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# ERROR ESTIMATES FOR EXTERNAL APPROXIMATION OF ORDINARY DIFFERENTIAL EQUATIONS AND THE SUPERCONVERGENCE PROPERTY

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*Summary.* A pointwise error estimate and an estimate in norm are obtained for a class of external methods approximating boundary value problems. Dependence of a superconvergence phenomenon on the external approximation method is studied. In this general framework, superconvergence at the knot points for piecewise polynomial external methods is established.

*Keywords:* ordinary differential operators, external approximation, superconvergence property

*AMS Subject Classification:* 65L10

## INTRODUCTION

Superconvergence of approximate solutions of differential and integral problems at the knot points is an interesting property investigated by many authors. Among other this property was established in [3] and [8] for the collocation method for solving second order ordinary differential equations and in [7], [9] for the Galerkin method for solving two-point boundary value problems and the heat equation ([14]). Moreover, superconvergence was proved for the Galerkin method for the Fredholm integral equation of the second kind (cf. [12], [5], [13]). A superconvergence phenomenon for the gradient of the finite element approximate solution was also analysed (cf. [15]). Extensive references concerning this problem can be found in [10].

It does not follow from the above papers what is the relation between the method of approximation and the presence of superconvergence. In the present paper a class of external approximations is investigated. This class, studied in part by Aubin [1], possesses some computational facilities and therefore is useful in practice. It is shown how superconvergence depends on the choice of a method from this class. The superconvergence results are established for a class of external approximations of boundary value problems for even order ordinary differential equations.

Up to now, a superconvergence result for external approximations was obtained for a special case of the method only (cf. [11]). In Section 1 this result together with the known superconvergence result for the Galerkin method [7] are quoted. In Section 2, a partial approximation convergence is proved. This theorem is based on the first Strang lemma ([4], Chap. 4.1) and has a general character. Next, our considerations are restricted to the case of the boundary value problems for ordinary differential equations of order 2m and to the external method generated by projections. The main theorem gives the dependence of a pointwise estimate on $t$ via the Green function $G(t, \cdot)$ of the operator $(-1)^m y^{(2m)}$ on $H_0^m$. The theorem is formulated in a general form in order to be applicable to methods generated by nonorthogonal projections (cf. Section 5). An $L^2$ error estimate is also given. In Section 4, on the basis of this theorem, superconvergence at the knot points is established for approximations generated by finite element subspaces and orthogonal projections. Some remarks concerning nonorthogonal projections are included.

## 1. EXAMPLE

Let us consider the following model problem:

find $u \in H_0^1(I)$ such that

$$(u', v') + (bu, v) = (f, v) \quad \forall v \in H_0^1(I)\,, \tag{1.1}$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(I)$, $I = (0, 1)$. Let $b \geqq b_0 > 0$ and $f, b \in H^{r-1}(I)$ $(r \geqq 1)$. Then $u \in H^{r+1}(I)$. For the approximate solving this problem let us apply a finite element method generated by the subspaces $S_h(H_0^1, r) \subset H_0^1$ corresponding to a partition $\Delta = \Delta(h) = \{ih\rangle_{i=0}^n$, $h = 1/n$. So

$$S_h(H_0^1, r) = \{v \in H_0^1(I) |\; v|_{(x_i, x_{i+1})} \in P_r \text{ for } x_i \in \Delta(h)\}$$

where $P_r$ denotes the set of polynomials of degree $\leqq r$. It is known that the solution $u_h^G$ of the Galerkin equation:

find $u_h^G \in S_h(H_0^1, r)$ such that

$$(u_h^{G'}, v_h') + (bu_h^G, v_h) = (f, v_h) \quad \forall v_h \in S_h(H_0^1, r)\,, \tag{1.2}$$

approximates $u$ with the error

$$\|u - u_h^G\|_0 + h\|u' - u_h^{G'}\|_0 \leqq ch^{r+1}\|u\|_{r+1}\,. \tag{1.3}$$

Moreover, if $r \geqq 2$ the rate of convergence increases at the knot points

$$|u(x_i) - u_h^G(x_i)| \leqq ch^{2r}\|u\|_{r+1} \quad \forall x_i \in \Delta(h)\,. \tag{1.4}$$

This is a superconvergence phenomenon described in [7]. In [11] it was shown that a similar property takes place also if instead of $u_h^G$ we take a solution $u_h$ of the following problem:

find $u_h \in S_h(H_0^1, r)$ such that

$$(u_h', v_h') + (b\varphi_h u_h, \varphi_h v_h) = (f, \varphi_h v_h) \quad \forall v_h \in S_h(H_0^1, r)\,, \tag{1.5}$$

where $\varphi_h$ is the orthogonal projection of $L^2(I)$ onto $S_h(L^2, r - 1)$. Namely,

$$(1.6) \qquad \|u - u_h\|_1 \leqq ch^r(\|b\|_{r-1} + \|f\|_{r-1})$$

and

$$|u(x_i) - u_h(x_i)| \leqq ch^{2r}(\|b\|_r + \|f\|_r) \quad \forall x_i \in \Delta(h) .$$

Moreover, it is known (for (1.6) this was shown in [11]) that on the right hand side of the above estimates (1.3), (1.4), (1.6) the norms of $H^{r-1}$ and $H^r$ can be replaced by the norms of $H^{r-1}_\Delta$ and $H^r_\Delta$, respectively; where

$$H^s_\Delta = \{v \in L^2(I) | \ v \in H^s(x_i, x_{i+1}) \text{ for } x_i \in \Delta\}$$

and

$$\|v\|_{\Delta s} = (\sum_{i=0}^{n-1} \|v\|^2_{H^s(x_i, x_{i+1})})^{1/2}.$$

Evidently, for $f \in H^s$ we have $\|f\|_s = \|f\|_{\Delta s}$. Nevertheless, the methods allow us to obtain the same order of convergence also in the case when $b, f \notin H^r(I)$ but $b, f \in H^r_{\Delta'}(I)$ (or $H^{r-1}_{\Delta'}(I)$), where $\Delta'$ is a certain finite set of points contained in the set $\Delta(h_0) = \{ih_0\}_{i=0}^{n_0-1} (h_0 n_0 = 1)$. In this case we consider such parameters $h(= 1/n)$ that $\Delta(h) \supset \Delta'$.

The estimates (1.6) show that the superconvergence phenomenon (1.4) occurs not only for the Galerkin method but also for its generalization (1.5). In the paper we show that besides (1.5) there exists a certain class of methods for which the superconvergence holds.

## 2. CONVERGENCE

Let $\Omega \subset R^d (d = 1, 2, \ldots)$ be a bounded domain with a Lipschitz boundary. Let $V_0 = L^2(\Omega)$ and let $V_1, \ldots, V_m$, $V$ be Sobolev spaces such that $V \hookrightarrow V_i \hookrightarrow V_0$ for $i = 1, \ldots, m$. Let $a_{ij}$ be bilinear forms on $V_i \times V_j$ generated by (ordinary or partial) differential operators $l_i$, $l_j$

$$a_{ij}(u, v) = (\alpha_{ij} l_i u, l_j v) \quad \forall u \in V_i , \quad v \in V_j , \quad i, j = 0, \ldots, m ,$$

where $l_i \in L(V_i, L^2)$, $l_0$ is the identity operator in $L^2$, and $\alpha_{ij}$ are real functions from $L^\infty(\Omega)$.

Let us consider the problem

find $u \in V$ such that

$$(2.1) \qquad a(u, v) := \sum_{i,j=0}^{m} a_{ij}(u, v) = (f, v) \quad \forall v \in V .$$

To seek for an approximate solution of the problem (2.1), consider a family $\{V_h\}$ of finite dimensional subspaces of $V$ and families $\{\varphi_{hi}\}$, $i = 0, \ldots, m$ of linear maps of $V$ into $V_i$. The parameter $h$ is the defining parameter of the families, $h \in \mathscr{H}$, and its limit is zero. Let us consider the following approximation of the problem (2,1):

find $u_h \in V_h$ such that

$$(2.2) \qquad \sum_{ij=0}^{m} a_{ij}(\varphi_{hi}u_h, \varphi_{hj}v_h) = (f, \varphi_{h0}v_h) \quad \forall v_h \in V_h .$$

It is a certain kind of partial approximation of (2.1) (cf. [1]), i.e. a special kind of external approximation. If $\varphi_{h0}, \ldots, \varphi_{hm}$ are identity operators on $V_h$ then (2.2) becomes a Galerkin equation. Generalized finite element methods introduced and discussed in [2] are similar to those considered here.

For the investigation of the approximation (2.2), the following notation will be convenient:

$$F = L^2 \times V_1 \times \ldots \times V_m \quad \text{with the norm} \quad \|\bar{u}\|_F = (\sum_{i=0}^{m} \|u_i\|_{V_i}^2)^{1/2} ;$$

$$F_h = \{(\varphi_{h0}v_h, \ldots, \varphi_{hm}v_h), v_h \in V_h\} ;$$

$$\omega: V \to F , \quad \omega u = (u, \ldots, u) ;$$

$$\omega_h: V \to F , \quad \omega_h u = (\varphi_{h0}u, \ldots, \varphi_{hm}u) ;$$

$$\bar{a}: F \times F \to R , \quad \forall \bar{u}, \bar{v} \in F \quad \bar{a}(\bar{u}, \bar{v}) = \sum_{ij=0}^{m} a_{ij}(u_i, v_j) ;$$

$$a_h: V_h \times V_h \to R , \quad \forall u_h, v_h \in V_h \quad a_h(u_h, v_h) = \bar{a}(\omega_h u_h, \omega_h v_h) .$$

We assume that

A1 – the form $a$ is $V$-elliptic,

A2 – the space $V$ is isomorphic to the subspace $\omega V$ of $F$,

A3 – $\exists c > 0 \ \|\omega_h u\|_F \leqq c\|u\|_V \quad \forall u \in V$,

A4 – the forms $a_h$ are uniformly $V_h$-elliptic.

Remark 1. If

i) the form $\bar{a}$ is $F$-elliptic and

ii) $\exists c > 0 \quad \forall u_h \in V_h \quad \forall h \in \mathscr{H} \quad c\|\omega_h u_h\|_F \geqq \|u_h\|_V$

then the assumption A4 is satisfied.

The condition ii) holds, for instance, in the case $V_m = V$ and $\varphi_{hm} = I$, since

$$\|\omega_h u_h\|^2 = \sum_{i=0}^{m-1} \|\varphi_{hi}u_h\|_{V_i}^2 + \|u_h\|_V^2 \geqq \|u_h\|_V^2 .$$

Let $u$ and $u_h$ be solutions of the problems (2.1) and (2.2), respectively. According to the first Strang Lemma ([4], Th. 4.1.1) there exists a constant $c$ independent of $h$ such that

$$(2.3) \qquad \|u - u_h\|_V \leqq c \{\inf_{v_h \in V_h} [\|u - v_h\|_V + \sup_{\substack{w_h \in V_h \\ \|w_h\| = 1}} |a(v_h, w_h) - a_h(v_h, w_h)|] + \\ + \sup_{\substack{w_h \in V_h \\ \|w_h\| = 1}} |(f, w_h - \varphi_{h0}w_h)|\} .$$

Estimating the terms appearing on the right-hand side of the above inequality we can obtain the following result:

**Proposition 1.** *If* A1 – A4 *are satisfied, then*

$$\|u - u_h\|_V \leqq c[\inf_{v_h \in V_h} \|u - v_h\|_V + \|(\omega - \omega_h) u\|_F + \sup_{\substack{v_h \in V_h \\ \|v_h\| = 1}} |\bar{a}(\omega u, (\omega - \omega_h) v_h)| + \\ + \sup_{\substack{v_h \in V_h \\ \|v_h\| = 1}} |(f, v_h - \varphi_{h0} v_h)|] .$$

Proof. Due to the definitions of $\bar{a}$, $\omega$ and $\omega_h$, the following identity holds

$$a(v_h, w_h) - a_h(v_h, w_h) = \bar{a}((\omega - \omega_h)(v_h - u), \omega w_h) + \\ + \bar{a}(\omega_h(v_h - u), (\omega - \omega_h) w_h) + \bar{a}((\omega - \omega_h) u, \omega w_h) + \\ + \bar{a}((\omega_h - \omega) u, (\omega - \omega_h) w_h) + \bar{a}(\omega u, (\omega - \omega_h) w_h) .$$

Since $l_i \in L(V_i, L^2)$, the form $\bar{a}$ is bounded. Thus, according to A2 – A4, there exists $c < \infty$ such that

$$|a(v_h, w_h) - a_h(v_h, w_h)| \leqq |\bar{a}(\omega u, (\omega - \omega_h) w_h)| + \\ + c\|w_h\|_V [\|u - v_h\|_V + \|\omega - \omega_h) u\|_F] .$$

Therefore, Proposition 1 is a consequence of the inequality (2.3).

Remark 2. The term $\bar{a}(\omega u, (\omega - \omega_h) w_h)$ can be easily estimated by $c\|(\omega - \omega_h) w_h\|_F$, but usually, such an estimate is not good enough to give an apropriate order of convergence, and a more detailed analysis is desired.

Let us assume that for $i = 0, \ldots, m$, $\varphi_{hi}$: $V \to V_i$ is a linear projection onto a subspace $V_{hi} \subset V_i$ with a domain $D\varphi_{hi} \supset V$ which is a subspace of $V_i$. In this case the following subspaces of $L^2$ will be useful for the error evaluation

$$W_{hi} = \{v \in L^2 : (v, l_i w) = 0 \ \forall w \in N(\varphi_{hi})\} , \tag{2.4}$$

where $N(\varphi_{hi})$ denotes the null space of $\varphi_{hi}$. Let $\varepsilon_{hi}(v)$ denote the distance of $v$ from $W_{hi}$ in $L^2$, i.e.

$$\varepsilon_{hi}(v) = \inf_{w \in W_{hi}} \|v - w\|_0 . \tag{2.5}$$

**Lemma 1.** *If* $\varphi_{hi}$: $V \to V_i$, $i = 0, \ldots, m$, *are projections then for any* $u, v \in V$

$$|\bar{a}(\omega u, (\omega - \omega_h) v)| \leqq c \sum_{j=0}^{m} \|(1 - \varphi_{hj}) v\|_{V_j} \sum_{i=0}^{m} \varepsilon_{hj}(\alpha_{ij} l_i u) .$$

Proof. Since $\varphi_{hj}$ is a projection, we have $(1 - \varphi_{hj}) u \in N(\varphi_{hj}) \ \forall u \in D\varphi_{hj}$. Thus, due to the definition of $W_{hj}$, for any $w_{hj} \in W_{hj}$ we have

$$(\alpha_{ij} l_i u, l_j(1 - \varphi_{hj}) v) = (\alpha_{ij} l_i u - w_{hj}, l_j(1 - \varphi_{hj}) v)$$

and

$$\bar{a}(\omega u, (\omega - \omega_h) v) = \sum_{j=0}^{m} \sum_{i=0}^{m} (\alpha_{ij} l_i u - w_{hj}, l_j(1 - \varphi_{hj}) v) .$$

Since $l_j \in L(V_j, L^2)$, taking the infimum of the right-hand side on $W_{h0} \times \ldots \times W_{hm}$ . . $(w_{hj} \in W_{hj}, j = 0, \ldots, m)$ we get Lemma 1.

To simplify the notation, let us introduce

$$\delta_h(v) = \inf_{v_h \in V_h} \|v - v_h\|_V, \quad \gamma_{hi} = \|(1 - \varphi_{hi})|_{V_h}\|_{L(V, V_i)} . \tag{2.6}$$

Combining Proposition 1 and Lemma 1, we obtain the main result of this section:

**Theorem 1.** *Let the assumptions* A1 – A4 *be satisfied. If* $\varphi_{h0}, \ldots, \varphi_{hm}$ *are projections, then*

$$\|u - u_h\|_V \leqq c\{\delta_h(u) + \|(\omega - \omega_h)\, u\|_F + \sum_{j=0}^{m} \gamma_{hj} \sum_{i=0}^{m} \varepsilon_{hj}(\alpha_{ij} l_i u) + \gamma_{h0}\varepsilon_{h0}(f)\} .$$

## 3. POINTWISE ERROR ESTIMATES

This section concerns the approximation (2.2) for ordinary differential problems only.

Let $k_0 = 0 < k_1 < \ldots < k_m$ ($k_i$ – integers). Let us consider the problem (2.1) for $V = H_0^{k_m}(I)$ $(I = (0, 1))$ and for the form

$$a(u, v) = (u^{(k_m)}, v^{(k_m)}) + \sum_{i=0}^{m-1} \sum_{j=0}^{m} (\alpha_{ij} u^{(k_i)}, v^{(k_j)}) . \tag{3.1}$$

Let $V_j = H_0^{k_j}(I)$ and $l_j = \mathrm{d}^{k_j}/dx^{k_j}$ for $j = 0, \ldots, m$ . Since $V_{k_m} = V$, we can assume that

$$\varphi_{hk_m} = I_V \quad (\text{identity on } V) .$$

Thus, the approximate problem (2.2) takes the form

find $u_h \in V_h$ such that

$$(u_h^{(k_m)}, v_h^{(k_m)}) + \sum_{i=0}^{m-1} \sum_{j=0}^{m} (\alpha_{ij}(\varphi_{hi}u_h)^{(k_i)}, \; (\varphi_{hj}v_h)^{(k_j)}) = (f, \varphi_{h0}v_h) \quad \forall v_h \in V_h . \tag{3.2}$$

We will assume that the matrix of coefficients $\mathscr{A}(t) := (\alpha_{ij}(t))_{ij=0}^{m}$ is such that $\mathscr{A}(t) + \mathscr{A}^T(t)$ is uniformly positive definite almost everywhere. Since this assumption implies F-ellipticity of the form $\bar{a}$, the forms $a_h$ are uniformly $V_h$-elliptic (A4) according to Remark 1. Adding the assumptions on the uniform boundedness of $\varphi_{hi}$, we get A1 – A4. So, the method (3.2) is convergent and the error bound is given by Proposition 1 or by Theorem 1, provided all $\varphi_{hi}$ are projections.

Let us observe that the problems (2.1) and (2.2) can be formulated as follows:

find $\quad u \in V$: $\bar{a}(\omega u, \omega v) = (f, v) \quad \forall v \in V$,

find $\quad u_h \in V_h$: $\bar{a}(\omega_h u_h, \omega_h v_h) = (f, \varphi_{h0} v_h) \quad \forall v_h \in V_h$ .

Putting $v = v_h$ in the first equation and subtracting from it the second, we get the relation

$$((u - u_h)^{(k_m)}, v_h^{(k_m)}) + \bar{b}(\omega u - \omega_h u_h, \omega_h v_h) = \tag{3.3}$$

$$= (f, v_h - \varphi_{h0} v_h) - \bar{a}(\omega u, (\omega - \omega_h) v_h) \quad \forall v_h \in V_h ,$$

where

$$\bar{b}(\bar{u}, \bar{v}) := \bar{a}(\bar{u}, \bar{v}) - (u_m^{(k_m)}, v_m^{(k_m)})$$

for $\bar{u} = (u_0, \ldots, u_m), \bar{v} = (v_0, \ldots, v_m) \in F$.

**Lemma 2.** *If $G(t, x)$ is the Green function of the operator $(-1)^{k_m} y^{(2k_m)}$ on $H_0^{k_m}$ then*

$$u(t) - u_h(t) = ((u - u_h)^{(k_m)}, G(t, \cdot) - v_h)^{(k_m)}) +$$

$$+ (f, (1 - \varphi_{h0}) v_h) - \bar{a}(\omega u, (\omega - \omega_h) v_h) - \bar{b}(\omega u - \omega_h u_h, \omega_h v_h)$$

$$\forall t \in (0, 1) \quad \forall v_h \in V_h .$$

Proof. Due to the properties of the Green function,

$$(y^{(k_m)}, G_x^{(k_m)}(t, \cdot)) = (-1)^{k_m} \int_0^1 y^{(2k_m)}(x) G(t, x) \, dx = y(t) \quad \forall y \in V .$$

Thus, for any $v_h \in V_h$,

$$u(t) - u_h(t) = ((u - u_h)^{(k_m)}, (G(t, \cdot) - v_h)^{(k_m)}) + ((u - u_h)^{(k_m)}, v_h^{(k_m)}) .$$

Applying now the relation (3.3), we get Lemma 2.

**Lemma 3.** *If $G(t, x)$ is the Green function of the operator $(-1)^{k_m} y^{(2k_m)}$ on $H_0^{k_m}$ and if $\psi_t(x)$ is the solution of the variational equation*

$$\bar{a}(\omega v, \omega \psi_t) = \bar{b}(\omega v, \omega \, G(t, \cdot)) \quad \forall v \in H_0^{k_m} , \tag{3.4}$$

*then for any $y_h \in V_h$,*

$$\bar{b}(\omega u - \omega_h u_h, \omega_h \, G(t, \cdot)) = \bar{a}(\omega u - \omega_h u_h, \omega \psi_t - \omega_h y_h) +$$

$$+ \bar{a}(\omega u - \omega u_h, (\omega_h - \omega) \, G(t, \cdot)) + \bar{a}((\omega - \omega_h) \, u_h, \omega(\psi_t + G(t, \cdot))) +$$

$$+ (f, y_h - \varphi_{h0} y_h) - \bar{a}(\omega u, (\omega - \omega_h) \, y_h) .$$

Proof. Since the form $\bar{a}$ is $F$-elliptic and $\bar{a}$, $\bar{b}$ are bounded, by the Lax-Milgram theorem there exists a unique solution $\psi_t$ of (3.4). For $v = u - u_h$, (3.4) implies

$$\bar{b}(\omega(u - u_h), \omega \, G(t, \cdot)) = \bar{a}(\omega(u - u_h), \omega \psi_t) .$$

Therefore, due to (3.3), for any $y_h \in V_h$ we have

$$\bar{b}(\omega(u - u_h), \omega \, G(t, \cdot)) = \bar{a}(\omega u - \omega_h u_h, \omega \psi_t - \omega_h y_h) + \bar{a}((\omega - \omega_h) \, u_h, \omega \psi_t) +$$

$$+ (f, y_h - \varphi_{h0} y_h) - \bar{a}(\omega u, (\omega - \omega_h) \, y_h) .$$

Combining the above and the identity

$$\bar{b}(\omega u - \omega_h u_h, \omega_h G(t, \cdot)) = \bar{a}(\omega u - \omega_h u_h, (\omega_h - \omega) G(t, \cdot)) +$$
$$+ \bar{a}((\omega - \omega_h) u_h, \omega G(t, \cdot)) + \bar{b}(\omega(u - u_h), \omega G(t, \cdot))$$

we get Lemma 3.

We are now in position to estimate the error at an arbitrary point $t \in (0, 1)$.

**Theorem 2.** *Let $u$ and $u_h$ be solutions of the problem (2.1) with the form (3.1) and the problem (3.2), respectively. If $\varphi_{h0}, \dots, \varphi_{hm-1}$ are projections then*

$$|u(t) - u_h(t)| \leqq c\{C_{1t}(h)\, \delta_h(u) + C_{2t}(h)\, \|(\omega - \omega_h)\, u\|_F + \tag{3.5}$$
$$+ \sum_{j=0}^{m-1} C_{3t}^j(h) \sum_{i=0}^{m} \varepsilon_{hj}(\alpha_{ij} u^{(k_i)}) + C_{3t}^0(h)\, \varepsilon_{h0}(f)\}\,,$$

*where $\delta_h$, $\varepsilon_h$ are given by (2.5) and (2.6) and*

$$C_{1t}(h) = \|(\omega - \omega_h)\, G_t\|_F + \|(\omega - \omega_h)\, \psi_t\|_F + \delta_h(\psi_t) + \delta_h(G_t) +$$
$$+ \sum_{i=0}^{m-1} \gamma_{hi} \sum_{j=0}^{m} \varepsilon_{hi}(\alpha_{ij}(\varphi_t + G_t)^{(k_j)})\,,$$

$$C_{2t}(h) = C_{1t}(h) + \sum_{i=0}^{m-1} \sum_{j=0}^{m} \varepsilon_{hi}(\alpha_{ij}(\varphi_t + G_t)^{(k_j)})\,,$$

$$C_{3t}^j(h) = \gamma_{hj}\, C_{1t}(h) + \|(\omega - \omega_h)\, \psi_t\|_F + \|(\omega - \omega_h)\, G_t\|_F\,, \quad j = 0, \dots, m - 1\,,$$

*for $\psi_t$ and $G_t (= G(t, \cdot))$ defined in Lemma 3 and $\gamma_{hi}$ given by (2.6). If the coefficients $\alpha_{ij}$ are sufficiently regular, namely*

$$\alpha_{ij} \in H_\Delta^{s+1}\,, \quad s \in \{0, 1, 2, \dots\}\,,$$

*then $C_{it}^j$ can be estimated as follows:*

$$C_{1t}(h) \leqq c\{\sup_{v:v^{(k_m)} \in B_t} [\delta_h(v) + \|(\omega - \omega_h)\, v\|_F] + \sum_{i=0}^{m-1} \gamma_{hi} \sup_{v \in B_t} \varepsilon_{hi}(v)\}\,,$$

$$C_{2t}(h) \leqq c\{\sup_{v:v^{(k_m)} \in B_t} [\delta_h(v) + \|(\omega - \omega_h)\, v\|_F] + \sum_{i=0}^{m-1} + \sup_{v \in B_t} \varepsilon_{hi}(v)\}\,, \tag{3.6}$$

$$C_{3t}^j(h) \leqq c\{\sup_{v:v^{(k_m)} \in B_t} [\gamma_{hj}\, \delta_h(v) + \|(\omega - \omega_h)\, v\|_F] + \sum_{i=0}^{m-1} \gamma_{hj} \gamma_{hi} \sup_{v \in B_t} \varepsilon_{hi}(v)\}\,,$$

*where $B_t$ denotes the unit ball in the space $H_\Delta^{s+1} \cap H_\Delta^{k_m - 1}$ for $t \notin \Delta(h)$ and in the space $H_\Delta^{s+1}$ for $t \in \Delta(h)$,.*

Proof. Due to Lemma 1 and the definition of $\gamma_{hj}$ we have

$$|\bar{a}(\omega u, (\omega - \omega_h)\, v_h)| \leqq \tag{3.7}$$
$$\leqq c \sum_{j=0}^{m-1} [\|(\omega - \omega_h)\, G_t\|_F + \|G_t - v_h\|_V\, \gamma_{hj}] \cdot \sum_{i=0}^{m-1} \varepsilon_{hj}(\alpha_{ij} u^{(k_i)})\,.$$

Similarly,

$$(3.8)\qquad |(f, (1 - \varphi_{h0})\, v_h)| \leqq [\|(\omega - \omega_h)\, G_t\|_F + \gamma_{h0}\|G_t - v_h\|_F]\, \varepsilon_{h0}(f)\,.$$

Moreover,

$$|((u - u)^{(k_m)}, (G_t - v_h)^{(k_m)})| + |\bar{b}(\omega u - \omega_h u_h, \omega_h(G_t - v_h))| \leqq$$
$$\leqq c\|G_t - v_h\|_V [\|u - u_h\|_V + \|(\omega - \omega_h)\, u\|_F]\,.$$

Thus, Lemma 2 and Theorem 1 yield

$$(3.9)\qquad |u(t) - u_h(t)| \leqq c\{\delta_h(u)\, \|G_t - v_h\|_V + \|(\omega - \omega_h)\, u\|_F\, \|G_t - v_h\|_V +$$
$$+ \sum_{j=0}^{m} [\|(\omega - \omega_h)\, G_t\|_F + \|G_t - v_h\|_V\, \gamma_{hj}] \sum_{i=0}^{m-1} \varepsilon_{hj}(\alpha_{ij} u^{(k_i)})] +$$
$$+ [\|(\omega - \omega_h)\, G_t\|_F + \gamma_{h0}\|\, G_t - v_h\|_F]\, \varepsilon_{h0}(f)\} + |\bar{b}(\omega u - \omega_h u_h, \omega_h G_t)|\,.$$

To the last term we apply Lemma 3. We have

$$|\bar{a}(\omega u - \omega_h u_h, \omega\psi_t - \omega_h y_h + (\omega_h - \omega)\, G_t)| \leqq$$
$$\leqq c[\|(\omega - \omega_h)\, \psi_t\|_F + \|\psi_t - y_h\|_V + \|(\omega - \omega_h)\, G_t\|_F]\, \|\omega u - \omega_h u_h\|_F\,.$$

Moreover, by the definition of the spaces $W_{hj}$, for any $w_{hj} \in W_{hj}$ and arbitrary $y, v \in V$ we have

$$|\bar{a}((\omega - \omega_h)\, y, \omega v)| = |\sum_{i=0}^{m-1} \sum_{j=0}^{m} ((y - \varphi_{hi} y)^{(k_i)}, \alpha_{ij} v^{(k_j)} - w_{hi})|$$

and thus

$$|\bar{a}((\omega - \omega_h)\, u_h, \omega)\psi_t + G_t))| \leqq c\{\|(\omega - \omega_h)\, u\|_F \sum_{i=0}^{m-1} \sum_{j=0}^{m} \varepsilon_{hi}(\alpha_{ij}(\psi_t + G_t)^{(k_j)}) +$$
$$+ \|u - u_h\|_V \sum_{i=0}^{m-1} \gamma_{hi} \sum_{j=0}^{m} \varepsilon_{hi}(\alpha_{ij}(\psi_t + G_t)^{(k_j)}\}\,.$$

From Theorem 1, the above estimates and the inequalities (3.7), (3.8) with $v_h$ and $G_t$ replaced by $y_h$ and $\psi_t$, we get (cf. Lemma 3)

$$(3.10)\qquad |\bar{b}(\omega u - \omega_h u_h, \omega_h G_t)| \leqq c\{C_{1t}(h)\, \delta_h(u) + C_{2t}(h)\|\, (\omega - \omega_h)\, u\|_F +$$
$$+ \sum_{i=0}^{m-1} C_{3t}^{j}(h) \sum_{j=0}^{m} \varepsilon_{hj}(\alpha_{ij} u^{(k_i)}) + C_{3t}^{0}(h)\, \varepsilon_{h0}(f)\}\,.$$

So, (3.9) and (3.10) imply the first part of Theorem 2. The estimates (3.6) follow from the regularity of $\psi_t$ and $G_t$. Indeed,

$$G_t \in \begin{cases} H^{2k_m - 1} & \text{for} \quad t \notin \Delta \\ H_\Delta^r & \text{for } r \text{ arbitrary and } t \in \Delta \end{cases}$$

and since $\psi_t$ is the solution of (3.4),

$$\psi_t \in \begin{cases} H_\Delta^{2k_m} \cap H_\Delta^{k_m+s+1} & \text{for} \quad t \notin \Delta \\ H_\Delta^{k_m+s+1} & \text{for} \quad t \in \Delta . \end{cases}$$

Moreover, for $i = 0, \ldots, m - 1$, $j = 0, \ldots, m$, we have

$$\alpha_{ij}(\psi_t + G)^{(k_j)} \in \begin{cases} H_\Delta^{s+1} \cap H_\Delta^{k_m-1} & \text{for} \quad t \notin \Delta \\ H_\Delta^{s+1} & \text{for} \quad t \in \Delta . \end{cases}$$

Remark 3. Theorem 2 indicates the possibility of superconvergence at the knot points $x_i$ in the case when $s \geqq k_m - 1$ and $V_h$ as well as the ranges of $\varphi_{h0}, \ldots, \varphi_{hm-1}$ are the spline subspaces corresponding to the partitions $\Delta(h)$.

Repeating the argumentation of the proof of Theorem 2 we can also obtain an $L^2$ error estimate.

**Theorem 3.** *If the assumptions of Theorem 2 are satisfied and* $\alpha_{ij} \in H_\Delta^{s+1}$, $\Delta' \subset \Delta(h)$, *then*

$$\|u - u_h\|_0 \leqq \{\tilde{C}_1(h)\, \delta_h(u) + \tilde{C}_2(h)\, \|(\omega - \omega_h)\, u\|_F +$$
$$+ \sum_{j=0}^{m-1} \tilde{C}_3^j(h) \sum_{i=0}^{m} \varepsilon_{hj}(\alpha_{ij} u^{(k_i)}) + C_3^0(h)\, \varepsilon_{h0}(f)\}$$

*and the coefficients* $\tilde{C}_i(h)$ *are estimated by* (3.6) *with* $B_t$ *replaced by the until ball* B *in the space* $H_\Delta^{s+1} \cap H_\Delta^{k_m}$.

Proof. The argument of the proof of the Aubin-Nitsche lemma yields

$$\|u - u_h\|_0 = \sup_{g \in L^2} \frac{\bar{a}(\omega(u - u_h), \omega\varphi_g)}{\|g\|_0},$$

where $\varphi_g \in V$ is the solution of the equation

$$\bar{a}(\omega v, \omega \varphi_g) = (v, g) \quad \forall v \in V .$$

Since

$$\bar{a}(\omega(u - u_h), \omega\varphi_g) =$$
$$= ((u - u_h)^{(k_m)}, \varphi_g^{(k_m)}) + \bar{b}(\omega(u - u_h), \omega\varphi_g - \omega_h u_h) + \bar{b}((\omega_h - \omega)\, u_h, \omega_h u_h) ,$$

an argumentation similar to that used in the proof of Theorem 2 yields the expected estimate. The ball $B_t$ is now replaced by $B$ due to the fact that instead of $G_t$ and $\psi_t$, only the function $\varphi_g \in H_\Delta^{2k_m}$ occurs.

## 4. SUPERCONVERGENCE

Let us consider the problem (2.1) for the form given by (3.1) and its approximation (3.4) generated by $V_h$ and the projections $\varphi_{h0}, \ldots, \varphi_{hm-1}, I$.

For application of the abstract results established in the previous section, some information about the approximation properties of the subspaces $W_{hj}$ is necessary.

For some special choices of the projections $\varphi_{hj}$, a characterization of $W_{hj}$ can be obtained.

Remark 4. If there are $\mu_{i\nu} \in L^2(I)$, $\nu = 0, \ldots, \nu_i$, such that

$$N(\varphi_{hi}) \subset \{u \in D\varphi_{hi} \mid (u, \mu_{i\nu}) = 0, \ \nu = 0, \ldots, \nu_i\} \tag{4.1}$$

then

$$W_{hi} \supset \{v \in L^2 \mid v^{(k_i)} \in \operatorname{span} \{\mu_{i\nu}\}_{\nu=0}^{\nu_i}\}.$$

Indeed, if $v^{(k_i)} \in \operatorname{span} \{\mu_{i\nu}\}_{\nu=0}^{\nu_i}$ then $(w, v^{(k_i)}) = 0$ for $w \in N(\varphi_{hi})$, and hence $v \in W_{hi}$. Let $\Delta(h) = \{x_i\}_{i=0}^{n}$ $(x_i = ih, h = 1/n)$ be a uniform partition of $I$, $I_i = (x_i, x_{i+1})$ and let $S_h(H^k, r)$, $k \leqq r$, denote a piecewise polynomial space

$$S_h(H^k, r) = \{v \in H^k \mid v|_{I_i} \in P_r(I_i) \ i = 0, \ldots, n-1\}$$

Remark 5. If $\mu_{i\nu}$ defined in (4.1) are such that

$$\operatorname{span} \{\mu_{i\nu}\}_{\nu=0}^{\nu_i} = S_h(H^p, r), \quad \text{then} \quad W_{hi} \supset S_h(H^{p+k_i}, r + k_i).$$

Let us now suppose that instead of (4.1) we have

$$N(\varphi_{hi}) \subset \{v \in V_i \mid v(x_\nu) = 0, \nu = 1, \ldots, n-1\}, \quad i \geqq 1. \tag{4.2}$$

Thus the Green formula implies

$$N(\varphi_{hi}) \subset \{v \in V_i \mid (\mu, v') = 0 \quad \forall \mu \in S_h(L^2, 0)\},$$

and for $w$ such that $w^{(k_i - 1)} \in S_h(L^2, 0)$ we have

$$(w, v^{(k_i)}) = 0 \quad \forall v \in N(\varphi_{hi}).$$

Remark 6. Let $i \geqq 1$. If (4.2) holds then $S_h(H^{k_i - 1}, k_i - 1) \subset W_{hi}$.

Finally, let us consider the case when $\varphi_{hi}$ is a projection onto $V_{hi}$ generated by the form $(u^{(k_i)}, v^{(k_i)})$, namely

$$((\varphi_{hi}u - u)^{(k_i)}, v^{(k_i)}) = 0 \quad \forall v \in V_{hi}. \tag{4.3}$$

It is easy to see that for $\varphi_{hi}$ defined above we have

$$W_{hi} = \{w \in L^2 \colon w = v^{(k_i)}, v \in V_{hi}\}, \tag{4.4}$$

and moreover,

$$\|(1 - \varphi_{hi}) v\|_{V_i} \leqq c|(1 - \varphi_{hi}) v|_{k_i} = c \inf_{v_h \in V_{hi}} |v - v_h|_{k_i}, \tag{4.5}$$

for $v \in H_0^{k_i}$ where $|\cdot|_{k_i}$ denotes the standard seminorm in $H^{k_i}$. For applications it is enough to find the image $\varphi_{hi}v$ of a basis of $V_h$.

Let us consider the approximation (3.2) given by (4.3), and the piecewise polynomial spaces

$$V_h = S_h(H_0^{k_m}, k_m + s), \quad V_{hi} = S_h(H_0^{k_i}, k_i + s) \tag{4.6}$$

corresponding to the uniform partition $\Delta(h)$ such that $\Delta(h) \supset \Delta'$. In this case we have the following result:

**Theorem 4.** *Let* (4.3) *and* (4.6) *hold and let $u$ and $u_h$ be the exact and approximate solution, respectively. If $\alpha_{ij} \in H_{\Delta'}^{s+1}$ and $f \in H_{\Delta'}^{s+1}$, $s \in \{0, 1, \ldots\}$, then*

$$\|u - u_h\|_V \leqq ch^{s+1},$$

$$\|u - u_h\|_0 \leqq \begin{cases} ch^{s+1+k_m} & \text{for } s > k_m - 1 \\ ch^{2(s+1)} & \text{for } s \leqq k_m - 1 . \end{cases}$$

*Moreover,*

$$|u(t) - u_h(t)| \leqq ch^{2(s+1)} \quad \textit{for} \quad t \in \Delta(h)$$

*which means that for $s > k_m - 1$ the method possesses the superconvergence property at the knot points $x_i \in \Delta(h)$.*

Proof. Let $J_\Delta^s v$ denote the spline interpolant of $v$ from $S_h(L^2, s)$ generated by the knots $x_{ij} = x_i + jh/(s+1)$ $j = 0, \ldots, s$. From Peano's kernel theorem it follows that (cf. Kowalewski's exact remainder for polynomial interpolation, [6]) for $v \in H^{k+l}(I_i)$

$$\|v^{(k)} - J_\Delta^s v^{(k)}\|_{L^2(I_i)} \leqq c(h^{s+1} + h^l) \|v^{(k)}\|_{H^l(I_i)} .$$

Since for any $v \in H_0^{k_i}$ we have

$$\inf_{v_h \in V_{hi}} \|v - v_h\|_{V_i}^2 \leqq c \inf_{v \in V_h} \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} |v^{(k_i)}(x) - v_h^{(k_i)}(x)|^2 \, dx \leqq$$

$$\leqq c \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} |v^{(k_i)}(x) - J_\Delta^s v^{(k_i)}(x)|^2 \, dx ,$$

then for any $v \in H_\Delta^{k_i} \cap H_\Delta^{k_i+l}$

$$\inf_{v_h \in V_{hi}} \|v - v_h\|_{V_i} \leqq c(h^{s+1} + h^l) \| v^{(k_i)}\|_{\Delta,l} . \tag{4.7}$$

This, by (4.5), implies that for any $v \in V$,

$$\begin{cases} \|(\omega - \omega_h) v\|_F \leqq c(h^{s+1} + h^l) \|v^{(k_m - 1)}\|_{\Delta,l} , \\ \delta_h(v) \leqq c(h^{s+1} + h^l) \|v^{(k_m)}\|_{\Delta,l} , \\ \gamma_{hj} \leqq c(h^{s+1} + h^{k_m - k_j}) . \end{cases} \tag{4.8}$$

Moreover, due to (4.4) we have $W_{hi} = S_h(L^2, s)$ and

$$\varepsilon_{hi}(v) \leqq c(h^{s+1} + h^l) \|v\|_{\Delta,l} . \tag{4.9}$$

By the assumption of regularity of $f$ and $\alpha_{ij}$, the solution belongs to $H_0^{k_m} \cap H_\Delta^{k_m+s+1}$. So, Theorems 1 and 3 and the estimates (4.8) and (4.9) imply the first part of Theorem 4. Moreover, according to (3.6), $C_{it}(h)$ can be estimated by $c(h^{s+1} + h^{k_m - 1})$ if $t \notin \Delta$ and by $ch^{s+1}$ if $t \in \Delta$. Thus, applying Theorem 2 we get the remaining part of Theorem 4.

Remark 7. Theorem 4 remains true if the uniform partition $\Delta(h)$ is replaced by a quasi-uniform one, i.e.

$$\max_{ij} \frac{x_{i+1} - x_i}{x_{j+1} - x_j} \leqq c .$$

## 5. REMARKS

Superconvergence at the knot points can also take place when non-orthogonal projections $\varphi_{h0}$ are applied. Due to the fact than then $W_{h0}$ may be a worse approximation of $H^1$ than $V_{h0}$, it is clear that the existence of the terms $\gamma_{hj}$ in the estimates (3.6) becomes important. As an example let us take $\varphi_h$: $H^1_\Delta(I) \to S_h(L^2, 1)$ given as follows:

$$\varphi_h u(x) = \sum_{i=0}^{n-1} \left[ \int_{ih}^{(i+1)h} \frac{1}{h} \{u'(t)(x - (i + \tfrac{1}{2}) h) + u(t)\} \, \mathrm{d}t \right] \chi\left(\frac{x}{h} - i\right),$$

where $\chi(t)$ is the characteristic function of $(0, 1)$. According to Remark 5, $W_{h0} \supset \supset S_h(L^2, 0)$. Thus

$$C_{2t}(h) \leqq \begin{cases} c & \text{for} \quad t \notin \Delta(h) \\ ch & \text{for} \quad t \in \Delta(h) ; \end{cases} \quad C_{1t}(h), C_{3t}(h) \leqq \begin{cases} ch & \text{for} \quad t \notin \Delta(h) \\ ch^2 & \text{for} \quad t \in \Delta(h) , \end{cases}$$

and by Theorems 1 and 2 $\|u - u_h\|_1 \leqq ch$ while $|u(t) - u_h(t)| \leqq ch^3$ for $t \in \Delta(h)$, where $u$ and $u_h$ are solutions of (1.1) and (1.5), respectively, with $V_h = S_h(H^1_0, 2)$ and $\varphi_h$ given above.

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Souhrn

## ODHAD CHYBY PRO VNĚJŠÍ APROXIMACI OBYČEJNÝCH DIFERENCIÁLNÍCH ROVNIC A VLASTNOST SUPERKONVERGENCE

TEREZA REGIŃSKA

Autorka odvozuje bodový odhad chyby a odhad v normě pro jistou třídu vnějších metod aproximujících okrajové úlohy a studuje závislost vnější aproximační metody na jevu superkonvergence. V tomto obecném rámci je dokázána superkonvergence v uzlových bodech pro bodové polynomiální externí metody.

Резюме

## ОЦЕНКА ОШИБКИ ДЛЯ ВНЕШНЕЙ АППРОКСИМАЦИИ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ И СВОЙСТВО СУПЕРСХОДИМОСТИ

TEREZA REGIŃSKA

Автором выведены точечная оценка ошибки и оценка в норме для некоторого класса внешных методов аппроксимирующих краевые задачи и изучена зависимость феномена суперсходимости от метода внешней аппроксимации. В этих общих рамках доказана суперсходимость в узловых точках для точечных полиномальних внешных методов.

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