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ON PERIODIC SOLUTIONS OF A SPECIAL TYPE OF THE BEAM EQUATION

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Summary. The paper deals with the existence of time-periodic solutions to the beam equation, in which terms expressing torsion and damping are also considered. The existence of periodic solutions is proved in the case of time-periodic outer forces by means of an apriori estimate and the Fourier method.

Keywords: Periodic solutions, beam equation, apriori estimate.

AMS Classification: 35B10, 73K12.

In this paper we shall prove the existence of functions u and v which satisfy the equations

$$u_{tt} - cv_{tt} + u_{xxxx} + \alpha u_t - \beta \left(\int_0^\pi u_x^2(\xi, \cdot)\, d\xi \right) u_{xx} = f^{(1)}, \tag{1}$$

$$-cu_{tt} + \gamma v_{tt} + \delta v_{xxxx} + \tilde{\alpha} v_t - \tilde{\beta} v_{xx} = f^{(2)}, \tag{2}$$

the boundary conditions

$$u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \quad t \in \langle 0, \pi \rangle, \tag{3}$$

$$v(0, t) = v(\pi, t) = v_{xx}(0, t) = v_{xx}(\pi, t) = 0, \quad t \in \langle 0, T \rangle, \tag{4}$$

and the periodicity conditions

$$u(\cdot, 0) = u(\cdot, T), \quad v(\cdot, 0) = v(\cdot, T), \tag{5}$$

$$u_t(\cdot, 0) = u_t(\cdot, T), \quad v_t(\cdot, 0) = v_t(\cdot, T), \tag{6}$$

where $T > 0$ is a real constant.

Coefficients $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, \delta$ are positive constants. We shall suppose $\gamma - c^2 > 0$. The physical meaning of the last inequality, as well as of the system (1), (2), may be found in Timoshenko [4]. The nonlinear term represents the change in the tension of the beam due to its extensibility.

This problem is connected with the initial-boundary value problem for the nonlinear beam equation that was discussed by De Andrade [1] and Ball [2].

As usual, functions $u, v: \langle 0, \pi \rangle \times \langle 0, T \rangle \to \mathbb{R}$ will be regarded as functions defined on $\langle 0, T \rangle$, whose values belong to some function space on $\langle 0, \pi \rangle$. Let $L^2(0, \pi)$ be the Hilbert space of real valued Lebesgue measurable functions $u = u(x)$ on $\langle 0, \pi \rangle$ with $|u| < \infty$, where $|u|^2 = (u, u)$ and $(u, v) = \int_0^\pi u(x)\, v(x)\, \mathrm{d}x$. For the sake of simplicity we shall write u, v instead of $u(t), v(t)$, unless we wish to stress the dependence on time. Thus we shall use the notation

$$|u|^2 = |u(t)|^2 = \int_0^\pi u^2(x, t)\, \mathrm{d}x\,,$$

$$(u, v) = (u(t), v(t)) = \int_0^\pi u(x, t)\, v(x, t)\, \mathrm{d}x\,.$$

First of all we shall establish a formal apriori estimate provided we have solutions u, v which are smooth enough.

Multiplying (1), (2) by $u_t + \varepsilon u$, $v_t + \varepsilon v$, respectively ($\varepsilon > 0$ will be fixed in the course of calculations), and integrating from 0 to π we obtain two equations. Their sum gives

$$\text{(7)} \qquad \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} N(u, v, u_t, v_t) + M(u, v, u_t, v_t) = (f^{(1)}, u_t + \varepsilon u) + (f^{(2)}, v_t + \varepsilon v)\,,$$

and N, M may be written in the form

$$\text{(8)} \qquad N(u, v, u_t, v_t) = |u_t|^2 + \gamma |v_t|^2 - 2c(u_t, v_t) + 2\varepsilon((u_t, u) + \gamma(v_t, v) - \\ - c(u_t, v) - c(v_t, u)) + \varepsilon\alpha|u|^2 + \varepsilon\tilde{\alpha}|v|^2 + |u_{xx}|^2 + \delta|v_{xx}|^2 + \\ + \frac{\beta}{2}|u_x|^4 + \tilde{\beta}|v_x|^2\,,$$

$$\text{(9)} \qquad M(u, v, u_t, N_t) = \alpha|u_t|^2 + \tilde{\alpha}|v_t|^2 + \varepsilon|u_{xx}|^2 + \varepsilon\delta|v_{xx}|^2 + \varepsilon\beta|u_x|^4 + \\ + \varepsilon\tilde{\beta}|v_x|^2 + \varepsilon(2c(u_t, v_t) - |u_t|^2 - \gamma|v_t|^2)\,.$$

We shall omit the terms u_t, v_t in the definition of N, M, if it cannot lead to misunderstanding.

First of all we estimate the term $N(u, v)$.

Lemma 1. *If $u(t), v(t) \in H_0^1(0, \pi) \cap H_2(0, \pi)$, $u_t(t), v_t(t) \in L^2(0, \pi)$ then there exist constants $A, B > 0$ such that*

$$A(|u_t|^2 + |v_t|^2 + |u_{xx}|^2 + |v_{xx}|^2 + |u_x|^4) \leqq N(u, v) \leqq \\ \leqq B(|u_t|^2 + |v_t|^2 + |u_{xx}|^2 + |v_{xx}|^2 + |u_x|^4)\,.$$

Proof. Let us make use of the fact that for $u \in H_0^1(0, \pi) \cap H^2(0, \pi)$, $L(|u|^2 + |u_x|^2) \leqq |u_{xx}|^2$ holds for some fixed $L > 0$, and of the Young inequality $2ab \leqq a^2/\eta^2 + b^2\eta^2$, $\eta > 0$. Then we obtain

$$N(u, v) \geqq |u_t|^2 (1 - c/\eta^2 - \varepsilon(1 + c)) + |v_t|^2 (\gamma - c\eta^2 - \varepsilon(\gamma + c)) + \\ + |u|^2 \left(\varepsilon\alpha + \frac{L}{2} - \varepsilon(1 + c)\right) + |v|^2 \left(\varepsilon\tilde{\alpha} + \frac{\delta L}{2} - \varepsilon(\gamma + c)\right) + \tfrac{1}{2}|u_{xx}|^2 + \\ + \frac{\delta}{2} |v_{xx}|^2 + \frac{\beta}{2} |u_x|^4 .$$

As $\gamma > c^2$, there exists $\eta > 0$ such that $c < \eta^2 < \gamma/c$, so that we may choose $\varepsilon > 0$ satisfying

(10) $$\varepsilon < L/(2 + 2c + 2\alpha),$$

(11) $$\varepsilon < \delta L/(2\gamma + 2c + 2\tilde{\alpha}),$$

(12) $$\varepsilon < (1 - c/\eta^2)/(1 + c),$$

(13) $$\varepsilon < (\gamma - c\eta^2)/(\gamma + c),$$

(14) $$\varepsilon < \min \{\alpha/(1 + c), \tilde{\alpha}/(\gamma + c)\},$$

(15) $$2\varepsilon^2 < \{\min (\varepsilon\tilde{\alpha} + \delta + \beta, \gamma(\varepsilon\alpha + 1))\}/(\gamma + c\sqrt{\gamma}).$$

The first inequality is proved. The inequality (15) is required to be satisfied for a later use in the main theorem. On the other hand, by direct calculation we obtain

$$N(u, v) \leqq |u_t|^2 (1 + c)(1 + \varepsilon) + |v_t|^2 (\gamma + c)(1 + \varepsilon) + \\ + |u_{xx}|^2 (1 + \varepsilon/L(1 + c + \alpha)) + |u_x|^4 \frac{\beta}{2} + |v_{xx}|^2 (\delta + \varepsilon/L(\gamma + c + \tilde{\alpha}) + \beta/L)$$

which completes the proof of Lemma 1.

A simple application of Lemma 1 provides

$$M(u, v) \geqq |u_t|^2 (\alpha - \varepsilon(1 + c)) + |v_t|^2 (\tilde{\alpha} - \varepsilon(\gamma + c)) + \\ + \varepsilon|u_{xx}|^2 + \varepsilon|\delta v_{xx}|^2 + \varepsilon|\beta u_x|^4 \geqq K\, N(u, v)$$

where

(16) $$K = B^{-1} \min \{\alpha - \varepsilon(1 + c), \tilde{\alpha} - \varepsilon(\gamma + c), \varepsilon, \varepsilon\delta, \varepsilon\beta\}.$$

Hence, by (7), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} N(u, v) + K\, N(u, v) \leqq \frac{\xi^2}{2} (|f^{(1)}|^2 + |f^{(2)}|^2) + \frac{1}{2\xi^2}\left(|u_t|^2 + |v_t|^2 + \frac{\varepsilon}{L}|u_{xx}|^2 + \right. \\ \left. + \frac{\varepsilon}{L}|v_{xx}|^2\right) \leqq \frac{\xi^2}{2} (|f^{(1)}|^2 + |f^{(2)}|^2) + \tfrac{1}{2}K\, N(u, v)$$

where $\xi > 0$ satisfies $\xi^2 = (1/AK) \max(1, \varepsilon/L)$. If we denote $\tilde{\varphi} = \xi^2(f^{(1)\,2} + |f^{(2)}|^2)$ we can finally write

$$\frac{\mathrm{d}}{\mathrm{d}t} N(u, v)(t) + K N(u, v)(t) \leqq \tilde{\varphi}(t) . \tag{17}$$

In the proof of the main theorem we shall make use of the following lemma (see Ball [2]).

Lemma 2. *Let X be a Banach space. If $f \in L^2(0, T; X)$ and $\dot{f} \in L^2(0, T; X)$ then f, possibly after changing it on a set of measure zero, is continuous from $\langle 0, T\rangle$ into X.*

Indeed, for almost every $s, t \in \langle 0, T\rangle$:

$$f(t) - f(s) = \int_s^t \dot{f}(\sigma)\, \mathrm{d}\sigma .$$

Now we can establish the main theorem. Let us denote $S = H^2(0, \pi) \cap H_0^1(0, \pi)$. For $u \in S'$ and $v \in S$ we shall write (u, v) instead of $u(v)$.

Theorem. *Let $f^{(1)}, f^{(2)} \in L^\infty(0, T; L^2(0, \pi))$. Then there exist $u, v \in L^\infty(0, T; S)$ such that $u_t, v_t \in L^\infty(0, T; L^2(0, \pi))$, $u_{tt}, v_{tt} \in L^\infty(0, T; S')$ and both u and v satisfy the periodicity conditions (5), (6) and the equations (1), (2) in the weak sense:*

$$(u_{tt}, w) - c(v_{tt}, w) + (u_{xx}, w_{xx}) + \alpha(u_t, w) - \beta|u_x|^2 (u_{xx}, w) = (f^{(1)}, w) , \tag{18}$$

$$-c(u_{tt}, w) + \gamma(v_{tt}, w) + \delta(v_{xx}, w_{xx}) + \tilde{\alpha}(v_t, \omega) - \tilde{\beta}(v_{xx}, w) = (f^{(2)}, w) \tag{19}$$

for all $w \in S$.

Remark. The boundary conditions (3) and (4) are included in the formulation of the weak solution.

Proof. System $\{\sin kx\}_{k=1}^\infty$ is a basis of S. Let us denote $V_n = \operatorname{lin}\{\sin kx\}_{k=1}^m$. Now we shall try to find $u^m, v^m: \langle 0, T\rangle \to V_m$,

$$u^m(t) = \sum_{k=1}^m g_k^m(t) \sin kx ,$$

$$v^m(t) = \sum_{k=1}^m h_k^m(t) \sin kx ,$$

satisfying (18), (19) for all $w \in V_m$.

First we solve the initial-boundary value problem for (18), (19) and then we show that the initial conditions may be chosen in such a way that u^m and v^m satisfy the periodicity conditions.

We take the following initial conditions for u, v:

$$u^m(0) = \sum_{k=1}^{m} \alpha_k \sin kx \,, \tag{20}$$

$$v^m(0) = \sum_{k=1}^{m} \beta_k \sin kx \,, \tag{21}$$

$$u_t^m(0) = \sum_{k=1}^{m} \tilde{\alpha}_k \sin kx \,, \tag{22}$$

$$v_t^m(0) = \sum_{k=1}^{m} \tilde{\beta}_k \sin kx \,. \tag{23}$$

If we substitute w for $\sin kx$, $k = 1, \ldots, m$, in (18), (19) we obtain a system of $2m$ non-linear ordinary differential equations for $g_k^m(t)$, $h_k^m(t)$, $k = 1, \ldots, m$ (for the sake of simplicity we omit the index m):

$$\ddot{g}k - c\ddot{h}_k + k^4 g_k + \alpha \dot{g}_k + \frac{4\beta}{\pi} \left(\sum_{j=1}^{m} g_j^2 j^2 \right) k^2 g_k = f_k^{(1)} \,, \tag{24}$$

$$-c\ddot{g}_k + \gamma \ddot{h}_k + \delta k^4 h_k + \alpha \dot{h}_k + \tilde{\beta} k^2 h_k = f_k^{(2)} \,. \tag{25}$$

These equations may be transformed into $4m$ equations of the first order and written in the form

$$\dot{x} = R(x) + S(t) \,,$$

where $R(x)$ is a polynomial and $S(t) \in L^1(0, T)$. According to the Carathéodory theory (see Kurzweil [3]), the system (24), (25) with the initial conditions (derived from (20)–(23))

$$g_i(0) = \alpha_i \,, \quad h_i(0) = \beta_i \,, \quad \dot{g}_i(0) = \tilde{\alpha}_i \,, \quad \dot{h}_i(0) = \tilde{\beta}_i \tag{26}$$

has a unique solution on $\langle 0, t_m \rangle$, $t_m > 0$.

Multiplying (24), (25) by $\dot{g}_k + \varepsilon g_k$, $\dot{h}_k + \varepsilon h_k$, respectively, we obtain an priori estimate for u^m, v^m, which is of the same form as (17) where we substitute u^m, v^m for u, v. Thus

$$\frac{d}{dt} N(u^m, v^m)(t) + KN(u^m, v^m)(t) \leqq \tilde{\varphi}(t) \,. \tag{27}$$

Multiplying by e^{Kt} and integrating from 0 to t we obtain

$$N(u^m, v^m)(t) \leqq N(u^m, v^m)(0)\, e^{-Kt} + \int_0^t e^{K(s-t)} \tilde{\varphi}(s)\, ds \,. \tag{28}$$

It shows among other that $t_m = T$. So we have functions g_k, h_k on $\langle 0, T \rangle$ with $\dot{g}_k$, $\dot{h}_k$ absolutely continuous on $\langle 0, T \rangle$. Then there exist $\ddot{g}_k$, $\ddot{h}_k$ almost everywhere. Hence we have the solutions u^m, v^m, satisfying the initial conditions (20)–(23).

In order to find solutions satisfying (5), (6) we define a mapping

$$\varkappa\colon \mathbb{R}^{4n} \to \mathbb{R}^{4n}$$

by

$$\varkappa\colon (\alpha_k, \beta_k, \tilde{\alpha}_k, \tilde{\beta}_k) \to (g_k(T), h_k(T), \dot{g}_k(T), \dot{h}_k(T))$$

where g_k, h_k, $\dot{g}_k$, $\dot{h}_k$ are solutions of the system (24), (25) with the initial conditions (26). If $N(u^m, v^m, u_t^m, v_t^m)(0) < R$ then

(29) $$N(u^m, v^m, u_t^m, v_t^m)(T) \leqq R\,\mathrm{e}^{-KT} + Q\,,$$

where

$$Q = \int_0^T \tilde{\varphi}(s)\,\mathrm{d}s\,.$$

We shall denote

$$u = \sum_1^m \alpha_k \sin kx\,, \quad v = \sum_1^m \beta_k \sin kx\,,$$

$$w = \sum_1^m \tilde{\alpha}_k \sin kx\,, \quad z = \sum_1^m \tilde{\beta}_k \sin kx\,,$$

$$P_R = \{(\alpha_k, \beta_k, \tilde{\alpha}_k, \tilde{\beta}_k) \in \mathbb{R}^{4m}\,, \quad k = 1, \ldots, m\,; \quad N(u, v, w, z) \leqq R\}\,.$$

Vectors from P satisfy the relation

$$\sum_{k=1}^m \{\tilde{\alpha}_k^2 + \gamma\tilde{\beta}_k^2 - 2c\tilde{\alpha}_k\tilde{\beta}_k + 2\varepsilon(\alpha_k\tilde{\alpha}_k + \gamma\beta_k\tilde{\beta}_k - c\beta_k\tilde{\alpha}_k - c\alpha_k\tilde{\beta}_k) +$$
$$+ \alpha_k^2(\varepsilon\alpha + k^4) + \beta_k^2(\varepsilon\tilde{\alpha} + \delta k^4 + \beta k^2)\} + \frac{\pi\beta}{4}\left(\sum_{k=1}^m k^2\alpha_k^2\right)^2 \leqq R\,.$$

Using the transformation in the form

$$\alpha_k = \sqrt{(\gamma)}\,(a_k + b_k)\,,$$
$$\beta_k = a_k - b_k\,,$$
$$\tilde{\alpha}_k = \sqrt{(\gamma)}\,(\tilde{a}_k + \tilde{b}_k)\,,$$
$$\tilde{\beta}_k = \tilde{a}_k - \tilde{b}_k$$

and the assumption on ε given by (15), we can prove, after some calculations, that P_R is a convex set. As P_R is also closed and bounded, we can take advantage of the Brouwer theorem. From (29) it is easy to see that for $R > Q \setminus (1 - \exp(-KT))$ we have $\varkappa(P_R) \subset P_R$. So there exists a fixed point of the mapping $\varkappa$ and thus $g_k(0) = g_k(T)$, $h_k(0) = h_k(T)$ and $\dot{g}_k(0) = \dot{g}_k(T)$, $\dot{h}_k(0) = \dot{h}_k(T)$. Consequently, the corresponding u^m, v^m satisfy the periodicity conditions (5), (6).

Inequality (29) also gives $N(u^m, v^m)(t) \leqq \tilde{C}, \tilde{C} > 0$ for $t \in \langle 0, T \rangle$. From Lemma 1 we can see that

$$|u_t^m(t)|, |u_x^m(t)|, |u_{xx}^m(t)|, |u^m(t)| \leqq C,$$

where C is independent of m, t, and analogously for v. This implies that $\{u^m\}$, $\{v^m\}$ are bounded in $L^\infty(0, T; S)$ and $\{u_t^m\}$, $\{v_t^m\}$, $\{u_{xx}^m\}$, $\{v_{xx}^m\}$ and $\{|u_x^m|^2 u_{xx}^m\}$ are bounded in $L^\infty(0, T; L^2(0, \pi))$.

We can choose subsequences with the following properties (we write m instead of m_n):

$$u^m \to u, \quad v^m \to v \quad \text{weakly* in} \quad L^\infty(0, T; S),$$

$$u_t^m \to u_t, \quad u_{xx}^m \to u_{xx} \quad \text{weakly* in} \quad L^\infty(0, T; L^2(0, \pi)),$$

$$v_t^m \to v_t, \quad v_{xx}^m \to v_{xx} \quad \text{weakly* in} \quad L^\infty(0, T; L^2(0, \pi)),$$

$$|u_x^m|^2 u_{xx}^m \to \varphi \quad \text{weakly* in} \quad L^\infty(0, T; L^2(0, \pi)).$$

Moreover, as the injection $H^1((0, \pi) \times (0, T)) \to L^2((0, \pi) \times (0, T))$ is compact, $u^{m_k} \to u$, $v^{m_k} \to v$ a.e. in $L^2((0, \pi) \times (0, T))$. In the sequel we shall write u^m instead of u^{m_k}. The next step is to show that $\varphi = |u_x|^2 u_{xx}$.

For $\psi \in L^1(0, T; L^2)$ we have

$$\int_0^T (\varphi - |u_x|^2 u_{xx}, \psi) \, dt = \int_0^T (\varphi - |u_x^m|^2 u_{xx}^m, \psi) \, dt +$$

$$+ \int_0^T |u_x|^2 (u_{xx}^m - u_{xx}, \psi) \, dt + \int_0^T (|u_x^m|^2 - |u_x|^2)(u_{xx}^m, \psi) \, dt.$$

But

$$\left| \int_0^T (|u_x^m|^2 - |u_x|^2)(u_{xx}^m, \psi) \, dt \right| \leqq c \left| \int_0^T (u^m - u, u_{xx}^m + u_{xx}) \, dt \right| \leqq$$

$$\leqq C \left(\int_0^T |u^m - u|^2 \, dt \right)^{1/2} \to 0, \quad m \to \infty.$$

The other integrals also tend to zero and the arbitrariness of ψ implies that $\varphi = |u_x|^2 u_{xx}$.

Now let $k \in N$ be fixed. Then

$$(u_{xx}^m, \sin kx) \to (u_{xx}, \sin kx) \quad \text{weakly* in} \quad L^\infty(0, T)$$

$$(u_t^m, \sin kx) \to (u_t, \sin kx) \quad \text{weakly* in} \quad L^\infty(0, T)$$

$$(u_{tt}^m, \sin kx) \to (u_{tt}, \sin kx) \quad \text{weakly* in} \quad D'(0, T),$$

and analogously for v. Hence the equations (18), (19) hold for $w = \sin kx$ and thus for all $w \in S$. Moreover, equations (18), (19) after some calculations give $(u_{tt}^m, w) \to (u_{tt}, w)$ weakly in $L^\infty(0, T)$ for all $w \in S$ and thus $(u_{tt}, w) \in L^\infty(0, T)$. That implies $u_{tt} \in L^\infty(0, T; S')$.

Finally, we must show that u, v satisfy conditions (5), (6). As $u^m \to u$ and $u_t^m \to u_t$ weakly* in $L^\infty(0, T; L^2)$, according to Lemma 2 we obtain, after a redefinition if necessary, $(u^m(0), \varphi) \to (u(0), \varphi)$ and $(u^m(T), \varphi) \to (u(T), \varphi)$ for all $\varphi \in L^2(0, \pi)$. Since $U^m(0) = u^m(T)$, we can see that $u(0) = u(T)$.

From (18), (19) we have for $w \in S$, $(u_{tt}^m, w) \to (u_{tt}, w)$ weakly in $L^\infty(0, T)$, thus from Lemma 2 we have $(u_t^m(0), w) \to (u_t(0), w)$, $w \in S$, and since $u_t^m(0) = u_t^m(T)$ we may again conclude that $u_t(0) = u_t(T)$ nad analogously for v. This completes the proof.

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Souhrn

O PERIODICKÝCH ŘEŠENÍCH SPECIÁLNÍHO TYPU ROVNICE TYČE

Jan Řeháček

Tento článek se zabývá existencí periodických řešení rovnic pro kmitání nosníku, ve kterých se uvažuje i ohýbání nosníku a tlumení kmitů. Za předpokladu časově periodických vnějších sil je ukázána existence periodických řešení pomocí apriorního odhadu a Fourierovy metody.

Резюме

О ПЕРИОДИЧЕСКИХ РЕШЕНИЯХ УРАВНЕНИЙ УПРУГИХ КОЛЕБАНИЙ

Jan Řeháček

В статье изучается существование периодических решений системы дифференциальных уравнений для упругих колебаний, в которых рассматривается также скручиваемость и гашение колебаний. В случае периодических внешних сил доказывается существование решений путем априорной оценки решений и методом Фурье.

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