asymptotic variance matrices.

Let us test null hypothesis consisting in the nonperiodic structure of the given moving average process of an order q against the alternative hypothesis consisting in the periodic structure with a given periodicity d and orders $q_1 = q_2 = \ldots = q_d = q$ (as concerns the equality of the orders in the framework of the alternative hypothesis we can argue that when the nonperiodic structure is rejected against such alternative hypothesis then from the practical point of view it will be rejected also against a more general periodic structure).

When $d = 2$ the critical region of the test on the significance level ν can be constructed as

$$(4.1) \qquad (\xi_1 - \xi_2)' (\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2)^{-1} (\xi_1 - \xi_2) > \chi^2_q(\nu),$$

where the vectors ξ_1 and ξ_2 are defined in (2.14), the matrices $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$ in (2.15) and in the text following (2.15) and $\chi^2_q(\nu)$ is the critical value of the chi-squared distribution with q degrees of freedom for the significance level ν (see e.g. [2]).

Remark. In the simplest case with $d = 2$ and $q_1 = q_2 = 1$ we can construct the critical region of the test directly as

$$(4.2) \qquad |a_1 - b_1| > u(\nu) \left\{ \left(\frac{1}{1 + d_1^2 + c_2^2 + \dots} + \frac{1}{1 + c_1^2 + d_2^2 + \dots} \right) \Big/ N \right\}^{1/2},$$

where $u(\nu)$ is the critical value of the standard normal distribution on the significance level ν.

When $d > 2$ is would be theoretically possible to use the test suggested by Anderson [2], p. 211 for testing simultaneously the equality of mean vectors of several normal distributions (in our case we should have to work with vectors $\boldsymbol{\Omega}_1^{-1/2}\, \mathbf{b}^*(1), \dots, \boldsymbol{\Omega}_d^{-1/2}\, \mathbf{b}^*(d)$, where $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_d$ are the estimated asymptotic variance matrices of the vectors $\mathbf{b}^*(1), \dots, \mathbf{b}^*(d)$, see Section 3). However, for practical purposes it is more suitable to test all pairs $\mathbf{b}^*(i)$ and $\mathbf{b}^*(j)$ $(1 \leqq i < j \leqq d)$ by means of (4.1).

5. SIMULATION STUDY

The above results have been verified by means of simulations.

First fifty periodic moving average processes of the type

$$(5.1) \qquad \begin{aligned} X_{2t} &= \varepsilon_{2t} + 0{\cdot}6\varepsilon_{2t-1}, \\ X_{2t+1} &= \varepsilon_{2t+1} - 0{\cdot}4\varepsilon_{2t} \end{aligned}$$

with the length $T = 200$ and with the white noise $\varepsilon_t \sim \mathrm{N}(0, 1)$ were generated on the computer ADT 4100 at the Dept. of Statistics of Charles University (i.e. $d = 2$, $q_1 = q_2 = 1$, $\alpha_1 = 0{\cdot}6$, $\beta_1 = -0{\cdot}4$) and then their parameters were estimated by means of (2.16). The results for the particular generated processes are given in the second and third column of Tab. 1. The observed means and standard deviations of these fifty results are

$$(5.2) \qquad \begin{aligned} \bar{a}_1 &= 0{\cdot}6018, \quad s_{a_1} = 0{\cdot}0972, \\ \bar{b}_1 &= -0{\cdot}3938, \quad s_{b_1} = 0{\cdot}0752. \end{aligned}$$

Table 1. Results of a simulation study for the process $X_{2t} = \varepsilon_{2t} + 0{\cdot}6\varepsilon_{2t-1}$, $X_{2t+1} = \varepsilon_{2t+1} - 0{\cdot}4\varepsilon_{2t}$, $\varepsilon_t \sim N(0, 1)$

Simulation	a_1	b_1	$\lvert a_1 - b_1 \rvert$	Critical value
1	0·5566	−0·3573	0·9139	0·2353
2	0·6210	−0·4099	1·0309	0·2188
3	0·5613	−0·4193	0·9806	0·2298
4	0·6179	−0·4391	1·0570	0·2244
5	0·4452	−0·5259	0·9711	0·2275
6	0·6143	−0·3713	0·9856	0·2297
7	0·6012	−0·2677	0·8689	0·2386
8	0·6473	−0·3557	1·0030	0·2346
9	0·5837	−0·4448	1·0285	0·2253
10	0·5151	−0·4091	0·9242	0·2345
11	0·5509	−0·3432	0·8941	0·2414
12	0·5838	−0·3604	0·9442	0·2356
13	0·6424	−0·2999	0·9423	0·2316
14	0·5671	−0·3818	0·9489	0·2277
15	0·7671	−0·4644	1·2315	0·2051
16	0·7013	−0·4157	1·1170	0·2247
17	0·4872	−0·5643	1·0515	0·2277
18	0·5802	−0·4302	1·0104	0·2246
19	0·6710	−0·5607	1·2317	0·2107
20	0·4385	−0·2722	0·7107	0·2486
21	0·6039	−0·3374	0·9413	0·2346
22	0·7285	−0·3112	1·0397	0·2291
23	0·7636	−0·2944	1·0580	0·2198
24	0·5599	−0·4110	0·9803	0·2280
25	0·7430	−0·3757	1·1187	0·2748
26	0·6428	−0·4047	1·0475	0·2208
27	0·6424	−0·4962	1·1386	0·2174
28	0·6251	−0·3254	0·9505	0·2307
29	0·5130	−0·3278	0·8408	0·2401
30	0·5565	−0·3811	0·9376	0·2382
31	0·6320	−0·4399	1·0719	0·2280
32	0·5180	−0·3664	0·8844	0·2342
33	0·5547	−0·3593	0·9140	0·2280
34	0·5592	−0·5833	1·1425	0·2145
35	0·5401	−0·4645	1·0046	0·2307
36	0·6749	−0·2320	0·9069	0·2357
37	0·3540	−0·3961	0·7501	0·2466
38	0·6671	−0·4238	1·0909	0·2203
39	0·6208	−0·3937	1·0145	0·2311
40	0·5338	−0·3802	0·9140	0·2403
41	0·4122	−0·3035	0·7157	0·2428
42	0·6038	−0·3412	0·9450	0·2349

Table 1 — Continued

Simulation	a_1	b_1	$\|a_1 - b_1\|$	Critical value
43	0·7000	−0·4199	1·1199	0·2118
44	0·5916	−0·3801	0·9717	0·2309
45	0·6198	−0·4049	1·0247	0·2268
46	0·7180	−0·4825	1·2005	0·2158
47	0·8820	−0·2976	1·1796	0·2209
48	0·7240	−0·3813	1·1053	0·2247
49	0·4516	−0·3934	0·8450	0·2396
50	0·5989	−0·4764	1·0753	0·2256

Table 2. Property of the periodicity test under the null nonperiodic hypothesis for the process $X_t = \varepsilon_t - 0{\cdot}4\varepsilon_{t-1}$, $\varepsilon_t \sim$ N(0, 1), on the significance level 5%

Simulation	a_1	b_1	5) $\|a_1 - b_1\|$	Critical value
1	−0·4210	−0·3773	0·0437	0·2369
2	−0·4812	−0·4936	0·0124	0·2254
3	−0·3839	−0·4709	0·0870	0·2328
4	−0·5418	−0·4134	0·1284	0·2236
5	−0·4223	−0·3523	0·0700	0·2380
6	−0·4116	−0·2209	0·1907	0·2467
7	−0·3666	−0·4410	0·0744	0·2423
8	−0·3927	−0·2174	0·1753	0·2459
9	−0·4682	−0·4244	0·0438	0·2328
10	−0·4631	−0·3891	0·0470	0·2396
11	−·03653	−0·4330	0·0677	0·2391
12	−0·3679	−0·4533	0·0854	0·2364
13	−0·5151	−0·3518	0·1633	0·2288
14	−0·2526	−0·3792	0·1266	0·2507
15	−0·5134	−0·5502	0·0368	0·2236
16	−0·3750	−0·3543	0·0207	0·2389
17	−0·5423	−0·3101	0·2322	0·2356
18	−0·3688	−0·3639	0·0049	0·2416
19	−0·2618	−0·2641	0·0023	0·2515
20	−0·4557	−0·3254	0·1303	0·2381
21	−0·3388	−0·3110	0·0278	0·2326
22	−0·3514	−0·2883	0·0613	0·2396
23	−0·3492	−0·4566	0·1074	0·2371
24	−0·3649	−0·4249	0·0600	0·2351
25	−0·4765	−0·3589	0·1176	0·2351

Table 2 — Continued

Simulation	a_1	b_1	$\|a_1 - b_1\|$	Critical value
26	−0·4013	−0·3887	0·0126	0·2428
27	−0·3654	−0·4304	0·0650	0·2400
28	−0·4676	−0·4842	0·0166	0·2229
29	−0·4476	−0·4411	0·0065	0·2351
30	−0·3624	−0·4435	0·0811	0·2335
31	−0·4983	−0·4839	0·0144	0·2309
32	−0·6261	−0·5058	0·1203	0·2129
33	−0·3721	−0·4283	0·0562	0·2387
34	−0·4689	−0·2264	0·2425	0·2493
35	−0·4815	−0·3154	0·0661	0·2337
36	−0·2822	−0·3245	0·0423	0·2486
37	−0·5078	−0·4420	0·0658	0·2259
38	−0·4116	−0·5796	0·1680	0·2216
39	−0·4459	−0·3759	0·0700	0·2388
40	−0·3747	−0·4916	0·1169	0·2208
41	−0·5359	−0·4198	0·1161	0·2305
42	−0·4091	−0·5012	0·0921	0·2247
43	−0·5063	−0·4756	0·0307	0·2299
44	−0·4364	−0·5726	0·1362	0·2260
45	−0·5462	−0·5175	0·0287	0·2231
46	−0·4510	−0·4043	0·0467	0·2317
47	−0·3972	−0·3531	0·0441	0·2416
48	−0·3560	−0·4511	0·0951	0·2389
49	−0·4440	−0·3320	0·1120	0·2400
50	−0·3884	−0·2618	0·1266	0·2374

This speaks in favour of our estimation method. The estimated variances of the white noise constructed according to the simple method from Section 2 are no reported in Tab. 1 but the results are also satisfactory; e.g. for the first three simulations in Tab. 1 we obtained the estimates 0·8203, 0·8904 and 0·9894.

Table 2 contains results verifying the suggested test of periodicity under the null hypothesis. Fifty nonperiodic moving average processes of the type

$$X_t = \varepsilon_t - 0{\cdot}4\varepsilon_{t-1} \tag{5.3}$$

wiht the length $T = 200$ and with the white noise $\varepsilon_t \sim N(0, 1)$ were generated and the test of periodicity (4.2) on the significance level 5% was performed for each of them. The null hypothesis was rejected in none of the performed simulations although it was nearly rejected in the simulations 17 and 34. Thus the test worked well in this case and also in other simulation studies which are not reported here.

Finally, the power of the test (4.2) is demonstrated for the example (5.1) in the fourth and fifth columns of Tab. 1 where the values $|a_1 - b_1|$ and the critical values forming the righthand sides of (4.2) on the significance level 5% are given. In this example the test rejects the null nonperiodic hypothesis for all simulations performed but the results are satisfactory also in the cases when the theoretical values of the parameters are not so distinct.

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Souhrn

PERIODICKÝ PROCES KLOUZAVÝCH SOUČTŮ

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Periodické procesy klouzavých součtů jsou reprezentanty třídy periodických modelů vhodných pro popis některých sezónních časových řad a pro konstrukci mnohorozměrných modelů klouzavých součtů. Protože v poslední době se pozornost soustředila hlavně na periodické autoregresní procesy, jsou v tomto článku navrženy některé metody statistické analýzy procesů klouzavých součtů. Tyto metody zahrnují odhadovou proceduru (založenou na Durbinově konstrukci odhadů parametrů v procesech klouzavých součtů a na výsledcích Pagana pro periodické autoregresní procesy) a test periodické struktury. Výsledky jsou demonstrovány pomocí číselných simulací.

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