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SHAPE OPTIMIZATION OF AN ELASTO-PLASTIC BODY FOR THE MODEL WITH STRAIN-HARDENING

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Summary. The state problem of elasto-plasticity (for the model with strain-hardening) is formulated in terms of stresses and hardening parameters by means of a time-dependent variational inequality.

The optimal design problem is to find the shape of a part of the boundary such that a given cost functional is minimized.

For the approximate solutions piecewise linear approximations of the unknown boundary, piecewise constant triangular elements for the stress and the hardening parameter, and backward differences in time are used.

Existence and uniqueness of a solution of the approximate state problem and existence of a solution of the approximate optimal design problem are proved. The main result is the proof of convergence of the approximations to a solution of the original optimal design problem.

Keywords: domain optimization, time-dependent variational inequality, elasto-plasticity, finite elements.

AMS Subject Class: 65K10, 65N30, 73E99.

INTRODUCTION

The optimal design problem on the class of domains that we consider in this paper was studied first for the Poisson equation by Begis, Glowinski [1]. Optimization of an elasto-plastic body has been studied by Hlaváček for Hencky's model [12] and for Prandtl-Reuss's model [6]. This paper is an extension of the latter work to an elasto-plastic body with isotropic strain-hardening.

Following Johnson [9], we formulate the state problem as a time-dependent variational inequality in terms of the stress tensor and the hardening parameter. For the approximations we use backward differences in time and piecewise constant finite elements for the stress and the hardening parameter.

For given body forces and surface tractions we have to find the shape of the part of the boundary where the body is fixed, such that the given cost functional attains its minimum. The cost functional is an integral of a function of the stress over the time-space domain. Using piecewise linear approximations of the unknown part of the boundary and a quadrature formula in the cost functional we formulate the approximate optimal design problem.

Further we prove existence of a solution of the approximate optimal design problem and convergence of these approximate solutions to a solution of the original optimal design problem. As a consequence of the results mentioned above we obtain the existence of a solution of the optimal design problem.

1. FORMULATION OF THE STATE PROBLEM EXISTENCE AND UNIQUENESS OF A SOLUTION

Let $\Omega \subset R^2$ be a bounded domain with a Lpschitz boundary. Assume that $\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_g$, $\Gamma_u \cap \Gamma_g = \emptyset$, where Γ_u and Γ_g are open in $\partial\Omega$. On Γ_g surface tractions are prescribed, while on Γ_u the body is fixed.

Let R^4_{sym} be the space of symmetric 2×2 matrices; on this space we define a norm

$$\|\tau\|_{R_{\text{sym}}{}^4} = (\tau_{ij}\tau_{ij})^{1/2}$$

(where the repeated index denotes summation over the range 1, 2).

Let a yield function $f: R^4_{\text{sym}} \to R$ be given, which is convex, continuously differentiable in $R^4_{\text{sym}} \setminus \{0\}$ and positively homogeneous (i.e. $f(\lambda\sigma) = |\lambda| f(\sigma)$ for all $\lambda \in R$, $\sigma \in R^4_{\text{sym}}$). Consequently, f is Lipschitz continuous (with a Lipschitz constant denoted by L). These assumptions are fulfilled e.g. for the well-known von Mises function

$$f(\sigma) = (\sigma^{\text{D}}_{ij}\sigma^{\text{D}}_{ij} + \tfrac{1}{9}(\sigma_{kk})^2)^{1/2}$$

where $\sigma^{\text{D}}_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$ is the stress deviatoric, δ_{ij} is the Kronecker symbol.

Let the following vector functions be given:

$$F^0 \in [L^2(\Omega)]^2\,, \quad g^0 \in [L^2(\Gamma_g)]^2\,.$$

We consider the load process in the time interval $I = \langle 0, T\rangle$, $T < +\infty$. Assume that the body forces F and the surface tractions g are of the following particular form:

$$(1.1) \qquad \begin{aligned} F(t, x) &= \gamma(t)\, F^0(x) \quad \text{in} \quad I \times \Omega\,, \\ g(t, x) &= \gamma(t)\, g^0(x) \quad \text{in} \quad I \times \Gamma_g\,, \end{aligned}$$

where $\gamma \in C^2(I)$ is a given real function, $\gamma(t) = 0$ in a "small" interval $\langle 0, t_1\rangle$, $0 < t_1 < T$.

Definition of some spaces and scalar products

In the usual $L^2(\Omega)$ space we denote the scalar product by $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ and the norm by $\|\cdot\|_{L^2(\Omega)}$. We introduce the following space of stress or strain tensors

$$S(\Omega) = \{\tau\colon \Omega \to R^4_{\text{sym}};\ \tau_{ij} \in L^2(\Omega),\ i, j = 1, 2\}$$

with the scalar product and norm defined by

$$\langle \tau, \sigma \rangle_{S(\Omega)} = \int_\Omega \tau_{ij}\sigma_{ij}\,\mathrm{d}x\,, \quad \|\tau\|_{S(\Omega)} = \langle \tau, \tau \rangle_{S(\Omega)}^{1/2}\,.$$

We deal with the space of pairs

$$H(\Omega) = S(\Omega) \times L^2(\Omega)\,.$$

Elements of this space will be denoted e.g. by $\boldsymbol{\sigma} = (\sigma, \alpha)$, where σ and α are called the stress tensor and the hardening parameter, respectively.

In $R^4_{\text{sym}} \times R$ we define the scalar product

$$\langle \boldsymbol{\sigma}(x), \boldsymbol{\tau}(x) \rangle = \sigma_{ij}(x)\,\tau_{ij}(x) + \alpha(x)\,\beta(x)$$

and the norm

$$|\boldsymbol{\sigma}(x)| = \langle \boldsymbol{\sigma}(x), \boldsymbol{\sigma}(x) \rangle^{1/2}\,.$$

In $H(\Omega)$ we introduce the scalar product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_\Omega = \int_\Omega \langle \boldsymbol{\sigma}(x), \boldsymbol{\tau}(x) \rangle\,\mathrm{d}x = \langle \sigma, \tau \rangle_{S(\Omega)} + \langle \alpha, \beta \rangle_{L^2(\Omega)}$$

and the norm

$$\|\boldsymbol{\sigma}\|_{0,\Omega} = \langle \boldsymbol{\sigma}, \boldsymbol{\sigma} \rangle_\Omega^{1/2} = \left(\|\sigma\|^2_{S(\Omega)} + \|\alpha\|^2_{L^2(\Omega)}\right)^{1/2}\,.$$

Let the elastic strain-stress relation be given by the inverse generalized Hooke's law

$$e^e_{ij} = b_{ijkl}\,\sigma_{kl} \quad i, j = 1, 2\,.$$

We assume that $b_{ijkl} \in L^\infty(\Omega)$ and that there exists a positive constant b_0 such that

$$b_{ijkl}(x)\,\sigma_{ij}\sigma_{kl} \geqq b_0\|\sigma\|^2_{R_{\text{sym}}{}^4} \quad \forall \sigma \in R^4_{\text{sym}}\,, \quad \text{a.e. in } \Omega\,,$$

and $b_{ijkl} = b_{jikl} = b_{klij}$.

Let a positive function $\varkappa \in L^\infty(\Omega)$ be given and let there exist such constants $\varkappa_1, \varkappa_2$ that

$$0 < \varkappa_1 \leqq \varkappa(x) \leqq \varkappa_2 < +\infty \quad \text{a.e. in } \Omega\,.$$

In $H(\Omega)$ we introduce an extended energy scalar product

$$\{\boldsymbol{\sigma}, \boldsymbol{\tau}\}_\Omega = \int_\Omega b_{ijkl}\sigma_{ij}\tau_{kl}\,\mathrm{d}x + \int_\Omega \varkappa\alpha\beta\,\mathrm{d}x$$

and the norm

$$|||\boldsymbol{\sigma}|||_{\Omega} = \{\boldsymbol{\sigma}, \boldsymbol{\sigma}\}_{\Omega}^{1/2} .$$

Note that the norms $\|\cdot\|_{0,\Omega}$ and $|||\cdot|||_{\Omega}$ are equivalent.

We define the space of test functions

$$V(\Omega) = \{w \in [H^1(\Omega)]^2;\ w = 0 \text{ on } \Gamma_u\} ,$$

and the set of statically admissible stress fields at a moment $t \in I$:

$$E(\Omega, t) = \{\tau \in S(\Omega);\ \langle \tau, e(w)\rangle_{S(\Omega)} = L_{\Omega}(w, t) \quad \forall w \in V(\Omega)\} ,$$

where

$$e_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \quad i, j = 1, 2 ,$$

$$L_{\Omega}(w, t) = \gamma(t)\left[\int_{\Omega} F_i^0 w_i \,\mathrm{d}x + \int_{\Gamma_g} g_i^0 w_i \,\mathrm{d}s\right] =^{\mathrm{def}} \gamma(t)\, L_{\Omega}^0(w) .$$

Let us further define the set

$$B = \{(\tau, \beta) \in R^4_{\mathrm{sym}} \times R;\ f(\tau) \leqq \beta\} .$$

The set of plastically admissible pairs will be denoted by

$$P(\Omega) = \{\boldsymbol{\tau} = (\tau, \beta) \in \boldsymbol{H}(\Omega);\ (\tau(x), \beta(x)) \in B \text{ a.e. in } \Omega\} .$$

The set $P(\Omega)$ is convex due to the convexity of the function f.

For all $t \in I$ we define the set

$$K(\Omega, t) = (E(\Omega, t) \times L^2(\Omega)) \cap P(\Omega) \subset \boldsymbol{H}(\Omega) .$$

Let $C_0^1(I, S(\Omega))$ be the space of continuously differentiable functions on the interval I with values in $S(\Omega)$, which vanish at $t = 0$. We define $H_0^1(I, S(\Omega))$ as the closure of $C_0^1(I, S(\Omega))$ in the norm

$$\|\tau\|_{H_0^1(I,S(\Omega))} = \left(\int_0^T \|\dot{\tau}\|^2_{S(\Omega)}\right)^{1/2} \quad \text{where} \quad \dot{\tau} = \frac{\partial \tau}{\partial t} .$$

Similarly, we define the spaces $H^1(I, L^2(\Omega))$, $H^1(I, \boldsymbol{H}(\Omega))$.

The validity of the following lemma is well-known.

Lemma 1.1. *For all* $\tau \in H_0^1(I, S(\Omega))$ *we have*

$$\|\tau(t) - \tau(t')\|_{S(\Omega)} \leqq |t - t'|^{1/2}\, \|\tau\|_{H_0^1(I,S(\Omega))} \quad \forall t, t' \in I . \tag{1.2}$$

A similar assertion is true for the spaces $H^1(I, L^2(\Omega))$, $H^1(I, \boldsymbol{H}(\Omega))$.

Throughout the paper, C will denote a positive constant not necessarily the same at each occurrence, the symbols t and x will be used for the time variable and the space variable, respectively.

A weak solution of the plasticity problem with an isotropic strain-hardening is a pair of functions

$$\boldsymbol{\sigma} = (\sigma, \alpha) \in H_0^1(I, S(\Omega)) \times H^1(I, L^2(\Omega))$$

such that

$$(1.3)\qquad \begin{aligned} &\alpha(0) = \alpha_0 , \\ &\boldsymbol{\sigma}(t) \in K(\Omega, t) \quad \text{for a.e.} \quad t \in I , \\ &\{\dot{\boldsymbol{\sigma}}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t)\}_\Omega \geqq 0 \quad \forall \boldsymbol{\tau} \in K(\Omega, t) , \quad \text{for a.e.} \quad t \in I . \end{aligned}$$

Here $\alpha_0 \in L^2(\Omega)$ is a given function such that

$$(1.4)\qquad \alpha_0 \geqq \alpha_1 \quad \text{a.e. in } \Omega ,$$

where α_1 is a positive constant.

Throughout the paper the following assumption will be needed:

There exists $\xi \in S(\Omega) \cap [C^{(0),1}(\bar{\Omega})]^4$ such that

$$(1.5)\qquad \langle \xi, e(w) \rangle_{S(\Omega)} = L_\Omega^0(w) \quad \forall w \in V(\Omega) .$$

Lemma 1.2. *Let* (1.5) *be fulfilled and let* $\delta_1 \in (0, \alpha_1/(1 + L))$ (*where* α_1 *occurs in* (1.4), L *is the Lipschitz constant for* f).

Then for each $t \in I$ *there exists a pair* $\boldsymbol{\xi}(t) = (\xi(t), \zeta(t)) \in K(\Omega, t)$ *such that*

$$\boldsymbol{\xi}(0) = (0, \alpha_0) ,$$

$$(1.6)\qquad \boldsymbol{\xi}(t, x) + \delta_1 \boldsymbol{\tau} \in B \quad \forall t \in I \quad \forall \boldsymbol{\tau} \in R_{\text{sym}}^4 \times R ; \quad |\boldsymbol{\tau}| \leqq 1 , \quad \text{a.e. in } \Omega$$

$$(1.7)\qquad \boldsymbol{\xi} \in C^2(I, S(\Omega)) \times C^2(I, L^2(\Omega)) ,$$

$$(1.8)\qquad \|\boldsymbol{\xi}(t)\|_{0,\Omega} \leqq C \quad \forall t \in I .$$

Sketch of the proof. Let us set

$$(1.9)\qquad \xi(t) = \gamma(t)\, \xi , \quad \zeta_1(t, x) = |\gamma(t)|\, C_3 + \alpha_0(x)$$

where ξ is from (1.5),

$$C_3 = \left[C_2 - \frac{\alpha_1 - \delta_1(L + 1)}{|\gamma(\bar{t})|} \right]^+ ,$$

where C_2 is the maximum of $f(\xi(x))$ over $\bar{\Omega}$, $\bar{t} \in I$ is the point at which $|\gamma(t)|$ attains its maximum and $[\ ]^+$ denotes the positive part. As we can prove, the pair $(\xi(t), \zeta_1(t))$ fulfils all requirements of the lemma except (1.7).

For the function γ we can find such $\tilde{\gamma} \in C^2(I)$ that

$$(1.10)\qquad \begin{aligned} &\tilde{\gamma}(t) \geqq |\gamma(t)| \quad \forall t \in I , \\ &\tilde{\gamma}(t) = 0 \quad \forall t \in \langle 0, t_2 \rangle , \end{aligned}$$

where t_2 is a fixed number from $(0, t_1)$.

It is readily seen that the pair $(\xi(t), \zeta(t))$, where

$$\zeta(t, x) = \tilde{\gamma}(t)\, C_3 + \alpha_0(x) \tag{1.11}$$

is the one we looked for in Lemma 1.2.

Remark 1.1. As in the paper of C. Johnson [9], we can prove the existence and uniqueness of a solution of the state problem (1.3) (supposing (1.5) and using also Lemma 1.2).

2 FORMULATION OF THE OPTIMAL DESIGN PROBLEM

Following Begis-Glowinski [1], we introduce the set of admissible design variables

$$\mathscr{U}_{\text{ad}} = \left\{ v \in C^{(0,1)}(\langle 0, 1\rangle);\ a \leqq v \leqq b,\ \left|\frac{\partial v}{\partial x_2}(x_2)\right| \leqq c_1 \right.$$
$$\left. \text{for a.e. } x_2 \in \langle 0, 1\rangle\,,\ \textstyle\int_0^1 v \, \mathrm{d}x_2 = c_2 \right\},$$

where $0 < a \leqq c_2 \leqq b < \infty$, $c_1 > 0$ are given constants.

Throughout the paper we shall consider a class of domains $\Omega = \Omega(v)$, where $v \in \mathscr{U}_{\text{ad}}$ and

$$\Omega(v) = \{(x_1, x_2) \in R^2;\ 0 < x_1 < v(x_2),\ 0 < x_2 < 1\}\,.$$

Let $\Gamma(v) = \{(x_1, x_2);\ x_1 = v(x_2), 0 < x_2 < 1\}$ denote the graph of the function v over $(0, 1)$; $\Gamma(v)$ is the part of the boundary to be optimized.

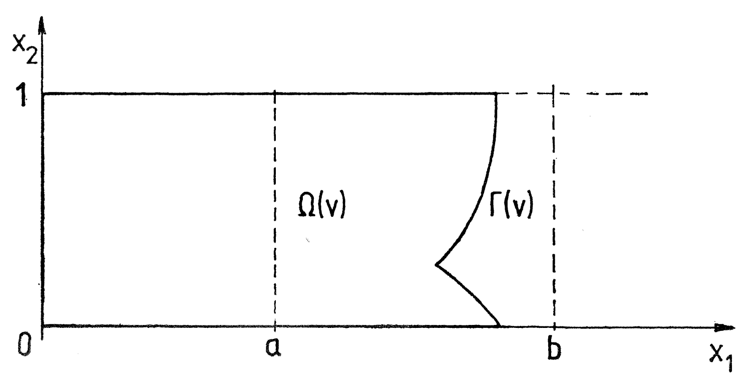

Fig. 1

Let us have a constant $\delta > b$, denote

$$\Omega_\delta = (0, \delta) \times (0, 1)\,,\quad \Gamma_\delta = \{(x_1, x_2);\ x_1 = \delta,\ 0 < x_2 < 1\}\,.$$

Convergence of functions which are defined on different domains $\Omega(v)$, will be defined by means of their extensions by zero to the domain $\Omega_\delta \setminus \Omega(v)$. Such extensions will be denoted by a bar over the fucntion (e.g. $\bar{\sigma}$).

We will consider only the following case of partition of the boundary:

$$\Gamma_u = \Gamma(v)\,,$$
$$\Gamma_g = \partial\Omega \setminus \Gamma(v)\,.$$

Let the functions $b_{ijkl}, \alpha_0, \varkappa$ be defined and fulfil the assumptions stated before on the domain Ω_δ. Let the functions $F^0 \in [L^2(\Omega_\delta)]^2$, $g^0 \in [L^2(\partial\Omega_\delta \setminus \Gamma_\delta)]^2$ be given. The state problem (1.3) is then defined for all domains $\Omega(v)$, $v \in \mathscr{U}_{\mathrm{ad}}$.

Let us assume:

There exists $\xi \in S(\Omega_0) \cap [C^{(0),1}(\bar{\Omega}_\delta)]^4$ such that

$$(2.1) \qquad \langle \xi, e(w) \rangle_{S(\Omega_\delta)} = L^0_{\Omega_\delta}(w) \quad \forall w \in V(\Omega_\delta)\,.$$

(We denote the Lipschitz constant for ξ by L_ξ.)

Obviously, also any restriction $\xi|_{\Omega(v)}$ fulfils analogous conditions on $\Omega(v)$ and the assertion of Lemma 1.2 is true for all domains $\Omega(v)$, $v \in \mathscr{U}_{\mathrm{ad}}$, provided (2.1) is fulfilled.

Remark 2.1. Let the boundary conditions be given by the following functions:

$$g^0 = (g_{1\mathrm{D}}(x_1), g_{2\mathrm{D}}(x_1))^{\mathrm{T}} \quad \text{on} \quad \{(x_1, x_2);\ x_1 \in \langle 0, \delta \rangle,\ x_2 = 0\}\,,$$
$$g^0 = (g_{1\mathrm{H}}(x_1), g_{2\mathrm{H}}(x_1))^{\mathrm{T}} \quad \text{on} \quad \{(x_1, x_2);\ x_1 \in \langle 0, \delta \rangle,\ x_2 = 1\}\,,$$
$$g^0 = (g_{1\mathrm{L}}(x_2), g_{2\mathrm{L}}(x_2))^{\mathrm{T}} \quad \text{on} \quad \{(x_1, x_2);\ x_1 = 0,\ x_2 \in \langle 0, 1 \rangle\}\,.$$

Let the functions $g_{1\mathrm{L}}, g'_{2\mathrm{L}}, g'_{1\mathrm{D}}, g_{2\mathrm{D}}, g'_{1\mathrm{H}}, g'_{2\mathrm{H}}, F^0_1, F^0_2$ be Lipschitz and let

$$g_{1\mathrm{D}}(0) = g_{2\mathrm{L}}(0)\,,$$
$$-g_{1\mathrm{H}}(0) = g_{2\mathrm{L}}(1)\,.$$

Then the condition (2.1) is fulfilled. (The function ξ can be constructed explicitly.)

If we assume (2.1), there exists a unique solution of the state problem (1.3) on the domain $\Omega(v)$ for all $v \in \mathscr{U}_{\mathrm{ad}}$. This solution will be denoted by $\boldsymbol{\sigma}(v) = (\sigma(v), \alpha(v))$.

Let a function $f_1\colon R^4_{\mathrm{sym}} \to R$ be given such that

$$(2.2) \qquad f_1(0) = 0\,,$$
$$|f_1(\sigma_1) - f_1(\sigma_2)| \leqq L_1 \|\sigma_1 - \sigma_2\|_{R_{\mathrm{sym}}^4} \quad \forall \sigma_1, \sigma_2 \in R^4_{\mathrm{sym}}\,.$$

For all $v \in \mathscr{U}_{\mathrm{ad}}$, $\sigma \in H^1_0(I, S(\Omega(v)))$ we define

$$j(v, \sigma) = \int_0^T \int_{\Omega(v)} f_1^2(\sigma(t, x))\,\mathrm{d}x\,\mathrm{d}t\,.$$

We shall now define the cost functional on the set $\mathscr{U}_{\text{ad}}$ as follows:

(2.3) $$\mathscr{J}(v) = j(v, \sigma(v)) .$$

Now we can state the *Optimal Design Problem*:
Find a function $u \in \mathscr{U}_{\text{ad}}$ such that

(2.4) $$\mathscr{J}(u) \leqq \mathscr{J}(v) \quad \forall v \in \mathscr{U}_{\text{ad}} .$$

3. FORMULATION OF THE APPROXIMATE PROBLEM

Let M be a positive integer and $h = 1/M$. We denote by e_j, $j = 1, \ldots, M$, the subintervals $\langle (j-1)\, h, jh \rangle$ and introduce the set

$$\mathscr{U}^h_{\text{ad}} = \{v_h \in \mathscr{U}_{\text{ad}};\ v_h|_{e_j} \in P_1(e_j),\ j = 1, \ldots, M\} ,$$

where $P_1(e_j)$ denotes the set of linear polynomials on e_j.

Let us recall the following lemma (see Begis-Glowinski [1]).

Lemma 3.1. *For any $v \in \mathscr{U}_{\text{ad}}$ there exists a sequence $\{v_h\}$ (h is of the type $1/M$) such that*

$$v_h \in \mathscr{U}^h_{\text{ad}} , \quad v_h \xrightarrow[h \to 0]{} v \quad in \quad C(\langle 0, 1 \rangle) .$$

Definition 3.1. *Let a sequence of positive parameters h_j, $h_j \xrightarrow[j \to \infty]{} 0$ be given. Let a polygonal domain Ω_j and a triangulation $\mathscr{T}_{h_j}(\Omega_j)$ be given for every j.*

A set of triangulations $\{\mathscr{T}_{h_j}(\Omega_j)\}$ is said to be a **system**, *if there exists a positive C such that* diam $Q \leqq Ch_j$ *for all j and for all triangles $Q \in \mathscr{T}_{h_j}(\Omega_j)$.*

A system of triangulations $\{\mathscr{T}_{h_j}(\Omega_j)\}$ is said to be **regular**, *if*

$$\exists \omega_0 > 0 \quad \forall_j \quad \forall Q \in \mathscr{T}_{h_j}(\Omega_j) ; \quad \omega(Q) \geqq \omega_0 ,$$

where $\omega(Q)$ is the minimal angle in Q.

A regular system of triangulations $\{\mathscr{T}_{h_j}(\Omega_j)\}$ is siad to be **strongly regular**, *if*

$$\exists C' > 0 \quad \forall_j ; \quad \frac{d_j}{d'_j} \leqq C' ,$$

where d_j, d'_j are lengths of arbitrary two sides of $\mathscr{T}_{h_j}(\Omega_j)$.

Triangulation of the domain

Let Ω_h denote the domain $\Omega(v_h)$. We choose $a_0 \in (0, a)$ (independently of h) and divide the rectangle $\mathscr{R} = \langle 0, a_0 \rangle \times \langle 0, 1 \rangle$ uniformly into $M_2 . M$ rectangles, where $M_2 = 1 + [a_0 M]$ (the square brackets denote the integer part). In the remaining part $\Omega_h \setminus \mathscr{R}$ let the nodal points divide the intervals $\langle a_0, v_h(jh) \rangle$ into M_1 equal segments, where $M_1 = 1 + [(b - a_0) M]$. Every quadrangle is divided into two triangles as shown in Fig. 2.

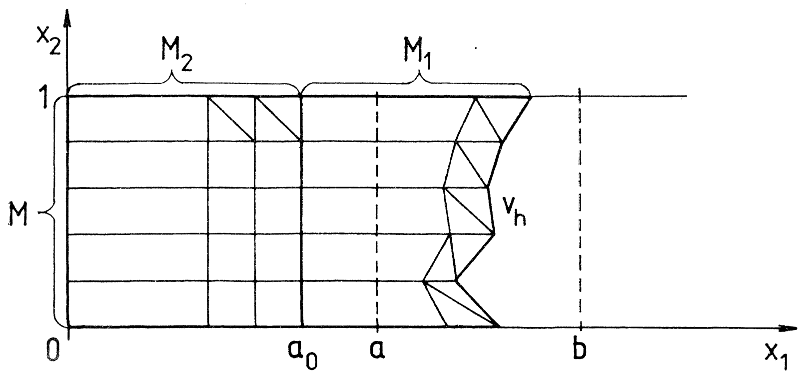

Fig. 2

In this way, for any $v_h \in \mathscr{U}^h_{ad}$ we construct a unique triangulation, which will be denoted by $\mathscr{T}_h(v_h)$.

The set of triangulations $\{\mathscr{T}_h(v_h)\}$, $h \to 0$, $v_h \in \mathscr{U}^h_{ad}$ forms a strongly regular system of triangulations (see Hlaváček [5]).

Denoting the triangles of $\mathscr{T}_h(v_h)$ by Q and the space of polynomials of degree at most s on Q by $P_s(Q)$, we define the following finite element spaces:

$$S_h(\Omega_h) = \{\tau_h \in S(\Omega_h);\ \tau_h|_Q \in [P_0(Q)]^4 \quad \forall Q \in \mathscr{T}_h(v_h)\}\,,$$

$$Z_h(\Omega_h) = \{\beta_h \in L^2(\Omega_h);\ \beta_h|_Q \in P_0(Q) \quad \forall Q \in \mathscr{T}_h(v_h)\}\,,$$

$$V_h(\Omega_h) = \{w_h \in V(\Omega_h);\ w_h|_Q \in [P_1(Q)]^2 \quad \forall Q \in \mathscr{T}_h(v_h)\}\,,$$

and the external approximation of the set $E(\Omega_h, t)$:

$$E_h(\Omega_h, t) = \{\tau_h \in S_h(\Omega_h);\ \langle \tau_h, e(w_h)\rangle_{S(\Omega_h)} = L_{\Omega_h}(w_h, t) \quad \forall w_h \in V_h(\Omega_h)\}\,.$$

Finally, we define the external approximation of the set $K(\Omega_h, t)$:

$$K_h(\Omega_h, t) = (E_h(\Omega_h, t) \times Z_h(\Omega_h)) \cap P(\Omega_h)\,.$$

Let us define some orthogonal projections into the above finite element spaces:

$$r_h \colon S(\Omega_h) \to S_h(\Omega_h)\,;$$

(3.1) $$\langle \tau - r_h\tau, \sigma_h \rangle_{S(\Omega_h)} = 0 \quad \forall \sigma_h \in S_h(\Omega_h)\,,$$

$$r'_h \colon L^2(\Omega_h) \to Z_h(\Omega_h)\,;$$

(3.2) $$\langle \beta - r'_h\beta, \alpha_h \rangle_{L^2(\Omega_h)} = 0 \quad \forall \alpha_h \in Z_h(\Omega_h)\,,$$

$$\boldsymbol{r}_h \colon H(\Omega_h) \to S_h(\Omega_h) \times Z_h(\Omega_h)\,;$$

(3.3) $$\langle \boldsymbol{\tau} - \boldsymbol{r}_h\boldsymbol{\tau}, \boldsymbol{\sigma}_h \rangle_{\Omega_h} = 0 \quad \forall \boldsymbol{\sigma}_h \in S_h(\Omega_h) \times Z_h(\Omega_h)\,.$$

It is readily seen that $\boldsymbol{r}_h\boldsymbol{\tau} = (r_h\tau, r'_h\beta)$ for $\boldsymbol{\tau} = (\tau, \beta)$. These projections have the mean value property, i.e.

$$(r_h\tau)_{ij}|_Q = \frac{1}{\mu(Q)} \int_\Omega \tau_{ij}\,\mathrm{d}x \quad \forall Q \in \mathscr{T}_h(v_h) \quad \forall \tau \in S(\Omega_h)\,, \quad i, j = 1, 2\,,$$

where $\mu(Q)$ is the area of the triangle Q (and similarly for r'_h, $\boldsymbol{r}_h$).

Next we shall study approximative properties of finite elements for non-smooth functions. First we shall restrict the choice of partition of the interval $\langle 0, 1\rangle$. For a sequence of positive numbers $\mathscr{H} = \{h_j\}$, we introduce the following condition:

$$(3.4) \qquad h_{j+1} = \frac{1}{p_j}\,h_j\,, \quad h_0 = 1\,,$$

where p_j are positive integers greater than 1. (An example of such a sequence is $h_j = 1/2^j$.) For simplicity, we shall henceforth write only h.

Lemma 3.2. *Let a sequence $\mathscr{H}$ satisfying (3.4) be given. For every $h \in \mathscr{H}$ let us have a function $v_h \in \mathscr{U}^h_{\mathrm{ad}}$ and a triangulation $\mathscr{T}_h(v_h)$ on the domain $\Omega_h = \Omega(v_h)$.*

Let H be a fixed number from the sequence $\mathscr{H}$ and let $\Omega^H \subset \Omega_\delta$ be a fixed polygonal domain (with the corresponding part of the boundary piecewise linear on the intervals $e_i = \langle (i-1)\,H, iH\rangle$). We suppose the following condition for Ω^H:

$$(3.5) \qquad \exists h^{(0)} > 0\,; \quad \forall h \in \mathscr{H}\,, \quad h < h^{(0)}\,; \quad \Omega_h \subset \Omega^H\,.$$

For every $h \in \mathscr{H}$, $h < h^{(0)}$, we extend the triangulation $\mathscr{T}_h(v_h)$ to the domain Ω^H. This extended triangulation will be denoted by $\mathscr{T}^H_h$. This construction can be done in such a way that the set $\{\mathscr{T}^H_h\}$ forms a system.

On the triangulation $\mathscr{T}^H_h$ we can define finite element spaces $Z_h(\Omega^H)$, $S_h(\Omega^H)$ (analogously to those for $\mathscr{T}_h(v_h)$) and the orthogonal projection

$$r'^H_h \colon L^2(\Omega^H) \to Z_h(\Omega^H)$$

and similarly r^H_h, $\boldsymbol{r}^H_h$.

For every $\beta \in L^2(\Omega^H)$ we then have the following convergence:

$$(3.6) \qquad \lim_{h\to 0} \|r'^H_h\beta - \beta\|_{L^2(\Omega^H)} = 0\,.$$

Similar convergence results are true for the other projections.

Corollary. *Let us assume (3.4), (3.5). From the mean value property we see that for $\beta \in L^2(\Omega^H)$ the projection $r'^H_h\beta$ is an extension of $r'_h\beta$ to the domain Ω^H. We have (from (3.6))*

$$\lim_{h\to 0} \|r'_h\beta - \beta\|_{L^2(\Omega_h)} = 0$$

(and likewise for the other projections).

Proof of Lemma 3.2. We shall prove only the convergence (3.6). Let us have an arbitrary $\varepsilon > 0$. Obviously there exists $\beta_\varepsilon \in C_0^\infty(\Omega^H)$ such that $\|\beta - \beta_\varepsilon\|_{L^2(\Omega^H)} \leqq \varepsilon/2$. Using the mean value theorem we obtain

$$\forall Q_i \in \mathscr{T}_h^H \quad \exists x_i' \in Q_i \, ; \quad \frac{1}{\mu(Q_i)} \int_{Q_i} \beta_\varepsilon \, \mathrm{d}x = \beta_\varepsilon(x_i') \, .$$

The boundedness of derivatives of the function β_ε together with the property of the system of triangulations imply the estimate

$$|\beta_\varepsilon(x) - \beta_\varepsilon(x_i')| \leqq C_\varepsilon h \quad \forall x \in Q_i (Q_i \in \mathscr{T}_h^H) \, .$$

Finally, we arrive at the following approximation result for the smooth function β_ε:

$$\|\beta_\varepsilon - r_h'^H \beta_\varepsilon\|_{L^2(\Omega^H)}^2 = \sum_{Q_i \in \mathscr{T}_h^H} \int_{Q_i} \left[\beta_\varepsilon - \frac{1}{\mu(Q_i)} \int_{Q_i} \beta_\varepsilon \right]^2 \mathrm{d}x =$$

$$= \sum_{Q_i} \int_{Q_i} [\beta_\varepsilon(x) - \beta_\varepsilon(x_i')]^2 \, \mathrm{d}x \leqq \sum_{Q_i} \int_{Q_i} C_\varepsilon^2 h^2 \, \mathrm{d}x = C_\varepsilon^2 h^2 \mu(\Omega^H) \, .$$

If we set

$$h(\varepsilon) = \frac{\varepsilon}{2 C_\varepsilon \sqrt{(\mu(\Omega^H))}} \, ,$$

then for all $h \in \mathscr{H}$, $h \leqq h(\varepsilon)$ we obtain

$$\|r_h'^H \beta - \beta\|_{L^2(\Omega^H)} \leqq \|r_h'^H \beta_\varepsilon - \beta\|_{L^2(\Omega^H)} \leqq$$

$$\leqq \|r_h'^H \beta_\varepsilon - \beta_\varepsilon\|_{L^2(\Omega^H)} + \|\beta_\varepsilon - \beta\|_{L^2(\Omega^H)} \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(here also the minimal property of the orthogonal projection has been used).

The time discretization

Let N be a positive integer and $k = T/N$. We shall divide the interval $I = \langle 0, T \rangle$ into N subintervals $I_m = \langle (m - 1) k, mk \rangle$, $m = 1, \ldots, N$. Let us denote $t^m = mk$.

For the function γ we shall assume that there exists such a partition of the interval I (characterized by some N_0) that γ is piecewise monotone on this partition.

Further we shall consider the sequence of partitions characterised by the sequence of positive numbers $\mathscr{K} = \{k_j\}$ which satisfy the condition

$$k_j = \frac{T}{q_j N_0} \, , \tag{3.7}$$

where q_j is an increasing sequence of positive integers. (We will write only k instead of k_j sometimes.)

Let us put

$$\boldsymbol{\sigma}^m = \boldsymbol{\sigma}(t^m),$$

$$\partial\boldsymbol{\sigma}^m = (\boldsymbol{\sigma}^m - \boldsymbol{\sigma}^{m-1})/k.$$

Now the *Approximate State Problem* can be formulated (for fixed h, k) on the domain Ω_h as follows:

Find $\boldsymbol{\sigma}_{hk} = (\boldsymbol{\sigma}_{hk}^1, \ldots, \boldsymbol{\sigma}_{hk}^N)$ such that for any $m = 1, \ldots, N$ the following relations hold:

(3.8) $$\boldsymbol{\sigma}_{hk}^m \in K_h(\Omega_h, t^m),$$

$$\{\partial\boldsymbol{\sigma}_{hk}^m, \boldsymbol{\tau} - \boldsymbol{\sigma}_{hk}^m\}_{\Omega_h} \geqq 0 \quad \forall \boldsymbol{\tau} \in K_h(\Omega_h, t^m),$$

$$\text{where} \quad \boldsymbol{\sigma}_{hk}^0 = (0, r_h'\alpha_0).$$

In what follows we will use the notation $\boldsymbol{\sigma}_{hk}^m = (\sigma_{hk}^m, \alpha_{hk}^m)$.

Let a quadrature formula with $N + 1$ equidistant nodes $t^0, \ldots, t^N$ and with positive coefficients be given:

$$\int_I G(t)\,\mathrm{d}t \sim k \sum_{m=0}^{N} c_m\, G(t^m),$$

which is convergent for continuous functions (e.g. the trapezoidal or Simpson rule).

For all $v_h \in \mathscr{U}_{\mathrm{ad}}^h$, $\sigma_{hk} \in [S_h(\Omega_h)]^N$, we define

$$j_{hk}(v_h, \sigma_{hk}) = k \sum_{m=1}^{N} c_m \sum_{Q \in \mathscr{T}_h(v_h)} \mu(Q)\, f_1^2(\sigma_{hk}^m|_Q).$$

Let us define the approximate cost functional on the set $\mathscr{U}_{\mathrm{ad}}^h$ by

(3.9) $$\mathscr{J}_{hk}(v_h) = j_{hk}(v_h, \sigma_{hk}(v_h)),$$

where $\sigma_{hk}(v_h)$ is the first component of the solution of the approximate state problem (3.8) on the domain $\Omega(v_h)$.

Now we are able to state the *Approximate Optimal Design Problem* (for given h, k).

Find a function $u_h^{(k)} \in \mathscr{U}_{\mathrm{ad}}^h$ such that

(3.10) $$\mathscr{J}_{hk}(u_h^{(k)}) \leqq \mathscr{J}_{hk}(v_h) \quad \forall v_h \in \mathscr{U}_{\mathrm{ad}}^h.$$

In the following section we shall prove that the Approximate State Problem (3.8) has a unique solution, so that the problem (3.10) has sense.

4. EXISTENCE AND UNIQUENESS OF SOLUTION OF THE APPROXIMATE STATE PROBLEM

Lemma 4.1. *If $\tau \in P(\Omega_h)$ then $r_h\tau \in P(\Omega_h)$ as well.*

Proof. For every $\tau \in P(\Omega_h)$ and for every triangle $Q \in \mathscr{T}_h(v_h)$ we want to prove the following inequality:

$$f\left(\frac{1}{\mu(Q)} \int_Q \tau(x)\,\mathrm{d}x\right) \leqq \frac{1}{\mu(Q)} \int_Q \beta(x)\,\mathrm{d}x\,. \tag{4.1}$$

Let us choose an arbitrary triangle $Q \in \mathscr{T}_h(v_h)$. On this triangle we can construct a system of triangulations $\mathscr{T}_\ell(Q)$, $\ell \to 0$. Analogously as in Lemma 3.2, for the function $\tau \in S(Q)$ we can construct a sequence of functions

$$\tau_\ell \in S(Q)\,, \quad \tau_\ell|_{Q_r} \in [P_0(Q_r)]^4 \quad \forall Q_r \in \mathscr{T}_\ell(Q)$$

such that

$$\tau_\ell \to \tau \quad \text{in} \quad S(Q)\,. \tag{4.2}$$

From the convexity of the function f we can prove the following inequality:

$$f\left(\frac{1}{\mu(Q)} \int_Q \tau_\ell(x)\,\mathrm{d}x\right) \leqq \frac{1}{\mu(Q)} \int_Q f(\tau_\ell(x))\,\mathrm{d}x\,. \tag{4.3}$$

The convergence (4.2) and the continuity of f imply the convergence of the left-hand side:

$$f\left(\frac{1}{\mu(Q)} \int_Q \tau_\ell(x)\,\mathrm{d}x\right) \xrightarrow[\ell \to 0]{} f\left(\frac{1}{\mu(Q)} \int_Q \tau(x)\,\mathrm{d}x\right).$$

We can consider f as a mapping from $S(Q)$ to $L^2(Q)$ ($f\colon \tau \mapsto f(\tau)$). Then from the Lipschitz property of f we conclude that f is continuous. Together with (4.2) we have $f(\tau_\ell) \to f(\tau)$ in $L^2(Q)$ and therefore

$$\frac{1}{\mu(Q)} \int_Q f\,(\tau_\ell(x))\,\mathrm{d}x \xrightarrow[\ell \to 0]{} \frac{1}{\mu(Q)} \int_Q f\,(\tau(x))\,\mathrm{d}x\,.$$

Consequently, we can pass to the limit with $\ell \to 0$ in (4.3). We obtain the integral analogue of the Jensen inequality:

$$f\left(\frac{1}{\mu(Q)} \int_Q \tau(x)\,\mathrm{d}x\right) \leqq \frac{1}{\mu(Q)} \int_Q f(\tau(x))\,\mathrm{d}x\,.$$

The assumption $\tau \in P(\Omega_h)$ yields $f(\tau) \leqq \beta$ a.e. in Ω_h and so the inequality (4.1) holds.

Now we can proceed to the proof of the existence theorem.

Theorem 4.1. *Let the assumption* (2.1) *be fulfilled. Let a function* $v_h \in \mathscr{U}^h_{ad}$ *be given. Then the Approximate State Problem* (3.8) *has a unique solution on the domain* $\Omega_h = \Omega(v_h)$.

Proof.

First we show that $K_h(\Omega_h, t^m)$ is non-empty.

If we assume (2.1), there exists

$$\xi(t) = (\xi(t), \zeta(t)) \in K(\Omega_h, t) \quad \forall t \in I \quad (\text{see Lemma } 1.2)\,.$$

Lemma 4.1 yields

$$r_h\, \xi(t) \in P(\Omega_h) \quad \forall t \in I\,.$$

Let w_h be an arbitrary function from the space $V_h(\Omega_h) \subset V(\Omega_h)$. Then $e(w_h) \in S_h(\Omega_h)$. Using the property (3.1) and the definition of $E(\Omega_h, t)$, we obtain

$$\langle e(w_h), r_h\, \xi(t) \rangle_{S(\Omega_h)} = \langle e(w_h), \xi(t) \rangle_{S(\Omega_h)} = L_{\Omega_h}(w_h, t)\,.$$

Consequently, the projection $r_h\, \xi(t)$ is an element of $K_h(\Omega_h, t)$. In particular, $K_h(\Omega_h, t^m) \neq \emptyset$ for $m = 1, \ldots, N$.

The problem (3.8) is equivalent (at the m-th time level) to the minimization of the functional

$$\tau \mapsto \tfrac{1}{2} |||\tau|||^2_{\Omega_h} - \{\tau, \sigma^{m-1}_{hk}\}_{\Omega_h}$$

over the set $K_h(\Omega_h, t^m)$.

It is not difficult to show that this functional is strictly convex, coercive and weakly lower semicontinuous. The set $K_h(\Omega_h, t^m)$ is non-empty, convex and closed. There exists a unique minimizer σ^m_{hk} (which solves this minimization problem) for any fixed σ^{m-1}_{hk} (see Céa [2], Chap. IV, Theorems 0.2, 0.3).

Since $\sigma^0_{hk} = (0, r'_h \alpha_0)$ is unique, we obtain by induction that the Approximate State Problem (3.8) has a unique solution.

5. EXISTENCE OF SOLUTION OF THE APPROXIMATE OPTIMAL DESIGN PROBLEM

First we prove an abstract lemma on the continuous dependence of the minimal point of a functional on a parameter.

Lemma 5.1. *Let* H *be a Hilbert space with the scalar product* $(\cdot, \cdot)$ *and the norm* $\|\cdot\|$.

Let a metric space X *with a metric* $d(\cdot, \cdot)$ *and a set of admissible parameters* $U \subset X$ *be given. Let* u *be an arbitrary but fixed element of* U.

Let a functional J_v on the space H be given for every $v \in U$. We assume that this functional has the Gâteaux derivative everywhere in H with the following properties:

$$\exists c > 0\,; \quad (J_v'(\tau) - J_v'(\sigma), \tau - \sigma) \geqq c\|\tau - \sigma\|^2 \quad \forall \sigma, \tau \in H$$
$$(\textit{strong monotonicity}),$$
$$\exists L_0 > 0\,; \quad \|J_v'(\tau) - J_v'(\sigma)\| \leqq L_0\|\tau - \sigma\| \quad \forall \sigma, \tau \in H$$
$$(\textit{Lipschitz continuity}),$$

where the constants c, L_0 are independent of v.

Let a convex, closed, non-empty set $K_v \subset H$ be given for every $v \in U$.

Then for every $v \in U$ there exists a unique element $\sigma_v \in K_v$ such that

$$J_v(\sigma_v) \leqq J_v(\tau) \quad \forall \tau \in K_v\,.$$

Moreover, let the following assumptions be fulfilled:

$$(5.1) \qquad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall v \in U\,; \quad d(v, u) < \delta \Rightarrow \|J_v'(\sigma_u) - J_u'(\sigma_u)\| < \varepsilon\,,$$

$$(5.2) \qquad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall v \in U\,; \quad d(v, u) < \delta \Rightarrow \begin{cases} \exists \tau_v \in K_v\,; & \|\sigma_u - \tau_v\| < \varepsilon\,, \\ \exists \tau_u \in K_u\,; & \|\sigma_v - \tau_u\| < \varepsilon\,. \end{cases}$$

Then the minimal point σ_v depends continuously on the parameter v at the point u, i.e.:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall v \in U\,; \quad d(v, u) < \delta \Rightarrow \|\sigma_v - \sigma_u\| < \varepsilon\,.$$

Proof. The existence of the minimal point σ_v may be proved in the standard way (see Céa [2]).

Let us prove the continuous dependence. For any $\varepsilon > 0$ there exists δ such that

$$(5.3) \qquad d(v, u) < \delta \Rightarrow \begin{cases} \|J_v'(\sigma_u) - J_u'(\sigma_u)\| < \varepsilon^2\,, \\ \exists \tau_v \in K_v\,; \quad \|\sigma_u - \tau_v\| < \varepsilon^2\,, \\ \exists \tau_u \in K_u\,; \quad \|\sigma_v - \tau_u\| < \varepsilon^2\,. \end{cases}$$

We shall consider some $v \in U$ such that $d(v, u) < \delta$. The elements σ_u, σ_v satisfy the following variational inequalities:

$$(J_v'(\sigma_v), \tau - \sigma_v) \geqq 0 \quad \forall \tau \in K_v\,,$$
$$(J_u'(\sigma_u), \tau - \sigma_u) \geqq 0 \quad \forall \tau \in K_u\,.$$

Substituting $\tau = \tau_v$ into the first inequality and $\tau = \tau_u$ into the second and adding them we obtain

$$(J_v'(\sigma_u) - J_v'(\sigma_v), \sigma_u - \sigma_v) \leqq (J_v'(\sigma_v), \tau_v - \sigma_u) + (J_u'(\sigma_u), \tau_u - \sigma_v) +$$
$$+ (J_v'(\sigma_u) - J_u'(\sigma_u), \sigma_u - \sigma_v)\,.$$

Making use of the strong monotonicity of J'_v, the Schwarz inequality and (5.3), we derive the estimate

$$c\|\sigma_u - \sigma_v\|^2 \leqq (\|J'_v(\sigma_v)\| + \|J'_u(\sigma_u)\| + \|\sigma_u - \sigma_v\|)\,\varepsilon^2 .$$

Using again (5.3) and the Lipschitz continuity of J'_v, we obtain

$$\|J'_v(\sigma_v)\| \leqq \|J'_v(\sigma_v) - J'_v(\sigma_u)\| + \|J'_v(\sigma_u) - J'_u(\sigma_u)\| + \|J'_u(\sigma_u)\| \leqq$$
$$\leqq L_0\|\sigma_v - \sigma_u\| + \varepsilon^2 + \|J'_u(\sigma_u)\| .$$

Denoting $c_{1u} = \|J'_u(\sigma_u)\|$, we may write

$$c\|\sigma_u - \sigma_v\|^2 \leqq 2c_{1u}\varepsilon^2 + \varepsilon^4 + (1 + L_0)\,\|\sigma_u - \sigma_v\|\,\varepsilon^2 .$$

The inequality is true for all $v \in U$ satisfying $d(v, u) < \delta$. For such v the estimate $\|\sigma_u - \sigma_v\| \leqq c_{2u}$ holds and consequently,

$$\|\sigma_u - \sigma_v\| \leqq c_{3u}\varepsilon \quad \forall v \in U\,; \quad d(v, u) < \delta .$$

The mapping $v \mapsto \sigma_v$ is thus continuous at the point u.

Next we are going to prove the boundedness of the solution of the Approximate State Problem.

Lemma 5.2. *Let the assumption* (2.1) *be fulfilled.*

Let a sequence $\mathscr{H}$ and a fixed number $k < t_1$ be given. Let a function $v_h \in \mathscr{U}^h_{\mathrm{ad}}$ be given for every $h \in \mathscr{H}$. The solution of the Approximate State Problem (3.8) *on the domain $\Omega_h = \Omega(v_h)$ will be denoted by $\boldsymbol{\sigma}_{hk} = (\boldsymbol{\sigma}^1_{hk}, \ldots, \boldsymbol{\sigma}^N_{hk})$ and its extension by zero to the domain $\Omega_\delta \setminus \Omega_h$ by $\bar{\boldsymbol{\sigma}}_{hk}$.*

Then there exists a constant C (independently of h, v_h, m) such that

$$(5.4) \qquad \|\bar{\boldsymbol{\sigma}}^m_{hk}\|_{\Omega_\delta} \leqq C \quad \textit{for} \quad m = 1, \ldots, N .$$

Proof. We shall use the induction:

I. $m = 1$

Using the assumption $k < t_1$ we obtain the assertion that the pair $(0, r'_h\alpha_0)$ is in $K_h(\Omega_h, t^1)$. This pair also fulfils the inequality (3.8) and therefore

$$(5.5) \qquad \boldsymbol{\sigma}^1_{hk} = (0, r'_h\alpha_0) \quad \text{in} \quad \Omega_h .$$

From the property of the orthogonal projection we have

$$\|(0, r'_h\alpha_0)\|_{0,\Omega_\delta} = \|r'_h\alpha_0\|_{L^2(\Omega_h)} \leqq \|\alpha_0\|_{L^2(\Omega_h)} \leqq \|\bar{\alpha}_0\|_{L^2(\Omega_\delta)} = C .$$

Using the equivalence of norms, we obtain

$$\|\bar{\boldsymbol{\sigma}}^1_{hk}\|_{\Omega_\delta} \leqq C_1 \quad (\text{with } C_1 \text{ independent of } h, v_h) .$$

II. The induction step

Let us assume that $|||\bar{\sigma}_{hk}^{m-1}|||_{\Omega_\delta} \leqq C_{m-1}$.

We shall substitute $\tau = r_h\, \xi(t^m) \in K_h(\Omega_h, t^m)$ in the inequality (3.8), where $\xi(t)$ is the function from Lemma 1.2 (obviously $|||r_h\, \xi(t^m)|||_{\Omega_h} \leqq C'$ holds, with C' independent of h, v_h, m).

We obtain

$$|||\sigma_{hk}^m|||_{\Omega_h}^2 \leqq \{\sigma_{hk}^{m-1}, \sigma_{hk}^m\}_{\Omega_h} + \{\sigma_{hk}^m - \sigma_{hk}^{m-1}, r_h\, \xi(t^m)\}_{\Omega_h} \leqq$$
$$\leqq C_{m-1}\, |||\sigma_{hk}^m|||_{\Omega_h} + C'(|||\sigma_{hk}^m|||_{\Omega_h} + C_{m-1}) =$$
$$= (C' + C_{m-1})\, |||\sigma_{hk}^m|||_{\Omega_h} + C_{m-1}C'\,.$$

Consequently, there exists a constant C_m (which depends on C_{m-1}) such that $|||\sigma_{hk}^m|||_{\Omega_h} \leqq C_m$. As k (and also N) is fixed, there exists a constant C (independent of h, v_h, m) which satisfies (5.4).

Lemma 5.3. *Let h, k be fixed numbers, $k < t_1$. Let u_h be an arbitrary (but fixed) function from $\mathscr{U}_{ad}^h$. We assume that (2.1) is fulfilled.*

Then, at every time level, the solution of the Approximate State Problem (3.8) $\sigma_{hk}^m(v_h)$ depends continuously on the control variable v_h at the point u_h.

Proof. For simplicity, we drop the subscripts h, k, whenever it is possible. We shall prove the lemma by induction over m.

I. $m = 1$

Assuming $k < t_1$, we have (as in (5.5)) $\sigma_{hk}^1(v) = (0, r'_{h(v)}\alpha_0)$, where $r'_{h(v)}$ is the projection corresponding to the triangulation $\mathscr{T}_h(v)$. The number of triangles in this triangulation will be denoted by $n = 2M(M_1 + M_2)$.

Let $v \to u$ in $C(\langle 0, 1\rangle)$. Let $Q_i(v)$ be a triangle in $\mathscr{T}_h(v)$, $Q_i(u)$ the corresponding triangle in $\mathscr{T}_h(u)$. Then we obviously have

$$\mu(Q_i(v)) \xrightarrow[v\to u]{} \mu(Q_i(u))\,, \tag{5.6}$$
$$\int_{Q_i(v)} \alpha_0\, \mathrm{d}x \xrightarrow[v\to u]{} \int_{Q_i(u)} \alpha_0\, \mathrm{d}x$$

and therefore

$$r'_{h(v)}\alpha_0|_{Q_i(v)} \xrightarrow[v\to u]{} r'_{h(u)}\alpha_0|_{Q_i(u)}\,, \quad i = 1, \ldots, n$$

(as the convergence of real numbers).

II. The induction step

We assume that $\sigma^{m-1}(v)$ depends continuously on v at the point u. The problem (3.8) is (at m-th time level) equivalent to searching an element $\sigma^m(v)$ for which the functional

$$J_v(\tau) = \tfrac{1}{2}|||\tau|||_{\Omega(v)}^2 - \{\tau, \sigma^{m-1}(v)\}_{\Omega(v)}$$

attains its minimum over the set $E_h(\Omega(v), t^m) \times Z_h(\Omega(v))$ with the additional conditions

$$(5.7) \qquad f(\tau|_{Q_i}) \leqq \beta|_{Q_i}, \quad i = 1, \ldots, n .$$

We shall formulate this problem using the vector notation. The functions from the space $S_h(\Omega(v))$ will be considered as vectors from R^{3n} and denoted by $\tau = (\tau_1, \ldots, \tau_n)^{\mathsf{T}}$, where $\tau_j = (\tau_{11j}, \tau_{12j}, \tau_{22j})^{\mathsf{T}}$ corresponds to the function τ on the triangle Q_j, $j = 1, \ldots, n$. Similarly, the elements from $Z_h(\Omega(v))$ will be denoted by $\beta = (\beta_1, \ldots, \beta_n)^{\mathsf{T}}$, $\beta_j \in R$. Let us denote $\boldsymbol{\tau} = (\tau, \beta)^{\mathsf{T}} \in R^{4n}$.

The functional $J_v(\boldsymbol{\tau})$ can be written in the form

$$(5.8) \qquad J_v(\boldsymbol{\tau}) = \tfrac{1}{2}\boldsymbol{\tau}^{\mathsf{T}}\, C(v)\, \boldsymbol{\tau} - \boldsymbol{\tau}^{\mathsf{T}}\, C(v)\, \boldsymbol{\sigma}^{m-1}(v) ,$$

where

$$C(v) = \begin{pmatrix} B(v) & 0 \\ 0 & D(v) \end{pmatrix},$$

and $B(v)$ is a block diagonal matrix of order $3n$ which has the following blocks on the diagonal:

$$B_i(v) = \begin{pmatrix} \int b_{1111} & 2\int b_{1112} & \int b_{1122} \\ 2\int b_{1211} & 4\int b_{1212} & 2\int b_{1222} \\ \int b_{2211} & 2\int b_{2212} & \int b_{2222} \end{pmatrix}$$

(all the integrals are over the domain $Q_i(v)$).

The matrix $D(v)$ is diagonal of order n, with the diagonal elements

$$D_{ii}(v) = \int_{Q_i(v)} \varkappa(x)\, \mathrm{d}x .$$

We can easily verify that $C(v)$ is symmetric and positive definite, i.e.

$$\exists C_2 > 0 \quad \forall v \in \mathcal{U}^h_{\mathrm{ad}} \quad \forall \boldsymbol{\tau} \in R^{4n}; \quad \boldsymbol{\tau}^{\mathsf{T}}\, C(v)\, \boldsymbol{\tau} \geqq C_2 \|\boldsymbol{\tau}\|^2_{R^{4n}} ,$$

where C_2 is independent of v. We can also prove that the elements of $C(v)$ depend continuously on v and that the norm of $C(v)$ is bounded by a constant independent of v.

The condition $\tau \in E_h(\Omega(v), t^m)$ can be written in the form

$$(5.9) \qquad A(v)\, \tau = L(v) ,$$

where $A(v)$ is an $n' \times 3n$ matrix, $L(v) \in R^{n'}$, $n' = 2(M+1)(M_1 + M_2)$ is the dimension of the space $V_h(\Omega(v))$. The elements of $A(v)$ are the scalar products

$$\langle \vartheta_j^{(11)}, e(w_i) \rangle_{S(\Omega(v))}, \ \langle \vartheta_j^{(12)}, e(w_i) \rangle_{S(\Omega(v))}, \ \langle \vartheta_j^{(22)}, e(w_i) \rangle_{S(\Omega(v))} ,$$

where $\vartheta_j^{(kl)}$ and w_i are the basis functions of $S_h(\Omega(v))$ and $V_h(\Omega(v))$, respectively. The elements of $A(v)$ and $L(v)$ depend continuously on v.

Finally, we have the following minimization problem:

$$(5.10)\qquad \begin{aligned} &J_v(\tau) = \tfrac{1}{2}\tau^{\mathsf T}\, C(v)\, \tau - \tau^{\mathsf T}\, C(v)\, \sigma^{m-1}(v) \to \min\,,\\ &A(v)\, \tau = L(v)\,,\\ &\tilde f(\tau_j) \leqq \beta_j\,,\quad j = 1, \dots, n\,, \end{aligned}$$

where $\tilde f\colon R^3 \to R$ corresponds to the function $f\colon R^4_{\mathrm{sym}} \to R$. Obviously, we have

$$|\tilde f(\tau_j) - \tilde f(\sigma_j)| \leqq \sqrt{(2)}\, L \|\tau_j - \sigma_j\|_{R^3}\,.$$

The problem has a unique solution (see Theorem 4.1), which will be denoted by $\boldsymbol{\sigma}^m(v)$.

Now the abstract Lemma 5.1 can be applied. Let us set $H = R^{4n}$ with the Euclidian scalar product, $X = C(\langle 0, 1\rangle)$, $U = \mathscr{U}^h_{\mathrm{ad}}$.

The functionals J_v have the Gâteaux derivatives $J'_v(\boldsymbol{\tau}) = C(v)\, \boldsymbol{\tau} - C(v)\, \boldsymbol{\sigma}^{m-1}(v)$ which fulfil the assumptions of Lemma 5.1. Let us set

$$\begin{aligned} E_v &= \{\tau \in R^{3n};\ A(v)\, \tau = L(v)\}\,,\\ P &= \{\boldsymbol{\tau} \in R^{4n};\ \tilde f(\tau_j) \leqq \beta_j,\ j = 1, \dots, n\}\,,\\ K_v &= (E_v \times R^n) \cap P\,. \end{aligned}$$

Obviously, K_v is a convex, closed and non-empty set. Using Lemma 5.2 and the positive definiteness of $C(v)$, we obtain

$$(5.11)\qquad \|\boldsymbol{\sigma}^m(v)\|_{R^{4n}} \leqq C \quad \forall v \in U\,,\quad m = 1, \dots, N\,.$$

Verification of the assumption (5.1)

$$\begin{aligned} &\|J'_v(\boldsymbol{\sigma}^m(u)) - J'_u(\boldsymbol{\sigma}^m(u))\|_{R^{4n}} =\\ &= \|C(v)\, \boldsymbol{\sigma}^m(u) - C(v)\, \boldsymbol{\sigma}^{m-1}(v) - C(u)\, \boldsymbol{\sigma}^m(u) + C(u)\, \boldsymbol{\sigma}^{m-1}(u)\|_{R^{4n}} \leqq\\ &\leqq \|C(v) - C(u)\|_*\, \|\boldsymbol{\sigma}^m(u)\|_{R^{4n}} + \|C(u)\|_*\, \|\boldsymbol{\sigma}^{m-1}(u) - \boldsymbol{\sigma}^{m-1}(v)\|_{R^{4n}} +\\ &+ \|C(u) - C(v)\|_*\, \|\boldsymbol{\sigma}^{m-1}(v)\|_{R^{4n}} \leqq\\ &\leqq 2C\|C(v) - C(u)\|_* + \|C(u)\|_*\, \|\boldsymbol{\sigma}^{m-1}(u) - \boldsymbol{\sigma}^{m-1}(v)\|_{R^{4n}} \end{aligned}$$

($\|\cdot\|_*$ is the spectral matrix norm).

The convergence of the first member to zero (for $v \to u$) follows from the continuous dependence of the entries of $C(v)$ on the parameter v. The convergence of the second member follows from the induction hypothesis.

Verification of the assumption (5.2)

First we shall prove that the matrices $A(v)$ have full rank (i.e. $r(A(v)) = n'$) for all $v \in U$.

Let the rows of $A(v)$ be linearly dependent, i.e. let there exist numbers $\lambda_1, \ldots, \lambda_{n'}$ (not all zeros) such that

$$\sum_{i=1}^{n'} \lambda_i \langle \vartheta_j^{(11)}, e(w_i) \rangle_{S(\Omega(v))} = 0 \quad \text{for} \quad j = 1, \ldots, n .$$

Consequently,

$$\langle \vartheta_j^{(11)}, e(w) \rangle_{S(\Omega(v))} = 0 \quad j = 1, \ldots, n ,$$

where $w = \sum_{i=1}^{n'} \lambda_i w_i$ is a non-zero element of $V_h(\Omega(v))$. A similar assertion holds for $\vartheta_j^{(12)}, \vartheta_j^{(22)}$.

The tensor $e(w) \in S_h(\Omega(v))$ is orthogonal to all elements of the basis of $S_h(\Omega(v))$, so that it must be zero. Finally, $w = 0$ in $\Omega(v)$ can be deduced, which is a contradiction.

We can choose n' linarly independent columns in the matrix $A(u)$. There exists $\delta_1 > 0$ such that for all $v \in \mathscr{U}_{\mathrm{ad}}^h$ with $d(u, v) < \delta_1$ the same choice of columns of $A(v)$ is linearly independent, too. By a suitable renumeration we can write the condition (5.9) in the form

$$A_1(v)\, \tau^{(1)} + A_2(v)\, \tau^{(2)} = L(v) ,$$

where $A_1(v)$ is a square matrix of order n', nonsingular for all $v \in \mathscr{U}_{\mathrm{ad}}^h$, $d(v, u) < \delta_1$, and $A_2(v)$ is an $n' \times (3n - n')$ matrix. The elements of these matrices depend continuously on v.

The vector $\boldsymbol{\sigma}(u) = (\sigma(u), \alpha(u))^{\mathrm{T}} \in K_u$ fulfils the following conditions:

$$A_1(u)\, \sigma^{(1)}(u) + A_2(u)\, \sigma^{(2)}(u) = L(u) ,$$

$$\tilde{f}(\sigma_i(u)) \leqq \alpha_i(u) , \quad i = 1, \ldots, n .$$

Let us set $\tau_v^{(2)} = \sigma^{(2)}(u)$. The vector $\tau_v^{(1)}$ can be determined from the condition

$$A_1(v)\, \tau_v^{(1)} + A_2(v)\, \tau_v^{(2)} = L(v) ,$$

so that we have

$$\tau_v^{(1)} = A_1^{-1}(v)\, L(v) - A_1^{-1}(v)\, A_2(v)\, \sigma^{(2)}(u) .$$

The norm of the difference can be estimated as follows:

$$\begin{aligned} &\|\sigma(u) - \tau_v\|_{R^{3n}} = \|\sigma^{(1)}(u) - \tau_v^{(1)}\|_{R^{n'}} \leqq \\ &\leqq \|A_1^{-1}(u)\, L(u) - A_1^{-1}(v)\, L(v)\|_{R^{n'}} + \\ &+ \|A_1^{-1}(v)\, A_2(v) - A_1^{-1}(u)\, A_2(u)\|_{**}\, C , \end{aligned}$$

where $\|\cdot\|_{**}$ is the standard norm of linear operators from $R^{3n-n'}$ to $R^{n'}$.

Let us choose $\varepsilon > 0$. As the matrices $A_1(u)$, $A_1(v)$ are nonsingular and the elements of $A_1(v)$, $A_2(v)$, $L(v)$ depend continuously on the parameter v at the point u, there exists $\delta < \delta_1$ such that for all $v \in \mathscr{U}^h_{ad}$ with $d(u, v) < \delta$ the following estimate holds:

$$\|\sigma(u) - \tau_v\|_{R^{3n}} \leqq \frac{\varepsilon}{\sqrt{(1 + 2L^2)}} .$$

Let us return to the original numbering of the components of vectors. Obviously, we have $\tau_v \in E_v$. For the vector τ_v we shall construct a vector β_v such that $\boldsymbol{\tau}_v = (\tau_v, \beta_v)^\mathsf{T} \in P$.

Let us denote by $\mathscr{M}_1$ the set of indices i for which $\tilde{f}(\tau_{vi}) \leqq \alpha_i(u)$,

by $\mathscr{M}_2$ the set of indices i for which $\tilde{f}(\tau_{vi}) > \alpha_i(u)$.

Let us set

$$\beta_{vi} = \alpha_i(u) \quad \text{for} \quad i \in \mathscr{M}_1 ,$$

$$\beta_{vi} = \tilde{f}(\tau_{vi}) \quad \text{for} \quad i \in \mathscr{M}_2 .$$

Obviously, $\tilde{f}(\tau_{vi}) \leqq \beta_{vi}$ holds for $i = 1, \ldots, n$, that is, $\boldsymbol{\tau}_v \in P$.

We estimate:

$$\|\beta_v - \alpha(u)\|^2_{R^n} = \sum_{i \in \mathscr{M}_2} |\tilde{f}(\tau_{vi}) - \alpha_i(u)|^2 \leqq \sum_{i \in \mathscr{M}_2} |\tilde{f}(\tau_{vi}) - \tilde{f}(\sigma_i(u))|^2 \leqq$$

$$\leqq 2L^2 \|\tau_v - \sigma(u)\|^2_{R^{3n}} \leqq 2L^2 \frac{\varepsilon^2}{1 + 2L^2} .$$

Finally, we have

$$\|\boldsymbol{\tau}_v - \boldsymbol{\sigma}(u)\|^2_{R^{4n}} = \|\tau_v - \sigma(u)\|^2_{R^{3n}} + \|\beta_v - \alpha(u)\|^2_{R^n} \leqq \varepsilon^2 .$$

Likewise, we can construct $\boldsymbol{\tau}_u \in K_u$ for $\boldsymbol{\sigma}(v) \in K_v$.

The continuous dependence $\boldsymbol{\sigma}^m(v)$ on the parameter v at the point u follows now from Lemma 5.1.

Theorem 5.1. *Let the assumption* (2.1) *be fulfilled. Then the Approximate Optimal Design Problem* (3.10) *has a solution for arbitrary* $h = 1/M$, $k = T/N$.

Proof. Let h, k be fixed.

There exists a unique solution $\boldsymbol{\sigma}_{hk}(v_h)$ of the problem (3.8) for all $v_h \in \mathscr{U}^h_{ad}$ (see Theorem 4.1). From the previous lemma and the Lipschitz continuity of f_1 we conclude that the approximate cost functional

$$\mathscr{J}_{hk}(v_h) = k \sum_{m=1}^{N} c_m \sum_{Q_i(v_h) \in \mathscr{T}_h(v_h)} \mu(Q_i(v_h)) f_1^2(\sigma^m_{hk}(v_h)|_{Q_i(v_h)})$$

depends continuously on v_h. The set $\mathscr{U}^h_{ad}$ is obviously compact and consequently the problem (3.10) has a solution.

Remark. We cannot say anything concerning the uniqueness of the solution of the Approximate Optimal Design Problem.

In this section we shall follow the arguments of Hlaváček [6]. The main idea is to insert some semidiscrete solution $\boldsymbol{\sigma}_k$ (discretized only in time) between $\boldsymbol{\sigma}_{hk}$ and $\boldsymbol{\sigma}$. We shall also use some results of Johnson [9].

Proposition 6.1. *Let the assumption (2.1) be fulfilled. Let a sequence $\mathscr{H}$ which satisfies (3.4) and a fixed $k < t_2$ be given. Then in particular $h \to 0$, and all limits will be considered for $h \in \mathscr{H}$ (if not stated otherwise). Let a function $v_h \in \mathscr{U}^h_{\mathrm{ad}}$ be given for all $h \in \mathscr{H}$ and let $v_h \to v$ in $C(\langle 0, 1 \rangle)$, where $v \in \mathscr{U}_{\mathrm{ad}}$. We shall denote the solution of the Approximate State Problem (3.8) on the domain $\Omega_h = \Omega(v_h)$ by $\boldsymbol{\sigma}_{hk} = (\boldsymbol{\sigma}^1_{hk}, \ldots, \boldsymbol{\sigma}^N_{hk})$ and its extension by zero into $\Omega_\delta \setminus \Omega_h$ by $\bar{\boldsymbol{\sigma}}_{hk}$.*

Then there exists $\bar{\boldsymbol{\sigma}}^m_k = (\bar{\sigma}^m_k, \bar{\alpha}^m_k) \in H(\Omega_\delta)$ for $m = 1, \ldots, N$ such that

$$(6.1) \qquad \bar{\boldsymbol{\sigma}}^m_{hk} \to \bar{\boldsymbol{\sigma}}^m_k \quad in \quad H(\Omega_\delta)\,,$$

and

$$(6.2) \qquad \bar{\boldsymbol{\sigma}}^m_k = (0, 0) \quad a.e. \ in \quad \Omega_\delta \setminus \Omega(v)\,.$$

If we use the notation $\bar{\boldsymbol{\sigma}}^m_k|_{\Omega(v)} = \boldsymbol{\sigma}^m_k$ then $\boldsymbol{\sigma}_k = (\boldsymbol{\sigma}^1_k, \ldots, \boldsymbol{\sigma}^N_k)$ is the solution of the following semidiscrete problem on $\Omega(v)$:

$$\begin{aligned} & for \quad m = 1, \ldots, N \quad we\ have \\ (6.3) \qquad & \boldsymbol{\sigma}^m_k \in K(\Omega(v), t^m)\,, \\ & \{\partial \boldsymbol{\sigma}^m_k, \boldsymbol{\tau} - \boldsymbol{\sigma}^m_k\}_{\Omega(v)} \geqq 0 \quad \forall \boldsymbol{\tau} \in K(\Omega(v), t^m)\,, \\ & where \quad \boldsymbol{\sigma}^0_k = (0, \alpha_0). \end{aligned}$$

Proof.

1)

According to Lemma 5.1, there exists a constant C (independent of h, v_h, m) such that

$$(6.4) \qquad |||\bar{\boldsymbol{\sigma}}^m_{hk}|||_{\Omega_\delta} \leqq C \quad \text{for} \quad m = 1, \ldots, N\,.$$

Hence, there exists a subsequence of $\{(\bar{\boldsymbol{\sigma}}^1_{hk}, \ldots, \bar{\boldsymbol{\sigma}}^N_{hk})\}_{h \in \mathscr{H}}$ (we shall denote it by the same symbol) and an N-tuple of functions $(\bar{\boldsymbol{\sigma}}^1_k, \ldots, \bar{\boldsymbol{\sigma}}^N_k)$, $\bar{\boldsymbol{\sigma}}^m_k \in H(\Omega_\delta)$ such that

$$(6.5) \qquad \bar{\boldsymbol{\sigma}}^m_{hk} \rightharpoonup \bar{\boldsymbol{\sigma}}^m_k \quad \text{(weakly) in } H(\Omega_\delta) \quad \text{for} \quad m = 1, \ldots, N\,.$$

2)

We shall prove by contradiction that $\bar{\boldsymbol{\sigma}}^m_k = (0, 0)$ a.e. in $\Omega_\delta \setminus \Omega(v)$.

Let there exist a set $\mathscr{M} \subset \Omega_\delta \setminus \Omega(v)$, $\mu(\mathscr{M}) > 0$, such that $|||\bar{\boldsymbol{\sigma}}^k_m|||_{\mathscr{M}} > 0$. From the convergence $v_h \to v$ in $C(\langle 0, 1 \rangle)$ it follows that $\mu(\Omega_h \cap \mathscr{M}) \to 0$. Denoting the characteristic function of $\mathcal{M}$ by $\chi_{\mathcal{M}}$ and using the weak convergence (6.5), we obtain

$$\{\bar{\boldsymbol{\sigma}}_{hk}^m, \chi_{\mathcal{M}}\bar{\boldsymbol{\sigma}}_k^m\}_{\Omega_\delta} \to \{\bar{\boldsymbol{\sigma}}_k^m, \chi_{\mathcal{M}}\bar{\boldsymbol{\sigma}}_k^m\}_{\Omega_\delta} = |||\bar{\boldsymbol{\sigma}}_k^m|||_{\mathcal{M}}^2 > 0 .$$

On the other hand, we have

$$|\{\bar{\boldsymbol{\sigma}}_{hk}^m, \chi_{\mathcal{M}}\bar{\boldsymbol{\sigma}}_k^m\}_{\Omega_\delta}| = |\{\bar{\boldsymbol{\sigma}}_{hk}^m, \bar{\boldsymbol{\sigma}}_k^m\}_{\Omega_h \cap \mathcal{M}}| \leqq |||\bar{\boldsymbol{\sigma}}_{hk}^m|||_{\Omega_\delta}\, |||\bar{\boldsymbol{\sigma}}_k^m|||_{\Omega_h \cap \mathcal{M}} \to 0 ,$$

which is a contradiction.

3)

We shall verify that $\boldsymbol{\sigma}_k^m \in K(\Omega(v), t^m)$ for all m.

Let an arbitrary $w \in V(\Omega(v))$ be given and let its extension by zero into $\Omega_\delta \setminus \Omega(v)$ be denoted by $\bar{w}$.

There exists a sequence $\{w_\ell\}$, $\ell \to 0$, such that $w_\ell \in [C^\infty(\bar{\Omega}_\delta)]^2$, supp $w_\ell \subset \bar{\Omega}(v)$, supp $w_\ell \cap \Gamma(v) = \emptyset$ and

$$w_\ell \xrightarrow[\ell \to 0]{} \bar{w} \quad \text{in} \quad [H^1(\Omega_\delta)]^2 . \tag{6.6}$$

For fixed ℓ there exists $h_0(\ell)$ such that supp $w_\ell \subset \bar{\Omega}_h$, supp $w_\ell \cap \Gamma_h = \emptyset$ holds for all $h \in \mathcal{H}$, $h \leqq h_0(\ell)$, and consequently $w_\ell|_{\Omega_h} \in V(\Omega_h)$.

Let π_h: $V(\Omega_h) \cap [C^\infty(\Omega_h)]^2 \to V_h(\Omega_h)$ be the standard interpolation using the piecewise linear finite elements. There exists $h_1(\ell) \leqq h_0(\ell)$ such that supp $\pi_h w_\ell \subset \bar{\Omega}(v)$ $\forall h \in \mathcal{H}$, $h \leqq h_1(\ell)$.

Denoting by $\bar{\pi}_h w_\ell$ the extension of $\pi_h w_\ell$ by zero into $\Omega_\delta \setminus \Omega_h$ and using the fact that $\sigma_{hk}^m \in E_h(\Omega_h, t^m)$, we obtain

$$\langle \bar{\sigma}_{hk}^m, e(\bar{\pi}_h w_\ell)\rangle_{S(\Omega_\delta)} = L_{\Omega_\delta}(\bar{\pi}_h w_\ell, t^m) . \tag{6.7}$$

Following Ciarlet [3], we can prove the estimate

$$\|w_\ell - \pi_h w_\ell\|_{1,\Omega_h} \leqq Ch|w_\ell|_{2,\Omega_h} ,$$

where $\|\cdot\|_{1,\Omega_h}$ denotes the norm in $[H^1(\Omega_h)]^2$ and $|\cdot|_{2,\Omega_h}$ is the usual seminorm in $[H^2(\Omega_h)]^2$. The constant C is independent of h , v_h.

Similarly, we have

$$\bar{\pi}_h w_\ell \to w_\ell \quad \text{in} \quad [H^1(\Omega_\delta)]^2 \tag{6.8}$$

and consequently

$$e(\bar{\pi}_h w_\ell) \to e(w_\ell) \quad \text{in} \quad S(\Omega_\delta) . \tag{6.9}$$

Using also the continuity of the trace operator (for the convergence of the right-hand sides) and the weak convergence (6.5), we can pass to the limit for $h \in \mathcal{H}$ in (6.7) obtaining

$$\langle \bar{\sigma}_k^m, e(w_\ell)\rangle_{S(\Omega_\delta)} = L_{\Omega_\delta}(w_\ell, t^m) .$$

Passing to the limit for $\ell \to 0$ and using the properties supp $\bar{w} \subset \bar{\Omega}(v)$, supp $\bar{w} \cap \Gamma(v) = \emptyset$, we conclude

$$(6.10) \qquad \langle \sigma_k^m, e(w) \rangle_{S(\Omega(v))} = L_{\Omega(v)}(w, t^m),$$

so that $\sigma_k^m \in E(\Omega(v), t^m)$ (as $w \in V(\Omega(v))$ was arbitrary).

Since $P(\Omega_\delta)$ is closed and convex, it is weakly closed. As $\bar{\boldsymbol{\sigma}}_{hk}^m \in P(\Omega_\delta)$ $\forall h \in \mathscr{H}$ and $\bar{\boldsymbol{\sigma}}_{hk}^m \rightharpoonup \bar{\boldsymbol{\sigma}}_k^m$ in $H(\Omega_\delta)$, we have $\bar{\boldsymbol{\sigma}}_k^m \in P(\Omega_\delta)$ and consequently $\boldsymbol{\sigma}_k^m \in P(\Omega(v))$.

4)

We shall verify the inequality occurring in (6.3) and the strong convergence (6.1) by induction.

I. $m = 1$

From the weak convergence (6.5) we obtain

$$(6.11) \qquad \langle \bar{\sigma}_{hk}^1, \bar{\tau} \rangle_{S(\Omega_\delta)} + \langle \bar{\alpha}_{hk}^1, \bar{\beta} \rangle_{L^2(\Omega_\delta)} \to \langle \bar{\sigma}_k^1, \bar{\tau} \rangle_{S(\Omega_\delta)} + \langle \bar{\alpha}_k^1, \bar{\beta} \rangle_{L^2(\Omega_\delta)} \quad \forall \bar{\boldsymbol{\tau}} = (\bar{\tau}, \bar{\beta}) \in H(\Omega_\delta).$$

Choosing $\bar{\beta} \equiv 0$ and using (5.5), we have

$$0 = \langle \bar{\sigma}_k^1, \bar{\tau} \rangle_{S(\Omega_\delta)} \quad \forall \bar{\tau} \in S(\Omega_\delta)$$

and so $\bar{\sigma}_k^1 = 0$ a. e. in Ω_δ.

According to **2)**, $\bar{\alpha}_k^1 = 0$ holds a.e. in $\Omega_\delta \setminus \Omega(v)$. We shall use the notation $\alpha_k^1 = \bar{\alpha}_k^1|_{\Omega(v)}$. Let us choose a parameter $\varrho \in (0, a)$ and define the domain $\Omega^\varrho = \Omega(v - \varrho)$.

We shall prove by contradiction that $\alpha_k^1 = \alpha_0$ a.e. in Ω^ϱ. Let there exists a set $\mathscr{M} \subset \Omega^\varrho$, $\mu(\mathscr{M}) > 0$, such that $\|\alpha_0 - \alpha_k^1\|_{L^2(\mathscr{M})} > 0$. Obviously, $\Omega^\varrho \subset \Omega_h$ and also $\mathscr{M} \subset \Omega_h$ holds for sufficiently small h. From (6.11) and (5.5) we find that

$$r_h' \alpha_0 = \alpha_{hk}^1 \rightharpoonup \alpha_k^1 \text{ in } L^2(\mathscr{M}).$$

We may write

$$0 < \|\alpha_0 - \alpha_k^1\|_{L^2(\mathscr{M})} = \langle \alpha_0 - \alpha_k^1, \alpha_0 - r_h' \alpha_0 \rangle_{L^2(\mathscr{M})} + \\ + \langle \alpha_0 - \alpha_k^1, r_h' \alpha_0 - \alpha_k^1 \rangle_{L^2(\mathscr{M})}.$$

Recalling also the corollary of Lemma 3.2, we observe that the two members of the right-hand side converge to zero for $h \in \mathscr{H}$. Thus we arrive at contradiction.

Passing to the limit for $\varrho \to 0$, we obtain $\alpha_k^1 = \alpha_0$ a.e. in $\Omega(v)$ and so $\boldsymbol{\sigma}_k^1 = (0, \alpha_0)$ a.e. in $\Omega(v)$. Then $\partial \boldsymbol{\sigma}_k^1 = (0, 0)$ and the inequality in (6.3) holds for $m = 1$ trivially.

Let us prove the strong convergence:

$$\|\bar{\boldsymbol{\sigma}}_{hk}^1 - \bar{\boldsymbol{\sigma}}_k^1\|_{0,\Omega_\delta}^2 = \|\bar{\alpha}_{hk}^1 - \bar{\alpha}_k^1\|_{L^2(\Omega_\delta)}^2 = \\ = \int_{\Omega(v) \cap \Omega_h} (r_h' \alpha_0 - \alpha_0)^2 \, dx + \int_{\Omega(v) \setminus \Omega_h} \alpha_0^2 \, dx + \int_{\Omega_h \setminus \Omega(v)} (r_h' \alpha_0)^2 \, dx.$$

The convergence of the first term follows from the corollary of Lemma 3.2, the convergence of the second follows from the property $\mu(\Omega(v) \setminus \Omega_h) \to 0$. The convergence of the third term can be obtained from the inequality

$$\|r'_h\alpha_0\|_{L^2(\Omega_h\setminus\Omega(v))} \leqq \|r'_h\alpha_0 - \alpha_0\|_{L^2(\Omega_h\setminus\Omega(v))} + \|\alpha_0\|_{L^2(\Omega_h\setminus\Omega(v))}$$

by using the corollary of Lemma 3.2 and the property $\mu(\Omega_h \setminus \Omega(v)) \to 0$.

II. The induction step

Let m be fixed. Let us suppose

$$(6.12) \qquad \bar{\sigma}_{hk}^{m-1} \to \bar{\sigma}_k^{m-1} \quad \text{in} \quad H(\Omega_\delta)\,.$$

Let an arbitrary but fixed $\boldsymbol{\tau} = (\tau, \beta) \in K(\Omega(v), t^m)$ be given. We have to verify the inequality

$$(6.13) \qquad \{\partial\sigma_k^m, \boldsymbol{\tau} - \sigma_k^m\}_{\Omega(v)} \geqq 0\,.$$

Let us introduce a parameter $\lambda > 0$, $\lambda \leqq \delta - b$ and set $\Omega^\lambda = \Omega(v + \lambda)$.

First, we shall construct a function $\boldsymbol{\tau}^\lambda = (\tau^\lambda, \beta^\lambda)$ which satisfies the conditions

$$(6.14) \qquad \boldsymbol{\tau}^\lambda \in K(\Omega^\lambda, t^m)\,,$$

$$(6.15) \qquad \|\boldsymbol{\tau}^\lambda - \boldsymbol{\tau}\|_{0,\Omega(v)} \xrightarrow[\lambda\to 0]{} 0\,.$$

Let $\boldsymbol{\xi}(t) = (\xi(t), \zeta(t))$ be the function from Lemma 1.2 defined by (1.9), (1.11). Let us define the function

$$\omega = \tau - \xi(t^m) \in S(\Omega(v))\,.$$

Obviously, $\langle\omega, e(w)\rangle_{S(\Omega(v))} = 0\ \forall w \in V(\Omega(v))$. Let us denote by $\tilde{\omega}$ the extension of ω by zero to the negative half-plane. We shall define the following transformation of coordinates:

$$y_1 = x_1 - \lambda\,, \quad y_2 = x_2\,.$$

Then for $x \in \Omega^\lambda$ we have that $y \in \Omega(v) \cup (-\lambda, 0) \times (0,1) =^{\text{def}} \Omega^{\lambda *}$. We define the function

$$\omega^\lambda(x) = \tilde{\omega}(y)\,.$$

We shall prove that $\langle\omega^\lambda, e(w)\rangle_{S(\Omega^\lambda)} = 0\ \forall w \in V(\Omega^\lambda)$. For arbitrary $w \in V(\Omega^\lambda)$ we define $w^*(y) = w(x)$. Obviously $w^*|_{\Omega(v)} \in V(\Omega(v))$ holds and then

$$\langle\omega^\lambda, e(w)\rangle_{S(\Omega^\lambda)} = \int_{\Omega^{\lambda *}} \tilde{\omega}_{ij}(y)\, e_{ij}(w^*(y))\,\mathrm{d}y =$$
$$= \int_{\Omega(v)} \omega_{ij}(y)\, e_{ij}(w^*|_{\Omega(v)}(y))\,\mathrm{d}y = \langle\omega, e(w^*|_{\Omega(v)})\rangle_{S(\Omega(v))} = 0\,.$$

Finally, setting

$$\tau^\lambda = \xi(t^m) + \omega^\lambda\,,$$

we observe that $\tau^\lambda \in E(\Omega^\lambda, t^m)$.

We have to construct a function β^λ such that $f(\tau^\lambda) \leqq \beta^\lambda$ a.e. in Ω^λ.

Let us set

$$\beta^\lambda(x) = \begin{cases} \beta(y) + LL_\xi|\gamma(\bar{t})|\lambda & \text{for} \quad x \in \Omega^\lambda \setminus Q_\lambda, \\ \zeta(t^m, x) & \text{for} \quad x \in Q_\lambda, \end{cases}$$

where $Q_\lambda = (0, \lambda) \times (0, 1)$. We have $\tau^\lambda = \xi(t^m)$ in Q_λ and since $\xi(t^m) \in P(\Omega_\delta)$, we obtain

$$f(\tau^\lambda) = f(\xi(t^m)) \leqq \zeta(t^m) = \beta^\lambda .$$

In the domain $\Omega^\lambda \setminus Q_\lambda$ we may write

$$\tau^\lambda(x) = \xi(t^m, x) + \omega(y) = \xi(t^m, x) + \tau(y) - \xi(t^m, y) .$$

From the Lipschitz continuity of f and ξ we obtain

$$|f(\tau^\lambda(x)) - f(\tau(y))| \leqq L\|\tau^\lambda(x) - \tau(y)\|_{R_{\text{sym}}^4} =$$
$$= L\|\gamma(t^m)\,[\xi(x) - \xi(y)]\|_{R_{\text{sym}}^4} \leqq LL_\xi|\gamma(\bar{t})|\;\|x - y\|_{R^2} = LL_\xi|\gamma(\bar{t})|\;\lambda .$$

Finally, we have

$$f(\tau^\lambda(x)) \leqq f(\tau(y)) + LL_\xi|\gamma(\bar{t})|\;\lambda \leqq \beta(y) + LL_\xi|\gamma(\bar{t})|\;\lambda = \beta^\lambda(x) .$$

Let us set $\boldsymbol{\tau}^\lambda = (\tau^\lambda, \beta^\lambda)$. It remains to verify the condition (6.15).

Obviously, the following holds:

$$\|\tau^\lambda - \tau\|^2_{S(\Omega(v))} = \|\omega^\lambda - \omega\|^2_{S(\Omega(v))} \xrightarrow[\lambda \to 0]{} 0$$

(see Nečas [11], Chap. 2, Th. 1.1).

For the second component we have to study two cases.

$$\text{In } Q_\lambda: \quad \beta^\lambda - \beta = \zeta(t^m) - \beta \in L^2(Q_\lambda), \quad \mu(Q_\lambda) \xrightarrow[\lambda \to 0]{} 0$$

holds and then $\|\beta^\lambda - \beta\|_{L^2(Q_\lambda)} \xrightarrow[\lambda \to 0]{} 0.$

$$\text{In } \Omega(v) \setminus Q_\lambda: \quad \beta^\lambda(x) - \beta(x) = LL_\xi|\gamma(\bar{t})|\;\lambda + \beta(y) - \beta(x)$$

and then

$$\|\beta^\lambda - \beta\|_{L^2(\Omega(v)\setminus Q_\lambda)} \leqq LL_\xi|\gamma(\bar{t})|\;\lambda\,\sqrt{(\mu(\Omega(v)))} +$$
$$+ \int_{\Omega(v)\setminus Q_\lambda} \left[\beta\left(x - \begin{pmatrix}\lambda \\ 0\end{pmatrix}\right) - \beta(x)\right]^2 \mathrm{d}x \xrightarrow[\lambda \to 0]{} 0$$

(see again Nečas [11]).

The function $\boldsymbol{\tau}^\lambda$ has the required properties.

For sufficiently small h we have $\Omega_h \subset \Omega^\lambda$.

The restriction $\boldsymbol{\tau}^\lambda|_{\Omega_h}$ is obviously an element of $K(\Omega_h, t^m)$. Henceforth this restriction will be denoted only by $\boldsymbol{\tau}^\lambda$.

Let us construct the projection $\boldsymbol{r}_h\boldsymbol{\tau}^\lambda$. As in the proof of Theorem 4.1, we can prove that

(6.16) $\qquad \boldsymbol{r}_h\boldsymbol{\tau}^\lambda \in K_h(\Omega_h, t^m) .$

There exists $H \in \mathscr{H}$ (for a given λ) and a polygonal domain $\Omega^{\lambda,H}$ (with the same properties as the domain Ω^H in Lemma 3.2) such that

$$(6.17) \qquad \Omega^{\lambda/2} \subset \Omega^{\lambda,H} \subset \Omega^{\lambda} .$$

There exists $h^{(0)}(\lambda)$ such that for all $h < h^{(0)}(\lambda)$

$$(6.18) \qquad \Omega_h \subset \Omega^{\lambda/2}$$

holds and so $\Omega_h \subset \Omega^{\lambda,H}$. The assumption (3.5) of Lemma 3.2 is fulfilled. As in this lemma we shall construct the system of triangulations $\{\mathscr{T}_h^H\}_{h\in\mathscr{H}}$ of the domain $\Omega^{\lambda,H}$, the finite element spaces and the orthogonal projections. The lemma yields

$$(6.19) \qquad \lim \|\boldsymbol{r}_h^H \boldsymbol{\tau}^\lambda - \boldsymbol{\tau}^\lambda\|_{0,\Omega^{\lambda,H}} = 0 .$$

From the mean value property we observe that $\boldsymbol{r}_h^H \boldsymbol{\tau}^\lambda$ is an extension of $\boldsymbol{r}_h \boldsymbol{\tau}^\lambda$ onto $\Omega^{\lambda,H}$.

We can insert $\boldsymbol{r}_h \boldsymbol{\tau}^\lambda$ into (3.8) obtaining

$$(6.20) \qquad \{\partial \boldsymbol{\sigma}_{hk}^m, \boldsymbol{r}_h \boldsymbol{\tau}^\lambda - \boldsymbol{\sigma}_{hk}^m\}_{\Omega_h} \geqq 0 .$$

After some modifications we obtain

$$(6.21) \qquad \{\boldsymbol{\sigma}_{hk}^m, \boldsymbol{r}_h \boldsymbol{\tau}^\lambda\}_{\Omega_h} - \{\boldsymbol{\sigma}_{hk}^{m-1}, \boldsymbol{r}_h \boldsymbol{\tau}^\lambda\}_{\Omega_h} - |||\boldsymbol{\sigma}_{hk}^m|||_{\Omega_h}^2 + \{\boldsymbol{\sigma}_{hk}^{m-1}, \boldsymbol{\sigma}_{hk}^m\}_{\Omega_h} \geqq 0 .$$

The weak convergence (6.5) and the convergence (6.19) imply that

$$\{\boldsymbol{\sigma}_{hk}^m, \boldsymbol{r}_h \boldsymbol{\tau}^\lambda\}_{\Omega_h} = \{\bar{\boldsymbol{\sigma}}_{hk}^m, \boldsymbol{r}_h^H \boldsymbol{\tau}^\lambda\}_{\Omega^{\lambda,H}} \to \{\bar{\boldsymbol{\sigma}}_k^m, \boldsymbol{\tau}^\lambda\}_{\Omega^{\lambda,H}} = \{\boldsymbol{\sigma}_k^m, \boldsymbol{\tau}^\lambda\}_{\Omega(v)} ,$$

and similarly

$$\{\boldsymbol{\sigma}_{hk}^{m-1}, \boldsymbol{r}_h \boldsymbol{\tau}^\lambda\}_{\Omega_h} \to \{\boldsymbol{\sigma}_k^{m-1}, \boldsymbol{\tau}^\lambda\}_{\Omega(v)} .$$

From the weak convergence (6.5) and the induction hypothesis (6.12) we observe that

$$\{\boldsymbol{\sigma}_{hk}^{m-1}, \boldsymbol{\sigma}_{hk}^m\}_{\Omega_h} = \{\bar{\boldsymbol{\sigma}}_{hk}^{m-1}, \bar{\boldsymbol{\sigma}}_{hk}^m\}_{\Omega_\delta} \to \{\bar{\boldsymbol{\sigma}}_k^{m-1}, \bar{\boldsymbol{\sigma}}_k^m\}_{\Omega_\delta} = \{\boldsymbol{\sigma}_k^{m-1}, \boldsymbol{\sigma}_k^m\}_{\Omega(v)} .$$

Using again the weak convergence and the weak lower semicontinuity of the functional $|||\cdot|||_{\Omega_\delta}^2$, we conclude

$$(6.22) \qquad \liminf_{h\in\mathscr{H}} |||\boldsymbol{\sigma}_{hk}^m|||_{\Omega_h}^2 = \liminf_{h\in\mathscr{H}} |||\bar{\boldsymbol{\sigma}}_{hk}^m|||_{\Omega_\delta}^2 \geqq |||\bar{\boldsymbol{\sigma}}_k^m|||_{\Omega_\delta}^2 = |||\boldsymbol{\sigma}_k^m|||_{\Omega(v)}^2 .$$

Passing to the limit with $h \in \mathscr{H}$ in (6.21), we arrive at

$$(6.23) \qquad \{\boldsymbol{\sigma}_k^m, \boldsymbol{\tau}^\lambda\}_{\Omega(v)} - \{\boldsymbol{\sigma}_k^{m-1}, \boldsymbol{\tau}^\lambda\}_{\Omega(v)} + \{\boldsymbol{\sigma}_k^{m-1}, \boldsymbol{\sigma}_k^m\}_{\Omega(v)} \geqq |||\boldsymbol{\sigma}_k^m|||_{\Omega(v)}^2 .$$

Passing to the limit with $\lambda \to 0$ and using (6.15), we obtain

$$\{\partial \boldsymbol{\sigma}_k^m, \boldsymbol{\tau} - \boldsymbol{\sigma}_k^m\}_{\Omega(v)} \geqq 0 .$$

As $\boldsymbol{\tau} \in K(\Omega(v), t^m)$ was arbitrary, we observe that $\boldsymbol{\sigma}_k^m$ is a solution of the semidiscrete problem (6.3).

Finally, we have to prove the strong convergence.

Let us insert $\tau = \sigma_k^m$ into (6.21):

$$\{\sigma_{hk}^m, r_h(\sigma_k^m)^\lambda\}_{\Omega_h} - \{\sigma_{hk}^{m-1}, r_h(\sigma_k^m)^\lambda\}_{\Omega_h} + \{\sigma_{hk}^{m-1}, \sigma_{hk}^m\}_{\Omega_h} \geqq |||\sigma_{hk}^m|||_{\Omega_h}^2 .$$

In a parallel way we can pass to the limit with $h \in \mathscr{H}$ and then with $\lambda \to 0$ on the left-hand side:

$$\{\sigma_k^m, \sigma_k^m\}_{\Omega(v)} - \{\sigma_k^{m-1}, \sigma_k^m\}_{\Omega(v)} + \{\sigma_k^{m-1}, \sigma_k^m\}_{\Omega(v)} \geqq \limsup_{h \in \mathscr{H}} |||\sigma_{hk}^m|||_{\Omega_h}^2 .$$

Using (6.22), we conclude

$$\limsup_{h \in \mathscr{H}} |||\sigma_{hk}^m|||_{\Omega_h}^2 \leqq |||\sigma_k^m|||_{\Omega(v)}^2 \leqq \liminf_{h \in \mathscr{H}} |||\sigma_{hk}^m|||_{\Omega_h}^2 .$$

The weak convergence (6.5) and the convergence of squares of norms yield the strong convergence (6.1).

We can easily prove that the semidiscrete problem has a unique solution and then the whole sequence $\{\bar{\sigma}_{hk}^m\}_{h \in \mathscr{H}}$ converges (strongly) to $\bar{\sigma}_k^m$ in $H(\Omega_\delta)$ for $m = 1, \ldots, N$.

Proposition 6.2. *Let the assumption (1.5) be fulfilled in the domain $\Omega(v)$. Let $\sigma_k = (\sigma_k^1, \ldots, \sigma_k^N)$ be the solution of the semidiscrete problem (6.3) in the domain $\Omega(v)$ for $k \in \mathscr{K}$ (where the set $\mathscr{K}$ fulfils the condition (3.7)). Let us denote by σ the solution of the State Problem (1.3) in $\Omega(v)$.*

Then there exist positive constants k_0, C such that for all $k \in \mathscr{K}$, $k \leqq k_0$ the following estimate holds:

$$\max_{1 \leqq m \leqq T/k} \|\sigma(t^m) - \sigma_k^m\|_{0,\Omega(v)} \leqq C k^{1/2} .$$

Proof. The poposition can be proved by modifying slightly the arguments of Johnson [7], [8] (for more details see also Hlaváček [4]). The most important step is an a priori estimate

$$\sum_{m=1}^{N} k |||\partial \sigma_k^m|||_{\Omega(v)}^2 \leqq C ,$$

which can be proved by the penalization method.

Theorem 6.1. *Let the assumption (2.1) be fulfilled. Let the sequence $\mathscr{H}$ and $\mathscr{K}$ fulfil the conditions (3.4) and (3.7), respectively. Let a function $v_h \in \mathscr{U}_{ad}^h$ be given for all $h \in \mathscr{H}$ and let $v_h \to v$ in $C(\langle 0, 1 \rangle)$, $v \in \mathscr{U}_{ad}$. Let us denote by $\mathscr{J}$ the cost functional defined in (2.3). Let $\mathscr{J}_{hk}$ be the approximate cost functional defined in (3.9) for given $h \in \mathscr{H}$, $k \in \mathscr{K}$.*

Then there exists a function $h(k)$: $R^+ \to R^+$ with $\lim_{k \to 0^+} h(k) = 0$, such that

$$\lim_{\substack{k \in \mathscr{K} \\ h \in \mathscr{H}, h \leqq h(k)}} \mathscr{J}_{hk}(v_h) = \mathscr{J}(v) .$$

Proof. Let us denote

$$\mathscr{J}(v) \quad = \int_0^T \mathscr{F}(t)\,\mathrm{d}t\,, \quad \text{where} \quad \mathscr{F}(t) = \int_{\Omega_\delta} f_1^2(\bar{\sigma}(t, x))\,\mathrm{d}x\,,$$

$$\mathscr{J}_{hk}(v_h) = k \sum_{m=1}^{N} c_m \mathscr{F}_{hk}^m\,, \quad \text{where} \quad \mathscr{F}_{hk}^m \quad = \int_{\Omega_\delta} f_1^2(\bar{\sigma}_{hk}^m(x))\,\mathrm{d}x\,.$$

Using the Lipschitz continuity of f_1, we obtain for a.e. $x \in \Omega_\delta$

$$|f_1^2(\bar{\sigma}_{hk}^m(x)) - f_1^2(\bar{\sigma}(t^m, x))| \leqq$$
$$\leqq L_1 \|\bar{\sigma}_{hk}^m(x) - \bar{\sigma}(t^m, x)\|_{R_{\mathrm{sym}}^4} (2|f_1(\bar{\sigma}(t^m, x))| +$$
$$+ L_1 \|\bar{\sigma}_{hk}^m(x) - \bar{\sigma}(t^m, x)\|_{R_{\mathrm{sym}}^4}) \leqq$$
$$\leqq L_1^2 \|\bar{\sigma}_{hk}^m(x) - \bar{\sigma}(t^m, x)\|_{R_{\mathrm{sym}}^4}^2 + 2L_1 \|\bar{\sigma}_{hk}^m(x) - \bar{\sigma}(t^m, x)\|_{R_{\mathrm{sym}}^4} |f_1(\bar{\sigma}(t^m, x))|\,.$$

Consequently, we may write

$$|\mathscr{F}_{hk}^m - \mathscr{F}(t^m)| \leqq$$
$$\leqq L_1^2 \|\bar{\sigma}_{hk}^m - \bar{\sigma}(t^m)\|_{S(\Omega_\delta)}^2 + 2L_1 \|\bar{\sigma}_{hk}^m - \bar{\sigma}(t^m)\|_{S(\Omega_\delta)} \|f_1(\bar{\sigma}(t^m))\|_{L^2(\Omega_\delta)}\,.$$

The properties of f_1 and Lemma 1.1 yield

$$\|f_1(\bar{\sigma}(t^m))\|_{L^2(\Omega_\delta)} \leqq [\int_{\Omega_\delta} L_1^2 \|\bar{\sigma}(t^m, x)\|_{R_{\mathrm{sym}}^4}^2\,\mathrm{d}x]^{1/2} =$$
$$= L_1 \|\bar{\sigma}(t^m)\|_{S(\Omega_\delta)} \leqq L_1 (t^m)^{1/2} \|\bar{\sigma}\|_{H_0^1(I, S(\Omega_\delta))} \leqq C\,.$$

Finally, we have

$$|\mathscr{F}_{hk}^m - \mathscr{F}(t^m)| \leqq C(\|\bar{\sigma}_{hk}^m - \bar{\sigma}(t^m)\|_{S(\Omega_\delta)}^2 + \|\bar{\sigma}_{hk}^m - \bar{\sigma}(t^m)\|_{S(\Omega_\delta)})\,. \tag{6.24}$$

The assumptions of Propositions 6.1 and 6.2 are fulfilled so that we may conclude

$$\max_{m=1,\ldots,T/k} \|\bar{\sigma}_{hk}^m - \bar{\sigma}(t^m)\|_{S(\Omega_\delta)} \leqq \tag{6.25}$$
$$\leqq \max_{m=1,\ldots,T/k} \|\bar{\sigma}_{hk}^m - \bar{\sigma}_k^m\|_{S(\Omega)_\delta} + \max_{m=1,\ldots,T/k} \|\bar{\sigma}_k^m - \bar{\sigma}(t^m)\|_{S(\Omega_\delta)} \xrightarrow[k\in\mathscr{K}, h\in\mathscr{H}, h\leqq h(k)]{} 0\,.$$

From (6.24), (6.25) we derive

$$\lim_{\substack{k\in\mathscr{K}\\ h\in\mathscr{H}, h\leqq h(k)}} \max_{m=1,\ldots,T/k} |\mathscr{F}_{hk}^m - \mathscr{F}(t^m)| = 0\,. \tag{6.26}$$

In a way analogous to (6.24) we can derive that $\mathscr{F}(t)$ is a continuous function. Supposing that the quadrature formula is convergent for all continuous functions, we have

$$\lim_{k\in\mathscr{K}} |\int_0^T \mathscr{F}(t)\,\mathrm{d}t - k \sum_{m=0}^{N} c_m \mathscr{F}(t^m)| = 0\,. \tag{6.27}$$

Finally, we write

$$|\mathscr{J}(v) - \mathscr{J}_{hk}(v_h)| \leqq$$
$$\leqq \left|\int_0^T \mathscr{F}(t)\,\mathrm{d}t - k \sum_{m=1}^{N} c_m\, \mathscr{F}(t^m)\right| + \left|k \sum_{m=1}^{N} c_m\, \mathscr{F}(t^m) - k \sum_{m=1}^{N} c_m\, \mathscr{F}^m_{hk}\right| \leqq$$
$$\leqq \left|\int_0^T \mathscr{F}(t)\,\mathrm{d}t - k \sum_{m=1}^{N} c_m\, \mathscr{F}(t^m)\right| + C_1 T \max_{m=1,\ldots,T/k} \left|\mathscr{F}(t^m) - \mathscr{F}^m_{hk}\right| .$$

The assertion of the theorem follows now from (6.26), (6.27).

We can pass to the final convergence theorem.

Theorem 6.2. *Let the assumption* (2.1) *be fulfilled. Let an arbitrary real* $p > 1$ *be given. Let the sequences* $\mathscr{H}$ *and* $\mathscr{K}$ *fulfil the conditions* (3.4) *and* (3.7), *respectively.*

Let $\{u_h^{(k)}\}$, $k \in \mathscr{K}$, $h \in \mathscr{H}$, $h \leqq h(k)$ *be a sequence of solutions of the Approximate Optimal Design Problem* (3.10) (*these solutions exist according to Theorem* 5.1).

Then there exists a subsequence $\{u_{h'}^{(k')}\}$ *such that*

$$u_{h'}^{(k')} \rightharpoonup u \quad (\textit{weakly}) \ \textit{in} \ W^{1,p}(\langle 0, 1\rangle) ,$$

where $u \in \mathscr{U}_{\text{ad}}$ *is a solution of the original Optimal Design Problem* (2.4).

The following convergence holds for the subsequence of solutions of the Approximate State Problem:

$$(6.28) \qquad \max_{m=1,\ldots,T/k'} \left\|\bar{\sigma}^m_{h'k'}(u_{h'}^{(k')}) - \bar{\sigma}^m(u)\right\|_{0,\Omega_\delta} \xrightarrow[k'\in\mathscr{K},h'\in\mathscr{H},h'\leqq h(k')]{} 0 .$$

(*Here we have used the notation* $\bar{\sigma}(u)(t^m) = \bar{\sigma}^m(u)$.)

Moreover, every weak accumulation point of the sequence $\{u_h^{(k)}\}$ *in* $W^{1,p}(\langle 0, 1\rangle)$ *is a solution of the Optimal Design Problem* (2.4).

Proof. The set $\mathscr{U}_{\text{ad}}$ is obviously closed in $W^{1,p}(\langle 0, 1\rangle)$. As it is convex, $\mathscr{U}_{\text{ad}}$ is also weakly closed in $W^{1,p}(\langle 0, 1\rangle)$. From the boundedness we obtain the weak compactness of $\mathscr{U}_{\text{ad}}$ in $W^{1,p}(\langle 0, 1\rangle)$.

As $u_h^{(k)} \in \mathscr{U}^h_{\text{ad}} \subset \mathscr{U}_{\text{ad}}$, we can select from $\{u_h^{(k)}\}$ a weakly convergent subsequence

$$u_{h'}^{(k')} \rightharpoonup u \in \mathscr{U}_{\text{ad}} \quad \text{in} \quad W^{1,p}(\langle 0, 1\rangle) \quad \text{for} \quad k' \in \mathscr{K} , \quad h' \in \mathscr{H} , \quad h' \leqq h(k') .$$

From the theorem on the compact imbedding of $W^{1,p}$ into C (see [10], Th. 5.8.3) we obtain the strong convergence

$$u_{h'}^{(k')} \to u \quad \text{in} \quad C(\langle 0, 1\rangle) .$$

Let an arbitrary $v \in \mathscr{U}_{\text{ad}}$ be given. There exists a sequence $\{v_h\}$, $h \in \mathscr{H}$, such that $v_h \in \mathscr{U}^h_{\text{ad}}$, $v_h \xrightarrow[h\in\mathscr{H}]{} v$ in $C(\langle 0, 1\rangle)$ (see Lemma 3.1). Obviously, the selected subsequence $\{v_{h'}\}$ converges to v as well. Theorem 6.1 implies that

$$\lim_{\substack{k' \in \mathscr{K} \\ h' \in \mathscr{H}, h' \leqq h(k')}} \mathscr{J}_{h'k'}(u_{h'}^{(k')}) = \mathscr{J}(u),$$

$$\lim_{\substack{k' \in \mathscr{K} \\ h' \in \mathscr{H}, h' \leqq h(k')}} \mathscr{J}_{h'k'}(v_{h'}) = \mathscr{J}(v).$$

By the definition (3.10), we have

$$\mathscr{J}_{h'k'}(u_{h'}^{(k')}) \leqq \mathscr{J}_{h'k'}(v_{h'}).$$

Passing to the limit, we obtain

$$\mathscr{J}(u) \leqq \mathscr{J}(v)$$

so that u is a solution of the Optimal Design Problem (2.4). The convergence (6.28) follows from Propositions 6.1 and 6.2 (similarly as (6.25)).

The last assertion of the theorem is true, because the argument used above can be applied to every selected weakly convergent subsequence.

Corollary 1. *Let the assumptions of Theorem* 6.2 *be fulfilled. Then there exists a subsequence* $\{u_{h'}^{(k')}\}$ *such that*

$$u_{h'}^{(k')} \to u \quad \text{in} \quad C(\langle 0, 1 \rangle).$$

Proof is an immediate consequence of the compact imbedding of $W^{1,p}(\langle 0, 1 \rangle)$ into $C(\langle 0, 1 \rangle)$.

Corollary 2. *Let the assumption* (2.1) *hold. Then there exists at least one solution of the Optimal Design Problem* (2.4).

Proof follows from Theorems 5.1 and 6.2.

References

[1] *D. Begis, R. Glowinski:* Application de la méthode des éléments finis à l'approximation d'un problème de domaine optimal. Appl. Math. Optim. 2 (1975), 130—169.

[2] *J. Céa:* Optimization: Théorie et algorithmes. Dunod, Paris, 1971; (in Russian, Mir, Moskva, 1973).

[3] *P. G. Ciarlet:* The finite element method for elliptic problems. North Holland Publ. Comp., Amsterdam, 1978; (in Russian, Mir, Moskva, 1980).

[4] *I. Hlaváček:* A finite element solution for plasticity with strain-hardening. RAIRO Annal. Numér. 14 (1980), 347—368.

[5] *I. Hlaváček:* Optimization of the domain in elliptic problems by the dual finite element method. Apl. Mat. 30 (1985), 50—72.

[6] *I. Hlaváček:* Shape optimization of an elastic-perfectly plastic body. Apl. Mat. 32 (1987), 381—400.

[7] *C. Johnson:* Existence theorems for plasticity problems. J. Math. Pures Appl. 55 (1976), 431—444.
[8] *C. Johnson:* A mixed finite element method for plasticity with hardening. SIAM J. Numer. Anal. 14 (1977), 575—583.
[9] *C. Johnson:* On plasticity with hardening. J. Math. Anal. Appl. 62 (1978), 325—336.
[10] *A. Kufner, O. John, S. Fučík:* Function spaces. Academia, Praha, 1977.
[11] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Praha, 1967.
[12] *I. Hlaváček:* Shape optimization of elasto-plastic bodies obeying Hencky's law. Apl. Mat. 31 (1986), 486—499.
[13] *I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek:* Solution of Variational Inequalities in Mechanics. Springer-Verlag, New York, 1988.

Souhrn

OPTIMALIZACE TVARU PRUŽNĚ-PLASTICKÉHO TĚLESA PRO MODEL SE ZPEVNĚNÍM DEFORMACÍ

VLADISLAV PIŠTORA

Stavová úloha pružně-plastického tělesa pro model se zpevněním deformací je formulována v napětích a parametrech zpevnění pomocí evoluční variační nerovnice. Minimalizuje se účelový funkcionál (integrál z funkce napětí) vzhledem k části hranice, na níž je (dvojrozměrné) těleso upevněno. Pomocí metody konečných prvků se definuje aproximovaná úloha a dokazuje se existence přibližného řešení a konvergence k řešení původní optimalizační úlohy.

Резюме

ОПТИМИЗАЦИЯ ФОРМЫ УПРУГОПЛАСТИЧЕСКОГО ТЕЛА ДЛЯ МОДЕЛИ С УПРОЧНЕНИЕМ ПОСРЕДСТВОМ ДЕФОРМАЦИЙ

VLADISLAV PIŠTORA

Задача состояния упругопластического тела для модели с упрочнением посредством деформаций формулирована двойственным подходом в форме эволюционного вариационного неравенства. Неизвестными являются тензор нахряжений и параметр упрочнения. Минимизируется целевой функционал относительно части границы, на которой тело фиксировано.

При помощи метода конечных элементов определяется приближенное решение и доказывается существование этого приближенного решения и сходимость к решению исходной проблемы оптимизации.

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