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ONE-STEP METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS WITH PARAMETERS

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Summary. In the present paper we are concerned with the problem of numerical solution of ordinary differential equations with parameters. Our method is based on a one-step procedure for ODEs combined with an iterative process. Simple sufficient conditions for the convergence of this method are obtained. Estimations of errors and some numerical examples are given.

Keywords: Ordinary differential equations with parameters, numerical solution, one-step method.

1. INTRODUCTION

Let $C(I, R^q)$ denote the set of all continuous functions on I into R^q, $I = [\alpha, \beta]$, $\alpha < \beta$. Here R^q denotes some q-dimensional real linear space of elements $x = (x_1, x_2, \ldots, x_q)^T$ with a norm $\|\cdot\|$.

We consider the differential equation of the form

$$(1) \qquad y'(t) = f(t, y(t), \lambda), \quad t \in I,$$

with boundary conditions

$$(2) \qquad y(\alpha) = y_0,$$

$$(3) \qquad \tilde{M}\lambda + \tilde{N}y(\beta) = \tilde{K},$$

where $\lambda \in R^q$ is a parameter and $f: I \times R^q \times R^q \to R^q$. The condition (3) is linear in λ and $y(\beta)$. The square matrices $\tilde{M}$ and $\tilde{N}$ of order q are given and such that $\tilde{M} + \tilde{N}$ is nonsingular. The vector $\tilde{K} \in R^q$ is given too.

By a solution (φ, λ) of BVP $(1-3)$ we mean a function $\varphi \in C(I, R^q)$ and a parameter $\lambda \in R^q$ that satisfy BVP $(1-3)$. This boundary value problem $(1-3)$ is know also as an eigenvalue problem for ODEs or as a problem of terminal control (see [12]). Many special cases can be reduced to BVP $(1-3)$ (see [6]).

Existence and uniqueness theorems for $(1-3)$ were established in many papers (for example, see [6, 8, 10, 11]). We assume that BVP $(1-3)$ has a solution $(\varphi, \lambda) \in C(I, R^q) \times R^q$. Our task is to approximate it by a numerical procedure.

Let N be a positive integer and $h = (\beta - \alpha)/N$, $R_N = \{0, 1, \ldots, N\}$. In practical calculations the solution (φ, λ) is approximated by (y_h, λ_{hj}) where the approximate solution y_h corresponding to a step size h is defined on the set $\{t_{hi} = \alpha + ih: i \in R_N\}$. We use a onestep method for y_h with an iterative method for λ_{hj} for finding the numerical solution (y_h, λ_{hj}) of BVP $(1-3)$.

They are defined by

$$(4) \qquad \begin{cases} \lambda_{h0} = \lambda_0 , \\ \lambda_{h,j+1} = \lambda_{hj} - (\tilde{M} + \tilde{N})^{-1} [\tilde{M}\lambda_{hj} + \tilde{N} y_h(\beta; \lambda_{hj}) - \tilde{K}] , \quad j = 0, 1, \ldots \end{cases}$$

and

$$(5) \qquad \begin{cases} y_h(t_{h0}; \lambda_{hj}) = y_0 , \\ y_h(t_{h,n+1}; \lambda_{hj}) = y_h(t_{hn}; \lambda_{hj}) + h\Phi(t_{hn}, y_h(t_{hn}; \lambda_{hj}), \lambda_{hj}, h) , \\ n \in R_{N-1} , \quad j = 0, 1, \ldots . \end{cases}$$

Applying the above procedure for finding (y_h, λ_{hj}) we have to determine an initial value for λ_0. Having it we find y_h using (5) and then a new value for λ using (4) and so on.

The purpose of this paper is to give sufficient conditions for the convergence of the method $(4-5)$. This convergence is proved under the assumption that f satisfies the Lipschitz condition with suitable constants with respect to the last two variables. The above mentioned result may be weakened for a special kind of BVPs for which the convergence and the estimates of errors are established (see Theorem 2). Numerical examples are presented.

2. DEFINITION AND ASSUMPTIONS

We introduce the following definition.

Definition 1. *We say that the method* $(4-5)$ *is convergent to the solution* (φ, λ) *of* $BPV(1-3)$ *if*

$$\lim_{\substack{N\to\infty \\ j\to\infty}} \max_{i\in R_N} \|\varphi(t_{hi}; \lambda) - y_h(t_{hi}; \lambda_{hj})\| = 0 ,$$

$$\lim_{\substack{N\to\infty \\ j\to\infty}} \|\lambda_{hj} - \lambda\| = 0 .$$

Remark 1. Note that (4) is identical with

$$\lambda_{h,j+1} = (\tilde{M} + \tilde{N})^{-1} \{\tilde{K} - \tilde{N}[y_h(\beta; \lambda_{hj}) - \lambda_{hj}]\} ,$$

or

$$\lambda_{h,j+1} = -y_h(\beta; \lambda_{hj}) + (\tilde{M} + \tilde{N})^{-1} \{\tilde{K} + \tilde{N}\lambda_{hj} + \tilde{M} y_h(\beta; \lambda_{hj})\} .$$

This remark may be useful for such cases when the formula (4) is too costly in numerical calculations. Perhaps its other presentation would be less expensive.

Remark 2. There are two integers j and N in the definition $(4-5)$. Both j and

N tend to infinity. The independent index j may be eliminated and expressed by N, namely

$$\begin{cases} \lambda_N = \lambda_0\,, \quad \text{a fixed initial vector,} \\ \lambda_{j+1} = \lambda_j - (\tilde{M} + \tilde{N})^{-1}\,[\tilde{M}\lambda_j + \tilde{N}y_h(\beta; \lambda_j) - \tilde{K}]\,, \\ j = N, N+1, \dots. \end{cases}$$

As a matter of course, taking now $N + 1$ instead of N we have a new λ and for the new stepsize $h = (\beta - \alpha)/(N + 1)$ we are able to find the approximation y_h of our solution. Now only N approaches ∞.

Assumption $\mathbf{H_1}$. *Let*

1° $f: I \times R^q \times R^q \to R^q$, $\Phi: I \times R^q \times R^q \times H \to R^q$, $H = [0, h_0]$, $h_0 > 0$, *and* f *is continuous with respect to the first two variables uniformly with respect to the last variable,*

2° *there exist constants* $L_1, L_2 > 0$ *such that for* $t \in I$, $(x, \bar{x}, \mu, \bar{\mu}) \in R^{4q}$ *we have*

$$\|f(t, x, \mu) - f(t, \bar{x}, \bar{\mu})\| \leqq L_1\|x - \bar{x}\| + L_2\|\mu - \bar{\mu}\|\,,$$

3° $\tilde{M}$ *and* $\tilde{N}$ *are square matrices of order* q, $\tilde{M} + \tilde{N}$ *is nonsingular, and there is a constant* $m \in (0, 1)$ *such that*

$$\|(\tilde{M} + \tilde{N})^{-1}\,\tilde{N}\| \leqq m\,,$$

where the matrix norm is consistent with the vector norm,

4° *there exists a function* $\eta: H \to R_+ = [0, \infty)$, $\lim_{h \to 0} \eta(h) = 0$, *such that for every* $(t, y, \mu, h) \in I \times R^q \times R^q \times H$ *we have*

$$\|\Phi(t, y, \mu, h) - f(t, y, \mu)\| \leqq \eta(h)\,.$$

Remark 3. It is known that the matrix norm is consistent with the vector norm if

$$\|Cx\| \leqq \|C\|\,\|x\|\,,$$

where C is a square matrix of order q and $x \in R^q$.

As the matrix norm we can take the corresponding subordinate matrix norm defined by

$$\operatorname{lub}(C) = \max_{x \neq \theta} \frac{\|Cx\|}{\|x\|}\,, \quad \text{where } \theta \text{ is the zero vector in } R^q\,.$$

It is consistent with the vector norm used to define it:

$$\|Cx\| \leqq \operatorname{lub}(C)\,\|x\|\,.$$

Obviously $\operatorname{lub}(C)$ is the smallest of all the matrix norms $\|C\|$ which are consistent with the vector norm $\|x\|$, therefore

$$\operatorname{lub}(C) \leqq \|C\|\,.$$

Examples of subordinate matrix norms:

1) for the maximum norm

$$\|x\|_\infty = \max_{i \in R_q} |x_i|\,,$$

the subordinate matrix norm is the row-sum norm

$$\text{lub}_\infty(C) = \max_{i\in R_q} \sum_{j=1}^{q} |c_{ij}| , \quad C = [c_{ij}] ,$$

2) associated with the Euclidean norm

$$\|x\|_2 = \sqrt{(\sum_{i=1}^{q} |x_i|^2)}$$

we have the subordinate matrix norm

$$\text{lub}_2(C) = \sqrt{(\lambda_{\max}(C^T C))}$$

which is expressed in terms of the largest eigenvalue $\lambda_{\max}$ of the matrix $C^T C$.

Remark 4. If the condition 4° is satisfied only for the solution (φ, λ) of (1–3) then it is necessary to add that the function Φ satisfies the Lipschitz condition with respect to the second and third variables.

3. CONVERGENCE OF THE METHOD (4–5)

We are now in a position to establish the main convergence theorem and the associated error estimates. We have

Theorem 1. *If Assumption* H_1 *is satisfied and if*

1° *there exists a solution* (φ, λ) *of BVP* (1–3),

2° $d = m(1 + A) < 1$ *where*

$$A = \frac{L_2}{L_1} [\exp((\beta - \alpha) L_1) - 1] ,$$

then the method (4–5) *is convergent to the solution* (φ, λ) *of BVP* (1–3), *and the estimates*

$$\|\lambda_{hj} - \lambda\| \leqq u_j(h) , \quad j = 0, 1, \ldots, \tag{6}$$

and

$$\max_{n\in R_N} \|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda)\| \leqq A\, u_j(h) + B\, \chi(h) , \quad j = 0, 1, \ldots, \tag{7}$$

hold true. Here χ *approaches* 0 *as* h *approaches* 0, $B = A/L_2$ *and*

$$u_j(h) = d^j \|\lambda_0 - \lambda\| + mB\, \chi(h) \frac{1 - d^j}{1 - d} .$$

Proof. Put

$$v_{hn}^j = \|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda)\| .$$

By the mean value theorem we have

$$\varphi(t_{h,n+1}) = \varphi(t_{hn}) + hf(t_{hn} + \tau h, \varphi(t_{hn} + \tau h); \lambda) ,$$

where $0 < \tau < 1$. Now by subtracting the values $\varphi(t_{h,n+1};\lambda)$ from (5) we are able to get the relation

$$\begin{aligned} v^j_{h,n+1} &\leqq \|y_h(t_{hn};\lambda_{hj}) - \varphi(t_{hn};\lambda)\| + \\ &+ h\|\Phi(t_{hn}, y_h(t_{hn};\lambda_{hj}), \lambda_{hj}, h) - f(t_{hn}, y_h(t_{hn};\lambda_{hj}), \lambda_{hj})\| + \\ &+ h\|f(t_{hn}, y_h(t_{hn};\lambda_{hj}), \lambda_{hj}) - f(t_{hn}, \varphi(t_{hn};\lambda), \lambda)\| + \\ &+ h\|f(t_{hn}, \varphi(t_{hn};\lambda), \lambda) - f(t_{hn} + \tau h, \varphi(t_{hn} + \tau h;\lambda), \lambda)\| \leqq \\ &\leqq (1 + hL_1)\, v^j_{hn} + hL_2\|\lambda_{hj} - \lambda\| + h\,\chi(h), \end{aligned}$$

where $\chi(h) = \eta(h) + \chi_1(h)$ and

$$\chi_1(h) = \sup_{\substack{t\in I \\ 0\leqq s\leqq h \\ z,\lambda\in R^q}} \|f(t, z(t), \lambda) - f(t + s, z(t + s), \lambda)\| .$$

The function χ_1 tends to zero as $h \to 0$.

Using Lemma 1.2 of [5] we can write

$$v^j_{hn} \leqq [hL_2\|\lambda_{hj} - \lambda\| + h\,\chi(h)] \sum_{i=0}^{n-1} \exp(ihL_1), \quad n \in R_N,$$

or

$$v^j_{hn} \leqq A\|\lambda_{hj} - \lambda\| + B\,\chi(h), \quad n \in R_N, \quad j = 0, 1, \ldots.$$

Now the definition of $\{\lambda_{hj}\}$ directly implies

$$\begin{aligned} \|\lambda_{h,j+1} - \lambda\| &= \|\lambda_{hj} - \lambda - (\tilde{M} + \tilde{N})^{-1} [\tilde{M}\lambda_{hj} + \tilde{N}y_h(\beta;\lambda_{hj}) - \\ &- \tilde{K} - \tilde{M}\lambda - \tilde{N}\varphi(\beta;\lambda) + \tilde{K}]\| = \\ &= \|(\tilde{M} + \tilde{N})^{-1}\, \tilde{N}[\lambda_{hj} - \lambda - y_h(\beta;\lambda_{hj}) + \varphi(\beta;\lambda)]\| \leqq \\ &\leqq m[\|\lambda_{hj} - \lambda\| + v^j_{hN}], \quad j = 0, 1, \ldots, \end{aligned}$$

or

$$\|\lambda_{h,j+1} - \lambda\| \leqq m(1 + A)\,\|\lambda_{hj} - \lambda\| + mB\,\chi(h), \quad j = 0, 1, \ldots.$$

Using again Lemma 1.2 [5] we obtain the estimate (6). Now the relation (7) is satisfied, too. Hence, because of 2° we see that

$$\lim_{\substack{h\to 0 \\ j\to\infty}} u_j(h) = 0 .$$

It means that the method (4–5) is convergent to the solution (φ, λ) of BVP (1–3). The proof of Theorem 1 is completed.

Remark 5. It follows from the estimates (6–7) that the numerical method (4–5) is convergent to the solution (φ, λ) if the sequence $\{\lambda_{hj}\}$ is convergent to λ.

Remark 6. If the problem (1–3) has solutions (φ, λ_1), (φ, λ_2) such that $\varphi(\beta;\lambda_1) = \varphi(\beta;\lambda_2)$, and if $\det(\tilde{M}) \neq 0$, then $\lambda_1 = \lambda_2$. To prove it we suppose that $\lambda_1 \neq \lambda_2$. Now we see

$$\tilde{M}\lambda_1 + \tilde{N}\varphi(\beta;\lambda_1) = \tilde{M}\lambda_2 + \tilde{N}\varphi(\beta;\lambda_2),$$

so $\lambda_1 = \lambda_2$.

Remark 7. If there is a number $\gamma > 0$ with $m + (L_2/L_1)\,\gamma < 1$ such that

$$(\beta - \alpha)\, L_1 \leqq \ln\left(1 + \frac{\gamma}{m}\right),$$

then the condition 2° is satisfied.

Remark 8. Put

$$C(t; \lambda) = \frac{\partial \varphi(t; \lambda)}{\partial \lambda}.$$

Let us determine the function C as a solution of the initial problem

$$\begin{cases} (\mathrm{d}/\mathrm{d}t)\, C(t; \lambda) = J_1(t, \varphi(t; \lambda), \lambda)\, C(t; \lambda) + J_2(t, \varphi(t; \lambda), \lambda)\,, \\ C(\alpha; \lambda) = \theta\,, \end{cases}$$

where

$$J_1(t, y, \lambda) = \frac{\partial f(t, y, \lambda)}{\partial y}, \quad J_2(t, y, \lambda) = \frac{\partial f(t, y, \lambda)}{\partial \lambda}.$$

So J_1 and J_2 are the Jacobian matrices of partial derivatives of f.

Now instead of (4) we may apply the Newton method in the form

$$\lambda_{h,j+1} = \lambda_{hj} - [\tilde{M} + \tilde{N} C_h(\beta; \lambda_{hj})]^{-1}\, [\tilde{M}\lambda_{hj} + \tilde{N} y_h(\beta; \lambda_{hj}) - \tilde{K}]\,,$$

or its modified form with $C_h(\beta; \lambda_0)$ instead of $C_h(\beta; \lambda_{hj})$. Such method usually converges rapidly to the solution λ but as you see it is a little complicated in this case. To use it we would determine $C(\beta; \lambda_{hj})$ by a numerical method to get its approximation $C_h(\beta; \lambda_{hj})$. Moreover, it should be known that $\tilde{M} + \tilde{N} C_h(\beta; \lambda_{hj})$ is nonsingular and

$$\|[\tilde{M} + \tilde{N} C_h(\beta; \lambda_{hj})]^{-1}\, \tilde{N}\| \leqq m < 1\,.$$

4. NUMERICAL EXAMPLES WITH $M \neq \theta$

Here we present results for Euler, Heun (second order), and Runge-Kutta (fourth order) methods on numerical problems. The calculations were carried out for two different starting vectors λ_0 with a fixed stepsize h. Only the numerical solutions y_h for the last point t and the corresponding λ_{hj} are given. We obtain our numerical solutions (y_h, λ_{hj}) if $\|\lambda_{hj} - \lambda_{h,j-1}\| < \varepsilon = {\cdot}0001$. The computations were carried out on IBM PC.

Example 1. We consider the boundary-value problem

$$\begin{cases} y'(t) = 1 - t^2 - \sin(t + 1) - t\cos(t + 1) + 3\cos(t) + t y(t) - \\ \quad - \lambda\cos(t)\,, \quad t \in [0, \pi - 1]\,, \\ y(0) = \cos(1)\,, \quad 150\lambda + y(\pi - 1) = \pi + 448\,. \end{cases}$$

It has the exact solution

$$\varphi(t) = t + \cos(t + 1)\,, \quad \lambda = 3\,, \quad \varphi(\pi - 1) \approx 1{\cdot}1416\,.$$

Indeed, we have

$$L_1 = \pi - 1\,, \quad L_2 = 1\,, \quad (\tilde{M} + \tilde{N})^{-1}\,\tilde{N} = \tfrac{1}{151} = m\,,$$

$$m\left[1 + \frac{L_2}{L_1}\left(\exp\left((\pi - 1)^2\right) - 1\right)\right] \approx \cdot 3070 < 1\,.$$

All assumptions of Theorem 1 are satisfied, so the method (4–5) is convergent to (φ, λ). The calculations are made for the stepsize $h = (\pi - 1)/100$. The results are given in the following tables.

	Euler		Heun		Runge-Kutta	
j	λ_{hj}	$y_h(\pi - 1; \lambda_{hj})$	λ_{hj}	$y_h(\pi - 1; \lambda_{hj})$	λ_{hj}	$y_h(\pi - 1; \lambda_{hj})$
0	·0000	23·3495	·0000	24·6334	·0000	24·6407
1	2·8317	2·3830	2·8246	2·5154	2·8245	2·5162
2	2·9907	1·1950	2·9897	1·2219	2·9897	1·2220
3	2·9996	1·1284	2·9994	1·1462	2·9994	1·1463
4	3·0001	1·1246	3·0000	1·1418	3·0000	1·1419
5	3·0001	1·1244	3·0000	1·1416	3·0000	1·1416

	Euler		Heun		Runge-Kutta	
j	λ_{hj}	$y_h(\pi - 1; \lambda_{hj})$	λ_{hj}	$y_h(\pi - 1; \lambda_{hj})$	λ_{hj}	$y_h(\pi - 1; \lambda_{hj})$
0	5·0000	−13·8243	5·0000	−14·5197	5·0000	−14·5245
1	3·1124	·2854	3·1170	·2257	3·1170	·2252
2	3·0064	1·0773	3·0068	1·0880	3·0068	1·0880
3	3·0005	1·1218	3·0004	1·1384	3·0004	1·1385
4	3·0001	1·1243	3·0000	1·1413	3·0000	1·1414
5	3·0001	1·1244	3·0000	1·1415	3·0000	1·1416

Example 2. Now we consider the system

$$\begin{cases} y_1'(t) = t\,y_1(t) + y_2(t) + (1 - t^2)\,\lambda_1 + 2t\lambda_2 - t\sin(t) - t^2\,, \\ y_2'(t) = -y_1(t) + t\lambda_1 - 2t^2\lambda_2 + 2t\,, \quad \text{for} \quad t \in [0, 1]\,, \end{cases}$$

with the conditions

$$y_1(0) = 0\,, \quad y_2(0) = 1\,,$$

$$\tilde{M}\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix} + \tilde{N}\begin{bmatrix}y_1(1; \lambda_1, \lambda_2) \\ y_2(1; \lambda_1, \lambda_2)\end{bmatrix} = \tilde{K}\,,$$

where

$$\tilde{M} = \begin{bmatrix} -\cdot 02 & \cdot 99 \\ -\cdot 97 & 0 \end{bmatrix}, \quad \tilde{N} = \begin{bmatrix} \cdot 02 & \cdot 01 \\ -\cdot 03 & 0 \end{bmatrix},$$

$$\tilde{K} = \begin{bmatrix} \cdot 02\sin(1) + \cdot 01\cos(1) + 1\cdot 01 \\ -\cdot 03\sin(1) - \cdot 53 \end{bmatrix}$$

Our problem has the solution given by

$$\varphi_1(t) = \sin(t) + \cdot 5t + t^2 , \quad \varphi_2(t) = \cos(t) + t^2 - t^3 ,$$
$$\lambda_1 = \cdot 5 , \quad \lambda_2 = 1 , \quad \varphi_1(1) \approx 2\cdot 3415 , \quad \varphi_2(1) \approx \cdot 5403 .$$

Similarly as in the previous example we have

$$L_1 = 2 , \quad L_2 = 3 , \quad m = \cdot 03 ,$$
$$A = 1\cdot 5 (\exp(2) - 1) , \quad m(1 + A) = \cdot 03(1\cdot 5 \exp(2) - \cdot 5) < 1 .$$

The numerical values for this problem are given in the following tables for $h = \cdot 01$.

	Euler		Heun		Runge-Kutta	
j	$\lambda_{1,hj}$ $\lambda_{2,hj}$	$y_{1,h}(1; \lambda_{hj})$ $y_{2,h}(1; \lambda_{hj})$	$\lambda_{1,hj}$ $\lambda_{2,hj}$	$y_{1,h}(1; \lambda_{hj})$ $y_{2,h}(1; \lambda_{hj})$	$\lambda_{1,hj}$ $\lambda_{2,hj}$	$y_{1,h}(1; \lambda_{hj})$ $y_{2,h}(1; \lambda_{hj})$
0	·0000	·8410	·0000	·8414	·0000	·8415
	·0000	1·5346	·0000	1·5403	·0000	1·5403
1	·5300	2·3649	·5300	2·3715	·5300	2·3715
	1·0001	·5536	1·0000	·5403	1·0000	·5403
2	·5002	2·3350	·5000	2·3415	·5000	2·3415
	1·0000	·5536	1·0000	·5403	1·0000	·5403
3	·5002	2·3350	·5000	2·3415	·5000	2·3415
	1·0000	·5536	1·0000	·5403	1·0000	·5403

	Euler		Heun		Runge-Kutta	
j	$\lambda_{1,hj}$ $\lambda_{2,hj}$	$y_{1,h}(1; \lambda_{hj})$ $y_{2,h}(1; \lambda_{hj})$	$\lambda_{1,hj}$ $\lambda_{2,hj}$	$y_{1,h}(1; \lambda_{hj})$ $y_{2,h}(1; \lambda_{hj})$	$\lambda_{1,hj}$ $\lambda_{2,hj}$	$y_{1,h}(1; \lambda_{hj})$ $y_{2,h}(1; \lambda_{hj})$
0	5·0000	10·8099	5·0000	10·8417	5·0000	10·8415
	5·0000	−3·3702	5·0000	− 3·4598	5·0000	−3·4597
1	·3809	2·2155	·3800	2·2215	·3800	2·2215
	·9997	·5539	1·0000	·5403	1·0000	·5403
2	·5002	2·3350	·5000	2·3415	·5000	2·3415
	1·0000	·5536	1·0000	·5403	1·0000	·5403
3	·5002	2·3350	·5000	2·3415	·5000	2·3415
	1·0000	·5536	1·0000	·5403	1·0000	·5403

5. BVP WITHOUT PARAMETERS

It is clear that the problem $(1-3)$ may be expressed as BVP without parameters. It has then the form

$$\text{(8)} \qquad \begin{cases} Y'(t) = \tilde{f}(t, Y(t)) , & t \in I , \\ \tilde{A}\, Y(\alpha) + \tilde{B}\, Y(\beta) = \tilde{C} , \end{cases}$$

where

$$Y = [y_1, \dots, y_q, y_{q+1}, \dots, y_{2q}]^{\mathrm{T}}, \quad (y_{q+i}(t) = \lambda_i, \quad i = 1, 2, \dots, q),$$
$$\tilde{f} = [f_1, \dots, f_q, 0, \dots, 0]^{\mathrm{T}}, \quad \tilde{f} \in R^{2q},$$
$$\tilde{A} = \left[\begin{array}{c|c} I & \Theta \\ \hline \Theta & \Theta \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c|c} \Theta & \Theta \\ \hline \tilde{N} & \tilde{M} \end{array}\right], \quad \tilde{C} = \left[\begin{array}{c} y_0 \\ \hline \tilde{K} \end{array}\right].$$

The square matrices $\tilde{A}$ and $\tilde{B}$ are of order $2q$ while the unit matrix I and the zero matrix Θ are of order q. Now the numerical method is convergent if the matrix $\tilde{A} + \tilde{B}$ is nonsingular, and the conditions

(9) $$\|(\tilde{A} + \tilde{B})^{-1} \tilde{B}\| \leqq b,$$

and

(10) $$b(\exp(L(\beta - \alpha)) - 1) < 1,$$

are fulfilled. Here L is the Lipschitz constant for $\tilde{f}$ with respect to the second variable (see [3, 7, 12]). The conditions obtained in this paper are similar to (9–10). For some cases our result is better than that obtained by Keller. To explain it we consider a scalar problem, i.e. $q = 1$. The problem (1–3) has now the form (8) with

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ \tilde{N} & \tilde{M} \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} y_0 \\ \tilde{K} \end{bmatrix}.$$

Here $\tilde{N}$, $\tilde{M}$, y_0 and $\tilde{K}$ are given numbers.

To compare the results we take the maximum norm

$$\|x\|_\infty = \max(|x_1|, |x_2|).$$

Using the matrix norm we have

$$\|(\tilde{A} + \tilde{B})^{-1} \tilde{B}\|_\infty = 1 + \left|\frac{\tilde{N}}{\tilde{M}}\right| = b, \quad \tilde{M} \neq 0, \quad L = L_1 + L_2,$$

and the condition (10) is now of the form

(11) $$\left[1 + \left|\frac{\tilde{N}}{\tilde{M}}\right|\right] [\exp((L_1 + L_2)(\beta - \alpha)) - 1] < 1.$$

The corresponding condition 2° of Theorem 1 has the form

(12) $$\left|\frac{\tilde{N}}{\tilde{M} + \tilde{N}}\right| \left[1 + \frac{L_2}{L_1}(\exp(L_1(\beta - \alpha)) - 1)\right] < 1.$$

Now we assume that

1) $L_1 = L_2$,

2) $\tilde{M} = a\tilde{N}$ for $a \in \mathscr{A} = R \setminus [-1 - \sqrt{2}, -1 + \sqrt{2}]$,

3) $\sqrt{2} \leqq \exp(L_1(\beta - \alpha)) < |1 + a|$ for $a \in \mathscr{A}$.

We see that for such cases only the condition (12) is satisfied (the condition (11) is not satisfied). A similar result is obtained if we take another norm. It means that for the above cases our result is a little better than Keller's one. Both Keller's result and ours require $\tilde{M} \neq \Theta$. The last condition may be weakened but only for a special kind of the function f.

6. SPECIAL CASE OF THE FUNCTION f

Whereas initial-value problems are normally uniquely solvable, boundary-value problems can also have no solution or several solutions. Even separated linear boundary-value problems are too general. It is well known that the assumptions of Keller's theorem for the existence of solutions of BVP(8) are very restrictive and already in the case $q = 1$ they are not satisfied for such simple boundary conditions as

$$y_1(\alpha) = c_1 , \quad y_1(\beta) = c_2 .$$

A similar situation occurs with our problem. The conditions of Theorem 1 are only sufficient, and, for example, the assumption 2° is not satisfied if $\tilde{M} = \Theta$. For this reason we consider the equation (1) in which the function f has a special form, namely

$$y'(t) = g(t, y(t) + S(t)\,\lambda, \lambda) - f_1(t)\,(y(t) + S(t)\,\lambda) \equiv f(t, y(t), \lambda) . \tag{13}$$

Here, S is a square matrix function of order q, and $g: I \times R^q \times R^q \to R^q, f_1: I \to R$. We assume in addition that f_1 does not change its sign in I, so

$$f_1(t) \geqq 0 \quad \text{or} \quad f_1(t) \leqq 0 \quad \text{for all} \quad t \in I . \tag{14}$$

We see that the function f is written as a sum of a linear part (in y and λ) and a non-linear one of a special type. For such case the assumption 2° can be weakened. Now we are in a position to define the numerical method for finding the solution (φ, λ) of our problem. It is given by

$$\begin{cases} \lambda_{h0} = \lambda_0 , \\ \lambda_{h,j+1} = \lambda_{hj} - (\tilde{M} - \tilde{N}S(\beta))^{-1}\,[\tilde{M}\lambda_{hj} + \tilde{N}y_h(\beta; \lambda_{hj}) - \tilde{K}] , \\ j = 0, 1, \ldots, \end{cases} \tag{15}$$

and

$$\begin{cases} y_h(t_{h0}; \lambda_{hj}) = y_0 , \\ y_h(t_{h,n+1}; \lambda_{hj}) = y_h(t_{hn}; \lambda_{hj}) + h\Phi(t_{hn}, y_h(t_{hn}; \lambda_{hj}), \lambda_{hj}, h) , \quad n \in R_{N-1} . \end{cases} \tag{16}$$

Theorem 2. *If*

1° $\Phi: I \times R^q \times R^q \times H \to R^q, g: I \times R^q \times R^q \to R^q, f_1 \in C(I, R)$ *and g is continuous with respect to the first two variables uniformly with respect to the last variable,*

2° *the matrices $\tilde{M}$, $\tilde{N}$ and S are square of order q, S is a matrix function defined for $t \in I$, and $\tilde{K} \in R^q$,*

3° *the matrix $\tilde{M} - \tilde{N}\,S(\beta)$ is nonsingular and there is a constant m such that*

$$\|(\tilde{M} - \tilde{N}\,S(\beta))^{-1}\,\tilde{N}\| \leqq m ,$$

where the matrix norm is consistent with the vector norm and $\|I\| = 1$, 4° *there exist constants* L_3 *and* L_4 *such that*

$$\sup_{t\in I} \|S(t)\| \leqq L_3\,, \quad \sup_{t\in I} \|S'(t)\| \leqq L_4\,,$$

5° *the condition* (14) *is satisfied,*

6° *there exist constants* L_1, $L_5 \geqq 0$ *such that for* $t \in I$, $x, \bar{x}, \mu, \bar{\mu} \in R^q \times R^q \times \times R^q \times R^q$ *we have*

$$\|g(t, x, \mu) - g(t, \bar{x}, \bar{\mu})\| \leqq L_1\|x - \bar{x}\| + L_5\|\mu - \bar{\mu}\|\,,$$

7° *there exists a function* $\eta\colon H \to R_+$, $\eta(h) \to 0$ *as* $h \to 0$, *such that for* $t, y, \mu, h \in \in I \times R^q \times R^q \times H$ *we have*

$$\|\Phi(t, y, \mu, h) - f(t, y, \mu)\| \leqq \eta(h)\,,$$

where f *is defined in* (13),

8° *there exists a solution* (φ, λ) *of BVP* (13, 2, 3),

9° $d = mQ < 1$ *where*

$$Q = \begin{cases} L_3 + (\beta - \alpha)(L_4 + L_5) & \text{if } L = L_1 - L_2 = 0\,,\ L_2 = \inf\limits_{t\in I} f_1(t)\,, \\ \exp(L(\beta - \alpha))(L_3 + (L_4 + L_5)/L) - (L_4 + L_5)/L & \\ & \text{if } L > 0 \text{ or } L < 0,\ L_3 \neq 0 \text{ and } L_3 L \leqq -(L_4 + L_5)\,, \\ -(L_4 + L_5)/L & \text{if } -(L_4 + L_5) < L_3 L < 0 \text{ or } L_3 = 0\,, \end{cases}$$

then the method (15–16) *is convergent to the solution* (φ, λ) *of BVP* (13, 2, 3) *and we have the following estimates*:

(17) $$\|\lambda_{hj} - \lambda\| \leqq \tilde{u}_j(h)\,, \quad j = 0, 1, \ldots,$$

(18) $$\max_{n\in R_N} \|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda)\| \leqq p_2\, \tilde{u}_j(h) + p_1\, \chi(h)\,, \quad j = 0, 1, \ldots,$$

where

$$\tilde{u}_j(h) = d^j\|\lambda_0 - \lambda\| + mp_1\, \chi(h)\,\frac{1 - d^j}{1 - d}\,, \quad j = 0, 1, \ldots.$$

The constants p_1 *and* p_2 *are of the form*

$$p_1 = \begin{cases} \dfrac{a - 1}{L_1 - L_2} & \text{if } L_1 > L_2\,, \\ \beta - \alpha & \text{if } L_1 = L_2\,, \\ \dfrac{1}{L_2 - L_1} & \text{if } L_1 < L_2\,, \end{cases}$$

$$p_2 = \begin{cases} L_3(a + 1) + \dfrac{L_4 + L_5}{L}(a - 1) & \text{if } L_1 > L_2\,, \\ 2L_3 + (\beta - \alpha)(L_4 + L_5) & \text{if } L_1 = L_2\,, \\ 2L_3 + \dfrac{L_4 + L_5}{L_2 - L_1} & \text{if } L_1 < L_2\,, \end{cases}$$

with

$$a = \exp\left((L_1 - L_2)(\beta - \alpha)\right).$$

Moreover, if $L_2 > 0$ it is necessary to add the above conditions are satisfied for the step size h such that

$$h < \frac{1}{\sup_{t \in I} f_1(t)}.$$

Proof. BVP (13, 2, 3) has a solution, so using the definition of λ_{hj} we have

$$\|\lambda_{h,j+1} - \lambda\| = \|\lambda_{hj} - \lambda - (\tilde{M} - \tilde{N}\,S(\beta))^{-1}\,[\tilde{M}\lambda_{hj} + \tilde{N}y_h(\beta; \lambda_{hj}) - \\ - \tilde{M}\lambda - \tilde{N}\,\varphi(\beta; \lambda)]\| \leqq m w_N^j,$$

where

$$w_n^j = \|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda) + S(t_{hn})(\lambda_{hj} - \lambda)\|. \quad n \in R_N.$$

It is possible to prove that the elements w_n^j satisfy the inequalities

$$\text{(19)} \qquad \begin{cases} w_{n+1}^j \leqq c_n w_n^j + h\,\chi(h) + h(L_4 + L_5)\,\|\lambda_{hj} - \lambda\|, & n \in R_{N-1}, \\ w_0^j \leqq L_3\|\lambda_{hj} - \lambda\|, \end{cases}$$

where

$$c_n = |1 - h\,f_1(t_{hn})| + hL_1.$$

The relation (19) is obtained by applying the definition of $\{y_h\}$ and the assumptions. Here χ has a similar form as in Theorem 1.

We consider only the case when

$$\text{(20)} \qquad f_1(t) \geqq 0 \quad \text{for} \quad t \in I,$$

(for the other one the proof is similar). Now from this condition we see that

$$1 - h\,f_1(t_n) > 0,$$

and hence

$$c_n = 1 - h\,f_1(t_{hn}) + hL_1 \leqq 1 + hL.$$

In fact L may be positive, negative or zero.

Now (19) directly implies

$$w_n^j \leqq a_n\|\lambda_{hj} - \lambda\| + b_n\,\chi(h), \quad n \in R_N, \quad j = 0, 1, \ldots,$$

where

$$a_n = \begin{cases} (1 + hL)^n L_3 + \dfrac{L_4 + L_5}{L}\,[(1 + hL)^n - 1] & \text{if} \quad L \neq 0, \\ L_3 + (\beta - \alpha)(L_4 + L_5) & \text{if} \quad L = 0, \end{cases}$$

$$b_n = \begin{cases} \dfrac{(1 + hL)^n - 1}{L} & \text{if} \quad L \neq 0, \\ \beta - \alpha & \text{if} \quad L = 0. \end{cases}$$

Hence we obtain

$$\|\lambda_{h,j+1} - \lambda\| \leqq m w_N^j \leqq m a_N\|\lambda_{hj} - \lambda\| + m b_N\,\chi(h), \quad j = 0, 1, \ldots,$$

and $$\|\lambda_{hj} - \lambda\| \leqq (ma_N)^j \|\lambda_0 - \lambda\| + mb_N \chi(h) \frac{1 - (ma_N)^j}{1 - ma_N}, \quad j = 0, 1, \ldots$$

It means that the inequality (17) is satisfied.

Now the inequality

$$\|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda)\| \leqq$$
$$\leqq \|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda) + S(t_{hn})(\lambda_{hj} - \lambda) - S(t_{hn})(\lambda_{hj} - \lambda)\| \leqq$$
$$\leqq w_n^j + L_3\|\lambda_{hj} - \lambda\| \leqq a_n\|\lambda_{hj} - \lambda\| + b_n \chi(h) + L_3\|\lambda_{hj} - \lambda\|,$$

implies that the relation (18) holds. The proof is complete.

Remark 9. It is known that the subordinate matrix norm lub $(\cdot)$ is consistent with the vector norm and lub $(I) = 1$. Indeed, we can write the conditions 3° and 4° using the norm lub $(\cdot)$ as well.

Remark 10. The following estimate

$$\|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda) + S(t_{hn})(\lambda_{hj} - \lambda)\| \leqq a_n \tilde{u}_j(h) + b_n \chi(h),$$
$$n \in R_N, \quad j = 0, 1, \ldots,$$

follows from the proof of Theorem 2.

7. NUMERICAL EXAMPLES WITH $\tilde{M} = 0$

We consider now some examples for the case $\tilde{M} = 0$ to demonstrate the convergence behaviour of our methods.

Example 3.

$$\begin{cases} y'(t) = t^2 + 2t + 3 + \sin(t) + \cos(t) - y(t) - \lambda, & t \in [0, \pi], \\ y(0) = 1, \quad y(\pi; \lambda) = \pi^2 + 1. \end{cases}$$

The exact solution is

$$\varphi(t) = t^2 + 1 + \sin(t), \quad \lambda = 2, \quad \varphi(\pi) \approx 10{\cdot}8696.$$

Here

$$L_1 = 0, \quad L_5 = 0, \quad L_3 = 1, \quad L_4 = 0, \quad L_2 = 1, \quad m = 1,$$
$$Q = \exp(-\pi).$$

The calculations are made for the stepsize $h = \pi/100$.

	Euler		Heun		Runge-Kutta	
j	λ_{hj}	$y_h(\pi; \lambda_{hj})$	λ_{hj}	$y_h(\pi; \lambda_{hj})$	λ_{hj}	$y_h(\pi; \lambda_{hj})$
0	·0000	12·7655	·0000	12·7835	·0000	12·7832
1	1·8959	10·9475	1·9139	10·9524	1·9136	10·9523
2	1·9738	10·8728	1·9967	10·8732	1·9963	10·8732
3	1·9770	10·8697	2·0002	10·8698	1·9998	10·8698
4	1·9771	10·8696	2·0004	10·8696	2·0000	10·8696
5	1·9772	10·8696	2·0004	10·8696	2·0000	10·8696

j	λ_{hj}	Euler $y_h(\pi; \lambda_{hj})$	λ_{hj}	Heun $y_h(\pi; \lambda_{hj})$	λ_{hj}	Runge-Kutta $y_h(\pi; \lambda_{hj})$
0	5·0000	7·9710	5·0000	7·9997	5·0000	7·9992
1	2·1014	10·7505	2·1301	10·7455	2·1296	10·7456
2	1·9823	10·8647	2·0060	10·8642	2·0056	10·8642
3	1·9774	10·8694	2·0007	10·8694	2·0002	10·8694
4	1·9772	10·8696	2·0004	10·8696	2·0000	10·8696
5	1·9772	10·8696	2·0004	10·8696	2·0000	10·8696

Example 4. Let

$$\begin{cases} y'(t) = \cos^2\left(y(t) - (t^2/9)\,\lambda + 3t - 2\right) - (t + 3)\left(y(t) - (t^2/9)\,\lambda\right) - \\ \qquad - 3t^2 - 5t + 2\,, \quad t \in [0, 2]\,, \\ y(0) = 2\,, \quad y(2) = 0\,. \end{cases}$$

The exact solution is

$$\varphi(t) = t^2 - 3t + 2\,, \quad \lambda = 9\,.$$

In this case we have

$$L_1 = 1\,, \quad L_2 = 3\,, \quad L_3 = 4/9\,, \quad L_4 = 4/9\,, \quad L_5 = 0\,, \quad m = 9/4\,,$$
$$Q \approx \cdot 2263, \quad d \approx \cdot 5091\,.$$

The calculations are made for $h = \cdot 02$.

j	λ_{hj}	Euler $y_h(2; \lambda_{hj})$	λ_{hj}	Heun $y_h(2; \lambda_{hj})$	λ_{hj}	Runge-Kutta $y_h(2; \lambda_{hj})$
0	·0000	−3·3254	·0000	−3·3256	·0000	−3·3256
1	7·4822	− ·5549	7·4827	− ·5513	7·4826	− ·5515
2	8·7307	− ·1015	8·7231	− ·1001	8·7235	− ·1001
3	8·9591	− ·0190	8·9482	− ·0185	8·9487	− ·0186
4	9·0018	− ·0036	8·9899	− ·0034	8·9905	− ·0035
5	9·0097	− ·0007	8·9976	− ·0006	8·9982	− ·0006
6	9·0112	− ·0001	8·9991	− ·0001	8·9997	− ·0001
7	9·0115	− ·0000	8·9994	− ·0000	8·9999	− ·0000
8	9·0116	− ·0000	8·9994	− ·0000	9·0000	− ·0000

j	λ_{hj}	Euler $y_h(2; \lambda_{hj})$	λ_{hj}	Heun $y_h(2; \lambda_{hj})$	λ_{hj}	Runge-Kutta $y_h(2; \lambda_{hj})$
0	5·0000	−1·4657	5·0000	−1·4636	5·0000	−1·4637
1	8·2978	− ·2583	8·2931	− ·2561	8·2933	− ·2562
2	8·8790	− ·0479	8·8693	− ·0471	8·8698	− ·0471
3	8·9868	− ·0090	8·9752	− ·0087	8·9758	− ·0088
4	9·0069	− ·0017	8·9949	− ·0016	8·9955	− ·0016
5	9·0107	− ·0003	8·9986	− ·0003	8·9992	− ·0003
6	9·0114	− ·0001	8·9993	− ·0001	8·9998	− ·0001
7	9·0116	− ·0000	8·9994	− ·0000	9·0000	− ·0000
8	9·0116	− ·0000	8·9994	− ·0000	9·0000	− ·0000

Example 5. Now we take the problem

$$\begin{cases} y'(t) = \sin(y(t) + \lambda) - 3(y(t) + \lambda) - \sin(t^3 - 3t^2 + \frac{11}{4}t) + 3t^3 - \\ \qquad - 6t^2 + \frac{9}{4}t + \frac{11}{4}, \quad t \in [0, 2], \\ y(0) = -{\cdot}75, \quad y(2) = {\cdot}75. \end{cases}$$

It has the exact solution

$$\varphi(t) = t^3 - 3t^2 + \tfrac{11}{4}t - \tfrac{3}{4}, \quad \lambda = \tfrac{3}{4}.$$

We have

$$L_1 = 1, \quad L_2 = 3, \quad L_3 = 1, \quad L_4 = 0, \quad L_5 = 0, \quad m = 1,$$
$$Q = \exp(-4).$$

The calculations are made for $h = {\cdot}02$.

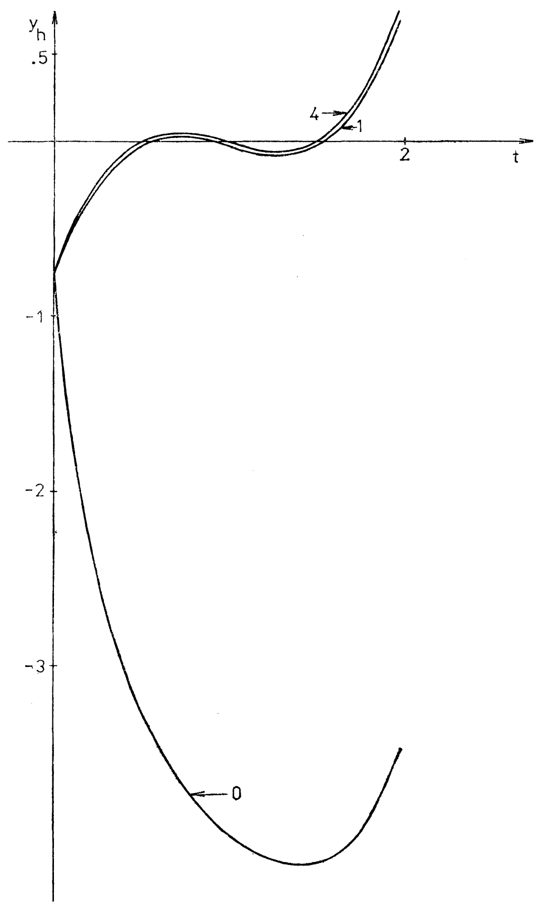

Fig. 1. Graph of y_h for $h = {\cdot}02$ and λ_{h0}, λ_{h1}, λ_{h4} with $\lambda_{h0} = 5$.

j	λ_{hj}	Euler $y_h(2; \lambda_{hj})$	λ_{hj}	Heun $y_h(2; \lambda_{hj})$	λ_{hj}	Runge-Kutta $y_h(2; \lambda_{hj})$
0	·0000	1·4782	·0000	1·4923	·0000	1·4918
1	·7282	·7572	·7423	·7581	·7418	·7581
2	·7354	·7501	·7504	·7500	·7499	·7501
3	·7355	·7500	·7505	·7500	·7500	·7500

j	λ_{hj}	Euler $y_h(2; \lambda_{hj})$	λ_{hj}	Heun $y_h(2; \lambda_{hj})$	λ_{hj}	Runge-Kutta $y_h(2; \lambda_{hj})$
0	5·0000	−3·4925	5·0000	−3·4747	5·0000	−3·3752
1	·7575	·7282	·7753	·7254	·7748	·7254
2	·7357	·7498	·7507	·7497	·7503	·7497
3	·7355	·7500	·7505	·7500	·7500	·7500
4	·7355	·7500	·7505	·7500	·7500	·7500

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Souhrn

JEDNOKROKOVÉ METODY PRO OBYČEJNÉ DIFERENCIÁLNÍ ROVNICE S PARAMETRY

TADEUSZ JANKOWSKI

Článek se zabývá numerickým řešením obyčejných diferenciálních rovnic s parametry. Předložená metoda je založena na jednokrokové proceduře pro ODR kombinované s iterativním postupem. Jsou dokázány jednoduché postačující podmínky pro konvergenci této metody, odvozeny odhady chyb a uvedeny numerické příklady.

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