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TESTING A TOLERANCE HYPOTHESIS BY MEANS OF AN INFORMATION DISTANCE

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Summary. In the paper a test of the hypothesis $\mu + c\sigma \leqq M$, $\mu - c\sigma \geqq m$ on parameters of the normal distribution is presented, and explicit formulas for critical regions are derived for finite sample sizes. Asymptotic null distribution of the test statistic is investigated under the assumption, that the true distribution possesses the fourth moment.

Keywords. Hypothesis testing, Fisher information matrix, concentration of the statistical population in prescribed tolerance limits, statistical quality control.

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1. INTRODUCTION AND THE MAIN RESULTS

Let us assume that a statistician has to decide whether at least $100(1 - \Delta)\,\%$ of the statistical population satisfies the relation

$$x \in \langle m, M \rangle \tag{1.1}$$

where x is an investigated quantity and $m < M$ are tolerance limits. This requirement corresponds to the hypothesis

$$H_0\colon \mathsf{P}[x \in \langle m, M \rangle] \geqq 1 - \Delta \tag{1.2}$$

which can, in principle, be tested in two ways. The simpler procedure consists in testing the hypothesis that the parameter $p = \mathsf{P}[x \in \langle m, M \rangle]$ of the binomial distribution satisfies the inequality

$$p \geqq 1 - \Delta\,. \tag{1.3}$$

As is well known, this test rejects the null hypothesis, if less than k amongst n randomly chosen elements satisfy (1.1). A disadvantage of this approach is that an element of the sample is declared to be defective regardless of the magnitude of the violation of (1.1). Thus part of information contained in the sample is not utilized and it is therefore logical to believe that a test based on numerical characteristics (mean, standard deviation) will be more powerful. The fact also is that in practice the tolerance limits m, M in (1.1) are often chosen subjectively. Since even a small change of limits can lead to a situation when the binomial test evaluates quality of the sample in a quite different way (from good to very bad), the test rejecting (1.2) on the basis of a "distance" (in some sense) from the ideal situation will be more appropriate than the binomial test.

Let the parametric set be

$$\Theta = \left\{ \begin{pmatrix} \mu \\ \sigma \end{pmatrix} ; \mu \in R, \sigma > 0 \right\}, \tag{1.4}$$

and for $\theta = (\mu, \sigma)' \in \Theta$ let

$$f(x, \theta) = (2\pi\sigma^2)^{-1/2} \exp\left[-(x - \mu)^2/2\sigma^2\right] \tag{1.5}$$

be the density of the normal $N(\mu, \sigma^2)$ distribution. Let $m < M$ be real numbers, $c > 0$ and

$$H = \left\{ \begin{pmatrix} \mu \\ \sigma \end{pmatrix} \in \Theta ; \mu + c\sigma \leqq M, \mu - c\sigma \geqq m \right\}. \tag{1.6}$$

If c is the $1 - \Delta/2$ quantile of the $N(0, 1)$ distribution and $\theta \in H$, then

$$\mathsf{P}_\theta[x \in \langle m, M \rangle] = \Phi\left(\frac{M - \mu}{\sigma}\right) - \Phi\left(\frac{m - \mu}{\sigma}\right) \geqq 2\Phi(c) - 1 = 1 - \Delta .$$

Although for such a c (1.6) only implies (but does not coincicide with) (1.2), from the point of view of the previous arguments it is of statistical interest to test this hypothesis under the normality assumption. We also remark that while the binomial test for a given sample size provides only a finite number of levels of significance, the set of levels of significance, available when the test (1.15) is used, is an interval, which is notable for small sample sizes.

The Fisher information matrix, defined in [3], p. 460 by the equality $J(\theta) = \operatorname{cov}\left(\frac{\partial \log f(x, \theta)}{\partial \theta_i}, \frac{\partial \log f(x, \theta)}{\partial \theta_j}\right)$, takes in the case of (1.5) the form

$$J(\theta) = \begin{pmatrix} \sigma^{-2}, & 0 \\ 0, & 2\sigma^{-2} \end{pmatrix}. \tag{1.7}$$

If $x_1, \ldots, x_n$ is a random sample from $N(\mu, \sigma^2)$ and

$$\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j, \quad s = \left[\frac{1}{n} \sum_{j=1}^{n} (x_j - \bar{x})^2\right]^{1/2}$$

then $s > 0$ with probability 1 and in such a case

$$\hat{J} = \begin{pmatrix} s^{-2}, & 0 \\ 0, & 2s^{-2} \end{pmatrix} \tag{1.8}$$

is an estimate of (1.7). The square of the sample information distance generated by the sample matrix (1.8) can be written in the form

$$(1.9) \qquad \hat{\varrho}^2(z, y) = \frac{\varrho^2(z, y)}{s^2}, \ \varrho(z, y) = |||z - y||| =$$

$$= [(z_1 - y_1)^2 + 2(z_2 - y_2)^2]^{1/2}.$$

If we denote for $C \subset R^2$ and $s > 0$

$$(1.10) \qquad \hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, C\right) = \inf\left\{\hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, y\right); \ y \in C\right\}$$

and for $\theta = (\mu, \sigma)' \in \Theta$ put

$$(1.11) \qquad \mathsf{P}_\theta(A) = \mathsf{P}\left[\begin{pmatrix}\bar{x}\\ s\end{pmatrix} \in A \mid N(\mu, \sigma^2)\right]$$

then the following assertion is true.

Theorem 1. *If $n > 1$, then for (1.6) and*

$$(1.12) \qquad G_n(t) = \sup \mathsf{P}_\theta\left[\hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H\right) > t; \ \theta \in H\right\}$$

the equality

$$(1.13) \qquad G_n(t) = \mathsf{P}\left[\hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H\right) > t \mid N\left(\frac{M + m}{2}, \left(\frac{M - m}{2c}\right)^2\right)\right] =$$

$$= \mathsf{P}\left[\hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, D\right) > t \mid N(0, c^{-2})\right]$$

holds, where

$$(1.4) \qquad D = \left\{\begin{pmatrix}\mu\\ \sigma\end{pmatrix} \in \Theta; \ \mu + c\sigma \leqq 1, \ \mu - c\sigma \geqq -1\right\}.$$

Hence the probability (1.12) depends only on n, t and c.

The theorem states, that if the test φ is defined by the formula

$$(1.15) \qquad \varphi(x_1, \ldots, x_n) = \begin{cases} \text{reject } H & \text{if } \hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H\right) > t \\ \text{accept } H & \text{if } \hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H\right) \leqq t \end{cases}$$

then the probability of rejection is maximized over H at its vertex $V = ((M + m)/2, (M - m)'/2c)$. Thus if $\alpha \in (0, 1)$ and

$$(1.16) \qquad G_n(t) = \alpha,$$

then α is the probability of the error of the first kind. The acceptance regions of this test are described in the following theorem.

Theorem 2. *Let* $t > 0$ *and*

$$A_t = \left\{\begin{pmatrix}\bar{x}\\ s\end{pmatrix};\ s > 0 \text{ and } \hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H\right) \leqq t\right\}. \tag{1.17}$$

(I) *If we denote*

$$B = B(m, M, t) = \sup\left\{s;\ \textit{there exists } \bar{x} \textit{ such that } \hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H\right) \leqq t\right\}, \tag{1.18}$$

$$L(s) = L(s, m, M, t) = \inf\{\bar{x};\ (\bar{x}, s)' \in A_t\}, \tag{1.19}$$

$$U(s) = U(s, m, M, t) = \sup\{\bar{x};\ (\bar{x}, s)' \in A_t\} \tag{1.20}$$

then

$$A_t = \left\{\begin{pmatrix}\bar{x}\\ s\end{pmatrix};\ 0 < s < +\infty,\ s \leqq B,\ \bar{x} \in \langle L(s), U(s)\rangle\right\} \tag{1.21}$$

where

$$B = \frac{M - m}{2} B(-1, 1, t) \tag{1.22}$$

$$L(s) = \frac{M + m}{2} - \frac{M - m}{2} U\left(\frac{2}{M - m} s, -1, 1, t\right), \tag{1.23}$$

$$U(s) = \frac{M + m}{2} + \frac{(M - m)}{2} U\left(\frac{2}{M - m} s, -1, 1, t\right).$$

(II) *The upper bound satisfies*

$$B(-1, 1, t) = \begin{cases} 2^{1/2}[c(2^{1/2} - t)]^{-1} & \textit{if } 0 < t < 2^{1/2} \\ +\infty & \textit{if } 2^{1/2} \leqq t. \end{cases} \tag{1.24}$$

(III) *Let us denote*

$$\delta = [2(1 + 2c^{-2})]^{1/2}, \quad s^* = [c(1 - t/\delta)]^{-1}. \tag{1.25}$$

If $0 < t < \delta$, *then* $0 < s^* < B(-1, 1, t)$ *and*

$$U(s, -1, 1, t) = \begin{cases} 1 + s\left[t\sqrt{\left(\frac{2 + c^2}{2}\right)} - c\right] & \textit{if } 0 < s \leqq s^* \quad (1.26) \\ [s^2t^2 - 2(s - c^{-1})^2]^{1/2} & \textit{if } s^* < s \leqq B(-1, 1, t). \quad (1.27) \end{cases}$$

(IV) *If* $t \geqq \delta$, *then*

$$U(s, -1, 1, t) = 1 + s[t^2 - 2]^{1/2}. \tag{1.28}$$

Theorem 2 enables us to perform the test (1.15), because according to (1.17) and (1.21) it is sufficient to verify whether $\bar{x} \in \langle L(s), U(s) \rangle$, where L, U are determined by (1.23). The shape of the acceptance regions is shown in Fig. 1.

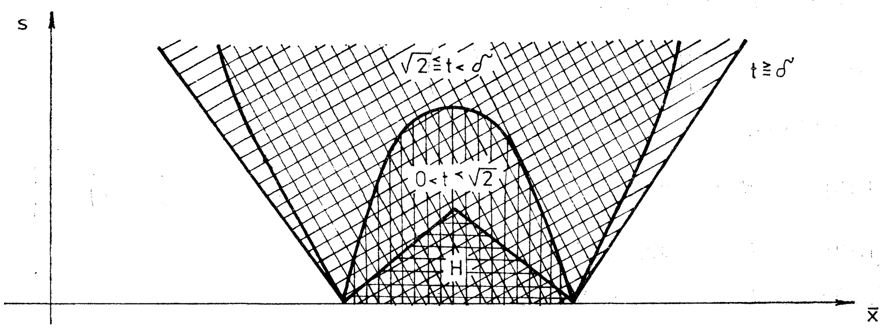

Fig. 1. Hypothesis 1.6 and its acceptance regions A_t

The case when the assumption of normality of the examined population is not valid, is dealt with (from the point of view of large samples) in the following assertion.

Theorem 3. *Let* Q *be a probability measure on* $(R^1, \mathscr{B}^1)$ *such that* $\int x^4 \, \mathrm{dQ}(x) < + \infty$. *Let us denote*

$$\mu = \int x \, d\mathrm{Q}(x), \quad \sigma^2 = \int (x - \mu)^2 \, \mathrm{dQ}(x), \quad \mu_j = \int (x - \mu)^j \, d\mathrm{Q}(x),$$

assume that $\sigma > 0$ *and put*

$$V = \begin{pmatrix} 1, & \mu_3/2\sigma^3 \\ \mu_3/2\sigma^3, & (\mu_4 - \sigma^4)/4\sigma^4 \end{pmatrix}. \tag{1.29}$$

$$K = \{y \in R^2;\ y_2 \leqq -|y_1|/c\}, \tag{1.30}$$

$$G(t, V) = \mathrm{P}[\varrho(z, K) > t \mid N(0, V)] \tag{1.31}$$

where ϱ *is the metric* (1.9). *Finally, let*

$$T_n \begin{pmatrix} \bar{x} \\ s \end{pmatrix} = n^{1/2} \hat{\varrho} \left(\begin{pmatrix} \bar{x} \\ s \end{pmatrix}, H \right). \tag{1.32}$$

If $(\mu, \sigma)' \in H$, *then for every* $t > 0$

$$\lim_{n \to +\infty} \mathrm{P}[T_n > t \mid \mathscr{L}(x) = \mathrm{Q}] \leqq G(t, V) \tag{1.33}$$

in the sense that the limit on the left-hand side of (1.33) *exists and equality in* (1.33) *holds if* $\mu = (M + m)/2$, $\sigma = (M - m)/2c$.

Thus, if the moments μ, σ^2, μ_3 and μ_4 of the true distribution are the same as in the normal case, then the limiting size of the test (1.15) is the same as under the normality assumption. Hence if we have no reason to assume that the true shape of the frequency curve differs from the normal curve in a significant way, we can expect that the test (1.15) with the constant t chosen so that (1.16) holds will have the size approximately α even for moderate n, because for these sample sizes the approximation based on the CLT usually yields good results.

If the hypothesis (1.6) is not true, i.e. at least one of the inequalities $\mu + c\sigma > M$, $\mu - c\sigma < m$ holds, the making use of $\bar{x} \to \mu$, $s \to \sigma$ we get $T_n \to +\infty$ and the power of the test based on the statistic T_n will tend to 1, if α is fixed.

We remark that for testing the hypothesis (1.6) also the likelihood ratio test statistic can be employed, but its critical regions are such that the computation of explicit bounds (especially the upper bound of the type (1.18)) leads to an expression which seems to be irresolvable, and therefore exact determination of the size of such a test cannot be performed for finite sample sizes. The asymptotic size of the LR test can be computed from the formulas derived in [5] and [6].

2. PROOFS OF ASSERTIONS FROM SECTION 1

Lemma 1. *The set* (1.17) *is convex and* (1.21) *holds.*

Proof. Since $\bar{H}$ is convex, for every $z \in R^2$ there exists a unique point $\pi(z) \in \bar{H}$ such that in the notation (1.9)

$$\varrho(z, \pi(z)) = \inf\{\varrho(z, y); y \in H\} .$$

Hence if $\tilde{\theta} = (\bar{x}, s)'$, $\theta^* = (\bar{x}_1, s_1)'$ belong to A_t, then (cf. (1.9))

$$|||\alpha\tilde{\theta} + (1 - \alpha)\, \theta^* - (\alpha\pi(\tilde{\theta}) + (1 - \alpha)\, \pi(\theta^*))||| \leqq$$
$$\leqq \alpha|||\tilde{\theta} - \pi(\tilde{\theta})||| + (1 - \alpha)\, |||\theta^* - \pi(\theta^*)||| \leqq (\alpha s + (1 - \alpha)\, s_1)\, t\, .$$

Since $\alpha\pi(\tilde{\theta}) + (1 - \alpha)\, \pi(\theta^*) \in \bar{H}$, by means of (1.9) we obtain that A_t is convex. This together with

$$\varrho\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H\right) = \varrho\left(\begin{pmatrix}M + m - \bar{x}\\ s\end{pmatrix}, H\right)$$

and the continuity of $\varrho(\cdot, A_t)$, (1.9) and (1.10) yields (1.21).

Lemma 2. (I) *If $a > 0$, b are real numbers and*

$$T\begin{pmatrix}\bar{x}\\ s\end{pmatrix} = \begin{pmatrix}t_1(\bar{x})\\ t_2(s)\end{pmatrix}, \quad t_1(\bar{x}) = a\bar{x} + b\, , \quad t_2(s) = as\, , \tag{2.1}$$

then

$$\mathsf{P}[A_t \mid N(\mu, \sigma^2)] = \mathsf{P}\left[\hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, T(H)\right) \leqq t \mid N(a\mu + b, (a\sigma)^2\right] \tag{2.2}$$

where

$$(2.3)\qquad T(H) = \left\{ \begin{pmatrix} \mu \\ \sigma \end{pmatrix} \in \Theta;\ \mu + c\sigma \leqq aM + b,\ \mu - c\sigma \geqq am + b \right\}.$$

(II) *If $\sigma > 0$ is a fixed number, then the function*

$$(2.4)\qquad \mathsf{P}(\mu) = \mathsf{P}(A_t \mid N(\mu, \sigma^2))$$

is non-decreasing on $(-\infty, (M + m)/2\rangle$ and non-increasing on $\langle (m + M)/2, +\infty)$.

Proof. It is well known, that $\bar{x}$ has the density

$$\frac{1}{\sigma} g_1\left(\frac{\bar{x} - \mu}{\sigma}\right), \quad g_1(y) = [n(2\pi)^{-1}]^{1/2} \exp[-ny^2/2].$$

Since $\xi = ns^2/\sigma^2$ has chi-square distribution with $(n - 1)$ degrees of freedom, making use of the formula for density of the transformed random variable (cf. [1], p. 47) we obtain that the density of s equals

$$\frac{1}{\sigma} g_2\left(\frac{s}{\sigma}\right), \quad g_2(s) = C(n)\, s^{n-2} \exp(-ns^2/2)\, \chi_{(0,+\infty)}(s)$$

where $C(n)$ is a constant depending on n. As is shown in Section 3.b.3 in [3], the random variables $\bar{x}$, s are independent, from which

$$(2.5)\qquad \mathsf{P}[A_t \mid N(\mu, \sigma^2)] = \int_{A_t} g_1\left(\frac{\bar{x} - \mu}{\sigma}\right) g_2\left(\frac{s}{\sigma}\right) \sigma^{-2}\, \mathrm{d}\bar{x}\, \mathrm{d}s.$$

(I) The equality

$$(2.6)\qquad \frac{\varrho(y, z)}{s} = \frac{\varrho(T(y), T(z))}{t_2(s)}$$

implies that

$$(2.7)\qquad T(A_t) = \left\{ \begin{pmatrix} \bar{x} \\ s \end{pmatrix};\ s > 0 \text{ and } \hat{\varrho}\left(\begin{pmatrix} \bar{x} \\ s \end{pmatrix}, T(H)\right) \leqq t \right\}$$

where $T(H)$ is the set (2.3). Since the Jacobian of the mapping (2.1) is a^2, (2.2) can be easily obtained from (2.5).

(II) If

$$(2.8)\qquad a = \frac{2}{M - m}, \quad b = -\frac{(M + m)}{M - m},$$

then according to (2.3) and (1.14)

$$(2.9)\qquad T(H) = D$$

and from (I) we obtain that

$$(2.10)\qquad \mathsf{P}(\mu) = \mathsf{P}[B_t \mid N(\mu^*, \tilde{\sigma}^2)] = \mathsf{P}_1(\mu^*)$$

where

$$(2.11) \qquad B_t = \left\{ \begin{pmatrix} \bar{x} \\ s \end{pmatrix}; s > 0, \hat{\varrho}\left(\begin{pmatrix} \bar{x} \\ s \end{pmatrix}, D \right) \leqq t \right\},$$

$$\mu^* = a\mu + b\,, \quad \tilde{\sigma} = a\sigma\,.$$

Since $\hat{\varrho}\left(\begin{pmatrix} \bar{x} \\ s \end{pmatrix}, D \right) = \hat{\varrho}\left(\begin{pmatrix} -\bar{x} \\ s \end{pmatrix}, D \right)$, according to Lemma 1 we have

$$(2.12) \qquad B_t = \left\{ \begin{pmatrix} \bar{x} \\ s \end{pmatrix}; s < +\infty, 0 < s \leqq B, \bar{x} \in \langle -U(s), U(s) \rangle \right\}$$

and (2.10) together with (2.5) yields

$$(2.13) \qquad \mathsf{P}_1(\mu^*) = \int_0^B \left[\Phi\left(\frac{n^{1/2}(U(s) - \mu^*)}{\tilde{\sigma}} \right) - \Phi\left(\frac{n^{1/2}(-U(s) - \mu^*)}{\tilde{\sigma}} \right) \right] . \; . \; \tilde{\sigma}^{-1} g_2(s/\tilde{\sigma})\, \mathrm{d}s\,.$$

Since the derivative of (2.13) can obviously be obtained by differentiating under the integration sign, denoting by f the density of Φ we get

$$(2.14) \qquad \frac{\partial \mathsf{P}_1(\mu^*)}{\partial \mu^*} = \int_0^B \left[f\left(\frac{n^{1/2}(-U(s) - \mu^*)}{\tilde{\sigma}} \right) - f\left(\frac{n^{1/2}(U(s) - \mu^*)}{\tilde{\sigma}} \right) \right] . \; . \; n^{1/2} \tilde{\sigma}^{-2} g_2\left(\frac{s}{\tilde{\sigma}} \right) \mathrm{d}s\,.$$

Let $\mu^* > 0$. Since $U(s) \geqq 0$,

$$|-U(s) - \mu^*| \geqq |U(s) - \mu^*|\,,$$

the derivative (2.14) is non-positive and P is non-increasing on $(M + m)/2, +\infty)$. The equality $\Phi(x) = 1 - \Phi(-x)$ together with (2.13) implies that

$$(2.15) \qquad \mathsf{P}_1(\mu^*) = \mathsf{P}_1(-\mu^*)\,,$$

and the lemma is proved.

Proof of Theorem 1. If we denote by ∂H the boundary of the set H, then making use of Lemma 2 (II) we obtain that

$$(2.16) \qquad G_n(t) = 1 - \inf\{\mathsf{P}_\theta(A_t); \theta \in \partial H \cap \Theta\}\,.$$

But (2.10) and (2.15) imply that

$$(2.17) \qquad \inf\{\mathsf{P}_\theta(A_t); \theta \in \partial H \cap \Theta\} = \inf\{F(\sigma); 0 < \sigma \leqq c^{-1}\}$$

where

$$F(\sigma) = \mathsf{P}\left[\hat{\varrho}\left(\begin{pmatrix} \bar{x} \\ s \end{pmatrix}, D \right) \leqq t \mid N(1 - c\sigma, \sigma^2) \right].$$

Putting in Lemma 2(I)

$$a = \sigma^{-1}, \quad b = -\sigma^{-1} + c$$

we obtain that

$$F(\sigma) = \mathsf{P}\left[\hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, D(\sigma)\right) \leqq t \mid N(0, 1)\right],$$

$$D(\sigma) = \left\{\begin{pmatrix}\mu\\ \sigma^*\end{pmatrix} \in \Theta; \mu + c\sigma^* \leqq c, \mu - c\sigma^* \geqq c - 2\sigma^{-1}\right\},$$

which implies the assertion of the theorem.

Lemma 3. *If* $\bar{x} \geqq 0$ *and* $s > 0$, *then in the notation* (1.10) *and* (1.14)

$$\hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, D\right)^2 = \begin{cases} \dfrac{(\bar{x}-1)^2}{s^2} + 2 & \text{if } \bar{x} > \dfrac{2}{c}s + 1, & (2.18)\\[2ex] \dfrac{(\bar{x}+cs-1)^2}{s^2}\,\dfrac{2}{2+c^2} & \text{if } \dfrac{2}{c}s + 1 \geqq \bar{x} \geqq \dfrac{2}{c}\left(s - \dfrac{1}{c}\right),\ \bar{x} + cs > 1, & (2.19)\\[2ex] 0 & \text{if } \dfrac{2}{c}s + 1 \geqq \bar{x} \geqq \dfrac{2}{c}\left(s - \dfrac{1}{c}\right),\ \bar{x} + cs \leqq 1, & (2.20)\\[2ex] \dfrac{\bar{x}^2 + 2(s - c^{-1})^2}{s^2} & \text{if } \dfrac{2}{c}\left(s - \dfrac{1}{c}\right) > \bar{x}. & (2.21) \end{cases}$$

Proof. If we denote

$$J = \begin{pmatrix}1, & 0\\ 0, & 2\end{pmatrix}$$

then in accordance with (1.9) we have

$$(2.22)\qquad |||x||| = (x'Jx)^{1/2}.$$

Since in the notation $C = \{x \in R^2; a'x + b \leqq 0\}$ and

$$(2.23)\qquad \pi(x) = \begin{cases} x - \dfrac{(a'x+b)}{a'J^{-1}a}J^{-1}a & \text{if } x \notin C,\\ x & \text{if } x \in C, \end{cases}$$

the inequality $(x - \pi(x))'\,J(\pi(x) - y) \geqq 0$ holds for every $y \in C$, (2.23) is the projection on C in the norm (2.22).

Let $C_j = \{x \in R^2; a_j'x + b_j \leqq 0\}$ and let $\pi_{i_1 \ldots i_k}$ be the projection on $\bigcap_{j=1}^{k} C_{i_j}^i$ in the norm (2.22). If we put $V = \{x \in R^2; a_j'x + b_j = 0, j = 1, \ldots, k\}$, then according to Lemma 4.2 in [4] the formula

$$\pi_{1\dots n}(x) = \begin{cases} \pi_{1\dots j-1\,j+1\dots n}(x) & \text{if} \quad \pi_{1\dots j-1\,j+1\dots n}(x) \in C_j\,, \\ \pi_V(x) & \text{if} \quad \pi_{1\dots j-1\,j+1\dots n}(x) \notin C_j\,, \quad j = 1, \dots, n \end{cases} \tag{2.24}$$

holds. In this notation π_V is the projection on V in the norm (2.22). We remark that (2.24) follows from the fact that $\varrho(z, K)$ is increasing on $\overline{y, \pi(x)}$ towards the point y for all $y \in K$, where π is the projection on the closed convex set K in the norm (2.22).

Obviously

$$\bar{D} = C_1 \cap C_2 \cap C_3$$

where

$$C_1 = \{x \in R^2; x_1 + cx_2 \leqq 1\}\,, \quad C_2 = \{x \in R^2; -x_1 + cx_2 \leqq 1\}\,,$$
$$C_3 = \{x \in R^2; -x_2 \leqq 0\}\,,$$

and according to (2.23)

$$\pi_1\begin{pmatrix}\bar{x}\\ s\end{pmatrix} = \begin{cases} \begin{pmatrix}\bar{x}\\ s\end{pmatrix} - \dfrac{(\bar{x} + cs - 1)}{2 + c^2}\begin{pmatrix}2\\ c\end{pmatrix} & \text{if} \quad \begin{pmatrix}\bar{x}\\ s\end{pmatrix} \notin C_1\,, \\ \begin{pmatrix}\bar{x}\\ s\end{pmatrix} & \text{if} \quad \begin{pmatrix}\bar{x}\\ s\end{pmatrix} \in C_1\,, \end{cases} \tag{2.25}$$

$$\pi_2\begin{pmatrix}\bar{x}\\ s\end{pmatrix} = \begin{cases} \begin{pmatrix}\bar{x}\\ s\end{pmatrix} + \dfrac{(\bar{x} - cs + 1)}{2 + c^2}\begin{pmatrix}-2\\ c\end{pmatrix} & \text{if} \quad \begin{pmatrix}\bar{x}\\ s\end{pmatrix} \notin C_2\,, \\ \begin{pmatrix}\bar{x}\\ s\end{pmatrix} & \text{if} \quad \begin{pmatrix}\bar{x}\\ s\end{pmatrix} \in C_2\,. \end{cases}$$

If $\bar{x} > \dfrac{2}{c}s + 1$, then

$$\begin{pmatrix}\bar{x}\\ s\end{pmatrix} \notin C_1\,, \quad \pi_1\begin{pmatrix}\bar{x}\\ s\end{pmatrix} \notin C_3$$

and from (2.24) we obtain that $\pi_{13}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}\right) = \begin{pmatrix}1\\ 0\end{pmatrix} \in C_2$. Proceeding in this way and making use of (2.24) and (2.25) we obtain after some calculation that for $\bar{x} \geqq 0$, $s > 0$

$$\pi_{123}\begin{pmatrix}\bar{x}\\ s\end{pmatrix} = \begin{cases} \begin{pmatrix}1\\ 0\end{pmatrix} & \text{if} \quad \bar{x} > \dfrac{2}{c}s + 1\,, \\ \pi_1\begin{pmatrix}\bar{x}\\ s\end{pmatrix} & \text{if} \quad \dfrac{2}{c}s + 1 \geqq \bar{x} \geqq \dfrac{2}{c}\left(s - \dfrac{1}{c}\right), \\ \begin{pmatrix}0\\ c^{-1}\end{pmatrix} & \text{if} \quad \dfrac{2}{c}\left(s - \dfrac{1}{c}\right) > x \geqq 0\,. \end{cases} \tag{2.26}$$

Taking into account (1.9) and (2.26) we easily obtain (2.18)–(2.21).

Proof of Theorem 2. (I) If a, b are the numbers (2.8), then taking into account (2.7), (2.3) and (2.12) we see that

$$T(A_t) = \left\{\binom{\bar{x}}{s};\ \hat{\varrho}\left(\binom{\bar{x}}{s}, D\right) \leqq t\right\}$$

(2.27) $$t_2(B) = B(-1, 1, t) \quad t_1(L(s)) = L(t_2(s), -1, 1, t)$$
$$t_1(U(s)) = U(t_2(s), -1, 1, t)$$

where according to (2.11) and (2.12)

(2.28) $$L(s, -1, 1, t) = -U(s, -1, 1, t)\,.$$

From (2.27) and (2.28) we can easily obtain (1.22) and (1.23).

(II) According to (2.21) for $s > c^{-1}$

$$\hat{\varrho}\left(\binom{0}{s}, D\right) = 2^{1/2}[1 - (cs)^{-1}]$$

which together with (I) implies (1.24).

(III) Obviously $0 < s^* < B(-1, 1, t)$. Let

$$0 < s \leqq s^*\,.$$

If $u = U(s, -1, 1, t)$ is the number (1.26), then

$$u = 1 - \frac{(2 + c^2)}{c^2}\frac{s}{s^*} + \frac{2}{c}s\,,$$

which leads to

(2.29) $$u > 0\,,\quad \frac{2}{c}s + 1 > u \geqq \frac{2}{c}\left(s - \frac{1}{c}\right).$$

Since (1.26) implies $u + cs > 1$, by virtue of (2.19) we have

(2.30) $$\hat{\varrho}\left(\binom{u}{s}, D\right)^2 = t^2\,.$$

But (2.29) and (2.19) imply that $\hat{\varrho}\left(\binom{\cdot}{s}, D\right)$ is increasing in a neighbourhood of u on the right from u, which together with (2.11) and (2.12) yields (1.26).

Let

(2.31) $$s^* < s \leqq B(-1, 1, t)\,.$$

The number (1.27) can be written in the form

(2.32) $$u = s[t^2 - 2(1 - (cs)^{-1})^2]^{1/2}\,,$$

which together with (2.31) and (1.24) yields $u \geqq 0$. From (2.31) and (2.32) we get

$$\left[u \Big/ \left(\frac{2}{c} s - \frac{1}{c}\right)\right]^2 < 1 .$$

Hence by (2.21) the equality (2.30) holds, and (1.27) is proved.

(III) Since $u \geqq 1 + 2c^{-1}s$, the relation (1.28) follows from (2.18).

Proof of Theorem 3. The central limit theorem yields

$$s^2 = \frac{1}{n} \sum_{j=1}^{n} (x_j - \mu)^2 - (\mu - \bar{x})^2 = \frac{1}{n} \sum_{j=1}^{n} (x_j - \mu)^2 + o_P(n^{-1/2}) .$$

Hence denoting

$$Y_n = \begin{pmatrix} \bar{x} \\ s^2 \end{pmatrix}$$

and making use of the multidimensional central limit theorem we obtain that

$$(2.33) \qquad \mathscr{L}\left[n^{1/2}\left(Y_n - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}\right)\right] \to N(0, W) , \quad W = \begin{pmatrix} \sigma^2, & \mu_3 \\ \mu_3, & \mu_4 - \sigma^4 \end{pmatrix} .$$

But if $\begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$ is an internal point of $U \subset R^2$ and

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : U \to R^2$$

is a function whose components f_1, f_2 possess in the interior U^0 of U all derivatives of the first order and these are continuous on U^0, then similarly as in [2], p. 366 one can prove that

$$(2.34) \qquad \mathscr{L}\left[n^{1/2}\left(f(Y_n) - f\begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}\right)\right] \to N(0, DWD')$$

where

$$D = \begin{pmatrix} \dfrac{\partial f_1(y)}{\partial y_1} , & \dfrac{\partial f_1(y)}{\partial y_2} \\ \dfrac{\partial f_2(y)}{\partial y_1} , & \dfrac{\partial f_2(y)}{\partial y_2} \end{pmatrix}_{y = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}} .$$

Combining (2.33) and (2.34) we get

$$(2.35) \qquad \mathscr{L}\left[n^{1/2}\left(\begin{pmatrix} \bar{x} \\ s \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma \end{pmatrix}\right)\right] \to N(0, \tilde{W}) , \quad \tilde{W} = \begin{pmatrix} \sigma^2, & \mu_3/2\sigma \\ \mu_3/2\sigma, & (\mu_4 - \sigma^4)/4\sigma^2 \end{pmatrix} .$$

In the notation (1.32) we have

$$(2.36) \qquad T_n = \frac{\varrho(\xi_n, C_n)}{s}$$

where

$$\xi_n = n^{1/2}\left[\begin{pmatrix} \bar{x} \\ s \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma \end{pmatrix}\right]$$

and ϱ is the distance (generated by the norm (2.22)) from the set

$$C_n = \{y \in R^2;\ y_2 \geqq -n^{1/2}\sigma,\ y_1 + cy_2 \leqq n^{1/2}M_1,$$
$$y_1 - cy_2 \geqq n^{1/2}M_2\},$$
$$M_1 = M - (\mu + c\sigma),\quad M_2 = m - (\mu - c\sigma).$$

If $\mu = (M + m)/2$, $\sigma = (M - m)/2c$, then $M_1 = M_2 = 0$ and obviously (cf. (1.30))

$$\varrho(z, C_n) \to \varrho(z, K) \tag{2.37}$$

from above. Combining (2.35)–(2.37) we obtain that (1.33) holds with the equality sign. Since

$$\varrho(z, C_n) \to \varrho(z, K^*)$$

from above, where

$$K^* = \begin{cases} \{y \in R^2;\ y_1 + cy_2 \leqq 0\} & \text{if } \mu + c\sigma = M,\ \mu - c\sigma > m, \\ \{y \in R^2;\ y_1 - cy_2 \geqq 0\} & \text{if } \mu + c\sigma < M,\ \mu - c\sigma = m, \\ R^2 & \text{if } \mu + c\sigma < M,\ \mu - c\sigma > m, \end{cases}$$

the theorem is proved.

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Súhrn

TESTOVANIE TOLERANČNEJ HYPOTÉZY POMOCOU INFORMAČNEJ VZDIALENOSTI

FRANTIŠEK RUBLÍK

V práci sa predkladá test hypotézy $\mu + c\sigma \leqq M$, $\mu - c\sigma \geqq$ o parametroch normálneho rozdelenia, založený na vzdialenosti, generovanej výberovou Fischerovou informačnou maticou, a sú odvodené explicitné vzorce pre kritické oblasti tohto testu.

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