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ON MESH INDEPENDENCE AND NEWTON-TYPE METHODS

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Summary. Mesh-independent convergence of Newton-type methods for the solution of nonlinear partial differential equations is discussed. First, under certain local smoothness assumptions, it is shown that by properly relating the mesh parameters H and h for a coarse and a fine discretization mesh, it suffices to compute the solution of the nonlinear equation on the coarse mesh and subsequently correct it once using the linearized (Newton) equation on the fine mesh. In this way the iteration error will be of the same order as the discretization error. The proper relation is found to be $H = h^{1/\alpha}$, where in the ideal case, $\alpha = 4$. This means that in practice the coarse mesh is very coarse. To solve the coarse mesh problem it is shown that under a Hölder continuity assumption, a truncated and approximate generalized conjugate gradient method with search directions updated from an (inexact) Newton direction vector, converges globally, i.e. independent of the initial vector. Further, it is shown that the number of steps before the superlinear rate of convergence sets in is bounded, independent of the mesh parameter.

Keywords: Nonlinear problems, Newton methods, mesh-independent convergence, two-level mesh method

AMS classification: 65H10, 65L60

1. Introduction

To solve nonlinear elliptic partial differential equations discretized by finite difference or finite element methods, for instance, a number of authors, such as [1, 2, 8], have investigated the question of mesh independent convergence of Newton iteration methods, i.e., they have considered whether the number of Newton steps, necessary to solve the problem to a certain accuracy depend on the finiteness of the discretization mesh.

In the present paper we consider this question from two different points of view.

In the first part we show that if we use the interpolant of the solution of the nonlinear problem on the coarse mesh as an initial approximation when solving the problem on the fine mesh, it suffices with one or two steps of the linearized (Newton) equation. This assumes that a certain relation, $H = h^{1/\alpha}$ holds between the coarse and fine mesh parameters for some positive α, $\alpha < 1$. The so computed solution has then an error of the same order as the discretization error. Certain local smoothness assumptions of the nonlinear function on the finite dimensional space are assumed to hold in some ball about the solution. In the ideal case, for piecewise linear finite element approximations, the relation is $H = h^{1/4}$, which means that, in practice, the coarse mesh contains very few nodepoints.

The method can be extended to a three-level (or, more generally, to a multilevel) method where the nonlinear problem is solved on the coarsest mesh, then corrected once with the linearized equation on the intermediate mesh and finally, the so found intermediate solution is corrected on the finest mesh. The relations between the mesh parameters in this case will typically be $h_0 = h_1^{1/2} = h_2^{1/4}$.

In the second part of the paper the solution of the problem on the coarse mesh is considered. As global convergence result is shown to hold if the nonlinear function F is differentiable and its derivative F' is nonsingular (except possibly at the solution) and Hölder continuous. To achieve the global convergence, a truncated generalized conjugate gradient type method is used to compute the successive approximations and the search directions which are then used are updated from a vector which is an approximate solution of the Newton (linearized) equation. The accuracy with which this latter equation must be solved is automatically monitored by the algorithm. In particular, at the initial stages the equation can be solved with much less precision then at the final stages. This can be expected to increase the efficiency of the method, because solving the Newton equations accurately when the approximate solution is far from the exact solution, is usually not justified. The computation of the solution is done by minimizing approximately the norm of the residual in a plane if two search directions are involved which latter is to be recommended for practical reasons. This minimization can be done approximately using two successive approximate line searches, for instance.

Finally, it is shown that the number of steps before the superlinear rate of convergence sets in, is bounded, independent of the mesh parameter.

2. A TWO-LEVEL MESH METHOD FOR NONLINEAR PROBLEMS

To illustrate the two-level mesh method we consider first the semilinear equation

$$-\Delta u + f(x, u) = 0, \quad x \in \Omega \subset \mathbf{R}^d, \ d \geqslant 2 \tag{2.1a}$$

with homogeneous boundary conditions. We assume for simplicity that f is twice continuously differentiable with respect to the variable u and that $\frac{\partial f}{\partial u}(x, u) \geqslant 0$. The variational formulation of (2.1a) is

$$a(u, v) + \big(f(\cdot, u), v\big) = 0 \quad \text{for all } v \in V = \mathring{H}^1(\Omega), \tag{2.1b}$$

where $a(u,v) = \int_\Omega \underline{\nabla} u \cdot \underline{\nabla} \mathrm{d}x$.

Assume for simplicity that Ω is piecewise polygonial and let $\Omega_H \subset \Omega_h \subset \Omega$, where Ω_H, Ω_h are two quasiregular finite element meshes. Let $V_H \subset V_h \subset V$, where V_H, V_h are two finite element subspaces spanned by piecewise continuous polynomial basis-functions of degree r on Ω_H and Ω_h, respectively.

The two-level mesh algorithm takes the following form:

Compute the Galerkin finite element solution on V_H, i.e. solve

$$a(u_H, v_H) + \big(f(\cdot, u_H)v_H\big) = 0 \quad \text{for all } v_H \in V_H \subset V, \tag{2.2}$$

to accuracy $O(h^{r+1})$.

Solve the Newton (linearized) equation on V_h,

$$a(u_h^0, v_h) + \big(f(\cdot, u_H), v_H\big) + \Big(\frac{\partial f}{\partial u}(\cdot, u_H)(u_h^0 - u_H), v_h\Big) = 0, \tag{2.3}$$

for all $v_h \in V_h$, and let u_h^0 be the approximate solution of u.

We want to compare the accuracy of u_h^0 with the accuracy of u_h, where u_h is the Galerkin finite element solution on V_h, that is,

$$a(u_h, v_h) + \big(f(\cdot, u_h), v_h\big) = 0 \quad \text{for all } v_h \in V_h. \tag{2.4}$$

The aim is to derive a relation $H = h^{1/\alpha}$, for which it holds that

$$\|u - u_h^0\|_1 = O\big(\|u - u_h\|_1\big), \quad h \to 0,$$

where u is the solution of (2.1b). To this end, we subtract (2.3) from (2.4), which shows that

$$a(u_h - u_h^0, v_h) + \big(f(\cdot, u_h), v_h\big) - \big(f(\cdot, u_H), v_h\big) - \Big(\frac{\partial f}{\partial u}(\cdot, u_H)(u_h^0 - u_H), v_h\Big) = 0,$$

for all $v_h \in V_h$, or

$$\begin{aligned} a(u_h - u_h^0, v_h) &+ \Big(\frac{\partial f}{\partial u}(\cdot, u_H)(u_h - u_h^0), v_h\Big) \\ &= -\Big(f(\cdot, u_h) - f(\cdot, u_H) - \frac{\partial f}{\partial u}(\cdot, u_H), v_h\Big) \\ &= -\Big(\frac{1}{2}\frac{\partial^2 f}{\partial u^2}(\cdot, \tilde{u}_h)(u_h - u_H)^2, v_h\Big) \end{aligned} \tag{2.5}$$

for some function $\tilde{u}_h$ between u_h and u_H. Taking $v_h = u_h - u_h^0$, using standard inequalities, (2.5) implies

$$\|u_h - u_h^0\|^2_{H^1(\Omega)} \leqslant C\|u_h - u_H\|^2_{L^2(\Omega)}\|u_h - u_h^0\|_{L^\infty(\Omega)} \tag{2.6}$$

where C is a bound of $\frac{1}{2}|\frac{\partial^2 f}{\partial u^2}|$.

Here we use the finite element inverse estimate,

$$\|v_h\|_{L^\infty(\Omega)} \leqslant C_0 h^{1-\frac{d}{2}-\varepsilon}\|v_h\|_{H^1(\Omega)}. \tag{2.7}$$

Here ε is an arbitrarily small but fixed positive number (Alternatively, we can let $\varepsilon = 0$ and $C_0 = O(\log h)$). In the following, C denotes a generic constant, independent of h. (2.6) and (2.7) show now

$$\|u_h - u_h^0\|_1 \equiv \|u_h - u_h^0\|_{H^1(\Omega)} \leqslant Ch^{1-\frac{d}{2}-\varepsilon}\|u_h - u_H\|^2_{L^2(\Omega)}$$

or, using the triangle inequality,

$$\|u - u_h^0\|_1 \leqslant \|u - u_h\|_1 + Ch^{1-\frac{d}{2}-\varepsilon}\left[\|u - u_h\|^2 + \|u - u_H\|^2\right].$$

Here $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. It remains to derive discretization error estimates in the $\|\cdot\|_1$ and $\|\cdot\|$ norms. This is done using standard finite element analysis and shows that for a sufficiently regular solution,

$$\|u - u_h\|_1 = Ch^r\|u\|_{r+1} \quad \text{and} \quad \|u - u_h\| = Ch^{r+1}\|u\|_{r+1},$$

where $\|u\|_{r+1}$ is the $(r+1)$st Sobolev norm. Hence

$$\|u - u_h^0\|_1 \leqslant C[h^r + h^{1-\frac{d}{2}-\varepsilon}H^{2(r+1)}]\|u\|_{r+1},$$

and to balance the two terms in the bracket we let

$$h^r = h^{1-\frac{d}{2}-\varepsilon}H^{2(r+1)},$$

that is,

$$H = h^{\frac{1}{2}(r-1+\frac{d}{2}+\varepsilon)/(r+1)}.$$

In this case

$$\|u - u_h^0\|_1 = O(\|u - u_h\|_1).$$

This shows that

$$H = \begin{cases} h^{\frac{1}{2}(r+\varepsilon)/(r+1)}, & d = 2 \\ h^{\frac{1}{2}(r+\varepsilon+\frac{1}{2})/(r+1)}, & d = 3 \end{cases}$$

so, for piecewise linear basis functions (where $r = 1$), we find approximately,

$$H \simeq \begin{cases} h^{1/4}, & d = 2 \\ h^{3/8}, & d = 3 \end{cases}$$

and for *any* r and $2 \leqslant d \leqslant 3$ we have

$$H \geqslant h^{\frac{1}{2}}.$$

Finally, using duality arguments, it can be shown that also

$$\|u - u_h^0\| = O(\|u - u_h\|).$$

It is readily seen that similar results can be derived when we use *three levels of meshes*, $\Omega_{h_0} \subset \Omega_{h_1} \subset \Omega_{h_2}$, in which case we solve in order:

(i) $a(u_{h_0}, v_{h_0}) + (f(\cdot, u_{h_0}), v_{h_0}) = 0$, for all $v_{h_0} \in V_{h_0}$

(ii) $a(u_{h_1}^0, v_{h_1}) + (f(\cdot, u_{h_0}), v_{h_1}) + (\frac{\partial f}{\partial u}(\cdot, u_{h_0})(u_{h_1}^0 - u_{h_0}), v_{h_1}) = 0$, for all $v_{h_1} \in V_{h_1}$

(iii) $a(u_{h_2}^0, v_{h_2}) + (f(\cdot, u_{h_1}^0), v_{h_2}) + (\frac{\partial f}{\partial u}(\cdot, u_{h_1}^0)(u_{h_2}^0 - u_{h_1}^0), v_{h_2}) = 0$, for all $v_{h_2} \in V_{h_2}$.

It can then be shown for *any* r that

$$\|u - u_{h_1}^0\| = O(\|u - u_{h_1}\|),$$
$$\|u - u_{h_2}^0\| = O(\|u - u_{h_2}\|),$$

provided $h_2 = h_1^{1/2} = h_0^{1/4}$.

It can also be seen that similar results can be derived when the mesh Ω_{h_2} is a local refinement of Ω_{h_1}.

We consider now the same method in a more general setting, namely for strongly monotone operator problems. This part is based on the presentation in [6].

Abstract boundary value problem

Let V be a Hilbert space with dual V' and consider a two-level solution procedure for nonlinear strongly monotone operator equations of the form:

Seek $u \in V$ satisfying

$$\langle F(u), v\rangle = \langle g, v\rangle, \quad \text{for all } v \in V. \tag{2.8}$$

Here $F\colon V \to V'$ and $g \in V'$ is given. F is assumed to be strongly monotone, locally Lipschitz and its Frechet derivative satisfies some additional smoothness to be specified later, around the solution.

A typical example is the quasilinear equation $\underline{\nabla} \cdot a(|\nabla u|^2)\underline{\nabla} u + f(u) = 0$ in Ω where either

$$0 < c_0 < a(\cdot) < C_1, \quad |a'(\cdot)| \leqslant C_2$$

for some constants c_0, C_1, C_2, which do not depend on the solution u, or

$$a(|\underline{\nabla} u|^2) = |\underline{\nabla} u|^{p-2}.$$

Here $\mathring{W}^{1,p}(\Omega)$, and $p = 2$ in the first case. Such equations occur in the mathematical modeling of torsion of a bar, electromagnetic field equations etc.

Letting $V_H \subset V_h \subset V$ be given finite dimensional subspaces as before, the two-level solution method takes the form

(a) Solve (2.8) on V_H,

$$\langle F(u_H), v_H\rangle = \langle g, v_H\rangle \quad \text{for all } v_H \in V_H.$$

(b) Correct once in V_h with the linearized (Newton) equation:

$$\left\langle F'(u_H)(u_h^0 - u_H), v_h\right\rangle = \langle g, v_h\rangle - \langle F(u_H), v_h\rangle,$$

for all $v_h \in V_h$.

The mapping $F\colon V \longrightarrow V'$ is assumed to satisfy the following three properties:

(i) F is *strongly monotone* on V_n in a ball about the solution u, i.e., there exists a function $\zeta(t)$ such that for all $u, v \in V_h$:

$$\langle F(u) - F(v), u - v\rangle \geqslant \zeta\big(\|u - v\|_V\big)\|u - v\|_V,$$

where $\zeta\colon [0, \infty) \longrightarrow R$ is an increasing function and $\zeta(0) = 0$, $\lim_{t\to\infty} \zeta(t) = \infty$.

(ii) F is *locally Lipschitz continuous* if there is a bounded function $\Gamma(r)$ such that, for all $r > 0$ and for all $u, v \in S(0, r) := \{v \in V; \|v - 0\|_V \leqslant r\}$,

$$\|F(u) - F(v)\|_{V'} \leqslant \Gamma(r)\|u - v\|_V.$$

(iii) F satisfies the following smoothness assumption:

$$\sup_{\chi^h} |\langle Q(u_h, u_H), \chi_h\rangle| \leqslant \sup_{\chi^h} K(H, h)\, \left|\left\langle (u_h - u_H)^2, \chi_h\right\rangle\right|,$$

where

$$\begin{aligned}\langle Q(u_h, u_H), \chi_H\rangle &\equiv \left\langle F(u_h) - \big(F(u_H) + F'(u_H)(u_h - u_H)\big), \chi_h\right\rangle \quad \text{for any } \chi_h \in V_h,\\ \langle F(u_h), v_h\rangle &= \langle g, v_h\rangle \quad \text{for all } v_h \in V_h\\ \langle F(u_H), v_H\rangle &= \langle g, v_H\rangle \quad \text{for all } v_H \in V_H\end{aligned}$$

and $K(H, h)$ is a problem dependent positive function (possibly constant).

If $V = \mathring{W}^{1,p}(\Omega)$, $a(|\underline{\nabla} u|^2) = |\underline{\nabla} u|^{p-2}$, $p \geqslant 2$ then it can be seen that $\zeta(s) = C(\frac{1}{2})^{p-2}s^{p-1}$ and for later use, we note that its inverse function is $\zeta^{-1}(s) = C^{1-q}2^{\frac{p-2}{p-1}}s^{q-1}$, where $\frac{1}{p}+\frac{1}{q}=1$. Further, in this case, it can be seen that

$$\Gamma(r) = (2p-3)r^{p-2}.$$

If F is twice continuously differentiable then (iii) holds with $K(H,h)$ a constant. In some applications however, this may not hold and then $K(H,h) \to \infty$ as H, $h \to 0$.

In some problems the strong monotonicity assumption may not hold for the variational formulation of the problem in the whole space but only in the finite dimensional subspace. This occurs for instance for the convection diffusion problem

$$-\underline{\nabla}(\varepsilon \underline{\nabla} u) + \underline{\nabla} \cdot \big(v(u)u\big) + f(u) = 0 \tag{2.9}$$

where $v(u)$ is vectorial function of u. It is well known that the standard Galerkin method applied on the original problem (2.9) is not stable uniformly in the parameter ε. When the problem is embedded in a certain third order differential equation problem it turns out that the Galerkin formulation of it becomes coercive (and hence the operator strongly monotone) in the finite element space, uniformly in the parameter ε for certain choices of the parameter involved in the embedding. For details in the linear operator case, see [5].

To find a ball about the origin in which the finite element solutions u_h, u_H can be found we consider first boundedness of the solution.

Lemma 2.1. *Suppose $F(\cdot)\colon V \to V'$ is strongly monotone on V_h and locally Lipschitz. Then*

$$\|u_h\|_V \leqslant r_0 := \zeta^{-1}\big(\|g\|_{V'} + \|F(0)\|_{V'}\big).$$

Proof. See [10]. □

This holds for any h. Assuming that $\|u_h - u\|_V \to 0$, it therefore also holds for the solution u. Next we derive a discretization error estimate.

Lemma 2.2. *Let $F(\cdot)\colon V \to V'$ be strongly monotone on V_h and locally Lipschitz continuous and let*

$$\langle F(u_h), v_h\rangle = \langle g, v_h\rangle \quad \textit{for all } v_h \in V_h.$$

Then, for all $\chi_h\colon \|u-\chi_h\| \leqslant r_0$, we have

$$\|u-u_h\|_V \leqslant \inf_{\chi_h \in V_h} \|u-\chi_h\|_V + \zeta^{-1}\big(\Gamma(2r_0) \inf_{\chi_h\in V_h} \|u-\chi_h\|_V\big) \tag{2.10}$$

where r_0 is defined in Lemma 2.1.

Proof. This follows readily from the variational formulations

$$\langle F(u), v\rangle = \langle g, v\rangle\,, \quad \text{for all } v \in V,$$
$$\langle F(u_h), v_h\rangle = \langle g, v_h\rangle\,, \quad \text{for all } v_h \in V_h,$$

and the orthogonality property,

$$\langle F(u) - F(u_h), v_h\rangle = 0, \quad \text{for all } v_h \in V_h.$$

□

In general, the second term converges slower, or, at best, with the same rate as the first term in (2.10).

Lemma 2.3. *Let $F(\cdot)\colon V \to V'$ be strongly monotone on V_h and differentiable in a ball about the solution u. Assume that $\lim_{r\to 0_+} \frac{1}{r}\zeta(r\|w_h\|_V) = \alpha\zeta(\|w_h\|_V)$ for some positive α, where $w_h \in V_h$. Then, for all $\xi_h \in V_h$ sufficiently close to u,*

$$\langle F'(\xi_h)w_h, w_h\rangle \geqslant \frac{1}{2}\alpha\zeta(\|w_h\|_V)\|w_h\|_V.$$

Proof. This follows readily from the definition of strong monotonicity. □

The following theorem gives now the basis for deriving the relation between H and h.

Theorem 2.4. *Suppose $F(\cdot)\colon V \to V'$ satisfies the assumptions* (i), (ii) *and* (iii). *Then, for $h(H)$ sufficiently small for Lemma* 2.3 *to hold,*

$$(2.11) \qquad \|u - u_h^0\|_V \leqslant \inf_{\chi_h\in V_h} \|u - \chi_h\|_V + \zeta^{-1}(\Gamma(2r_0) \inf_{\chi_h\in V_h} \|u - \chi_h\|_V) + \zeta^{-1}\Big(\frac{8}{\alpha}K(h, H)\|u - u_H)^2\|_{V'}\Big),$$

where r_0 is defined in Lemma 2.1.

Proof. By definition, u_h^0 satisfies

$$\big\langle F'(u_H)(u_h^0 - u_H), v_h\big\rangle = \langle g, v_h\rangle - \langle F(u_H), v_h\rangle\,.$$

Let u_h be the Galerkin finite element approximation, defined by

$$\langle F(u_h), v_h\rangle = \langle g, v_h\rangle\,, \quad \text{for all } v_h \in V_h.$$

Then

$$\langle F'(u_H)(u_h^0 - u_h), v_h \rangle = \langle F(u_h), v_h\rangle - \langle F(u_H), v_h\rangle \\ - \langle F'(u_H)(u_h - u_H), v_h\rangle \\ = \langle Q(u_h, u_H), v_h\rangle\,.$$

Taking $v_h = u_h^0 - u_h$ and using Lemma 2.3 shows now that

$$\alpha\frac{1}{2}\zeta(\|u_h^0 - u_h\|_V)\|u_h^0 - u_h\|_V \leqslant K(h,H)\|(u_h - u_H)^2\|_{V'}\|u_h^0 - u_h\|_V.$$

Hence

$$\|u - u_h^0\| \leqslant \|u - u_h\|_V + \zeta^{-1}\Big(\frac{4}{\alpha}K(h,H)(\|(u - u_H)^2\|_{V'} + \|(u - u_h)^2\|_{V'})\Big)$$

and Lemma 2.2 completes the proof. □

Remark. The proper relation between H and h to balance the (last two) terms in (2.11) depends on the particular problem considered. It is seen that we want

$$K(h,H)\|(u - u_H)^2\|_{V'} \leqslant \inf_{\chi_h \in V_h} \|u - \chi_h\|_V$$

to hold. For the semilinear problem (2.1), we have $K(h,H) = 1$ and the above relation takes the form

$$\|(u - u_H)^2\|_{V'} \leqslant Ch^r\|u\|_{r+1},$$

which holds, in particular, if

$$\|(u - u_H)^2\| \leqslant Ch^r\|u\|_{r+1}.$$

This requires an estimate of $\|u - u_H\|_{L^4(\Omega)}$, and such an estimate can be derived using inverse norm finite element estimates, in a similar way as was done above, which also leads to similar relations between H and h.

3. A GLOBALLY CONVERGENT APPROXIMATE NEWTON METHOD

Consider now the solution of the nonlinear problem on mesh Ω_H. In fact, the following presentation is more general, i.e., it is not restricted to problems with nonlinear strongly monotone operators arising from partial differential equations.

Let F be a differentiable mapping $\mathbf{R}^n \to \mathbf{R}^n$. The classical Newton method to solve $F(x) = 0$ has the form:

Given $x^0 \in \mathbf{R}^n$, for $k = 0, 1, \ldots$, solve for x^{k+1}:

$$F'(x^k)(x^{k+1} - x^k) = -F(x^k), \tag{3.1}$$

where x^{k+1} approximates a solution to the nonlinear equation.

The advantage with the method is that it converges fast, namely superlinearly in general (with a quadratic rate, if $F'(\cdot)$ is Lipschitz continuous), near the solution. However, except for special cases where F is globally convex for instance, the method may not converge unless the initial vector is sufficiently close to the solution. Hence the method must be modified to make it globally convergent.

There is another disadvantage of the classical method which can be improved. In practice, the "exact" solution of the linear equation (3.1) can be expensive and is also not justified when x^k is far from the solution. Therefore, it is efficient to solve the equation to some relative precision related to the present size of the norm of the residual. That is, one computes a vector p^{k+1} such that

$$\|F'(x^k)p^{k+1} + F(x^k)\| \leqslant \varrho_k \|F(x^k)\| \tag{3.2}$$

and lets

$$x^{k+1} = x^k + p^{k+1}.$$

This method is called *inexact Newton method.* Here $\{\varrho_k\}$ is a sequence of real numbers, $0 < \varrho_k < 1$, called forcing sequence. For convenience, in the following we let $\varrho_k \leqslant \frac{1}{2}$. In general, when one approaches a solution, ϱ_k should decrease. As we shall see, for the eventual superlinear convergence, ϱ_k must converge to zero sufficiently fast (a proper choice is $\varrho_k = O(\|F(x^k)\|^\gamma)$, for some γ, $0 < \gamma \leqslant 1$). Such a vector p^{k+1}, which is not uniquely defined, can be computed by some *inner* (linear) iterative method, for instance. In fact, the framework offered by inexact Newton method can be helpful even when solving linear equations $F(x) = Ax - b$, using variable preconditioners. In this case one lets $p^{k+1} = -M_k F(x^k)$, where M_k is a sequence of increasingly more accurate approximations of A^{-1}. One can also use some inner iteration steps with this preconditioner. Note that a variable preconditioner corresponds to a nonlinear operator. For a nonlinear problem, M_k is an approximate inverse of $F'(x^k)$.

Local convergence results (for the exact Newton method) i.e., results giving a ball around the solution in which the initial vector can be taken for convergence, can be found in [9], in [12] and in the references therein, for instance.

Several attempts have been made to make the method globally convergent, i.e., to make it converge for any choice of the initial vector. In [7] and in [3] it was shown that such global convergence could be achieved by using a damped (stepsize) version of the method, i.e., by letting

$$x^{k+1} = x^k + \tau_k p^{k+1},$$

where the stepsize τ_k was properly monitored (using back-tracking in [7] and based on the estimation of certain constants associated with F' in [3]). However, as it turned out in practice, the stepsizes τ_k were often unnecessarily small at the initial stages and the method converged therefore frequently too slowly.

In the present paper, the method will be coupled with an approximate minimization step over a low-dimensional subspace and it will be seen that it then converges globally, for any initial vector, even when the Jacobian matrix is singular at the limit point. *Furthermore, in this method there is no need to estimate any parameters because the stepsizes will be computed automatically by the algorithm.*

For the proof of the above result we shall assume that F' is Hölder continuous and $F'(x^k)$ is nonsingular. In a previous presentation of the above method by the author [4] it was assumed that the minimization problem on the subspace was solved exactly. This is relaxed here. Further, we clarify the situation when $F'(x)$ is singular at the solution.

Because the search direction vectors can be made orthogonal with respect to some inner product, or conjugate orthogonal with respect to some matrix, we call the combined method the approximate Newton direction nonlinear generalized conjugate gradient iteration method.

We give first a description of the algorithm and some recommendations for its implementation. The final section contains the proof of global convergence and a result on mesh independent convergence.

The approximate Newton direction nonlinear generalized conjugate gradient method. Given a non-increasing sequence $\{\varrho_k\}$, $0 \leqslant \varrho_k \leqslant \frac{1}{2}$, consider the iteration method:

Given x^0, for $k = 0, 1, \ldots$ compute p^{k+1} such that

$$\|F(x^k) + F'(x^k)p^{k+1}\| \leqslant \varrho_k \|F(x^k)\|. \tag{3.3}$$

Let

$$d^{k+1} = p^{k+1} + \sum_{j=1}^{s_k} \beta_j^{(k)} d^{k-j}, \quad 0 \leqslant s_k \leqslant r_k - 1 \tag{3.4}$$

where $\beta_j^{(k)}$ are computed to make

$$(d^{k+1}, d^{k-j})_1 = 0, \quad 0 \leqslant j \leqslant s_k.$$

Let

$$x^{k+1} = x^k + \sum_{j=0}^{r_k} \alpha_j^{(k)} d^{k+1-j}, \quad 0 \leqslant r_k \leqslant \min(r, k+1), \tag{3.5}$$

where $\alpha_j^{(k)}$, $0 \leqslant j \leqslant r_k$ are determined in such a way that

$$\|F(x^{k+1})\| \leqslant \min_{0<\tau\leqslant 1} \|F(x^k + \tau p^{k+1})\|.$$

Repeat until convergence.

Here $r \geqslant 0$. If $r = 0$ then we let $d^{k+1} = p^{k+1}$. $\|\cdot\|$ is in general based on a different inner product than $(\cdot,\cdot)_1$. Making the search direction vectors (conjugate) orthogonal is not required for convergence but only a matter of convenience. In practice it can be most efficient to make just $(d^{k+1}, d^k)_1 = 0$. Further, a practical choice of r is $r = 1$, in which case only two search directions are involved in the approximate minimization. In general, one can expect to get $\|F(x^{k+1})\| \leqslant \min_{0<\tau\leqslant 1} \|F(x^k + \tau p^{k+1})\|$ by first making some approximate line search for the minimum along the direction p^{k+1}. Starting from the point so found, this is followed by an approximate line search along the direction d^k. This gives a vector x^{k+1} which can be expected to be a reasonable approximation of the exact minimization vector $\hat{x}^{k+1} = \arg\min_{\alpha_0,\alpha_1} \|F(x^k + \alpha_0^{(k)} d^{k+1} + \alpha_1^{(k)} d^k)\|$. Note that since $p^{k+1} = d^{k+1} - \beta_0^{(k)} d^k$ (if $r = 1$), then

$$\|F(\hat{x}^{k+1})\| \leqslant \min_{0<\tau\leqslant 1} \|F(x^k + \tau p^{k+1})\|.$$

Clearly, there are other methods to compute x^{k+1}, repeatedly minimizing an interpolation function of $r+1$ variables, for instance, which is similar to popular methods used for line searches.

The vector p^{k+1} can be computed using an (inner) iteration method to approximately solve the linear problem $F'(x^k)p^{k+1} = -F(x^k)$. As we shall see, we shall let

$$\varrho_k = \min\left\{\|F(x^k)\|^{\xi}, \text{const}\, \kappa_k^{-1}\right\}$$

for some number ξ, $0 < \xi < 1$, where $\kappa_k = \|F'(x^k)\| \|F'(x^k)^{-1}\|$, i.e. κ_k is the condition number of $F'(x^k)$. If one uses a (generalized) conjugate gradient method to compute p^{k+1}, information gathered from the so computed coefficients in the conjugate gradient method can be used to estimate κ_k, and hence ϱ_k.

Furthermore, it can be efficient to "precondition" $F(x)$, i.e. letting $F(x^k) := M_k F(x^k)$, $F'(x^k) = M_k F'(x^k)$, where M_k is an approximation of the inverse of $F'(x^k)$, or $M_k = C_k^{-1}$ where C_k is an approximation of $F'(x^k)$. If we let $M_k = M$ be fixed, for $k = 0, 1, \ldots, k_0 - 1$ then this corresponds to working with the function $F(x) := MF(x)$. After every k_0 steps for some $k_0 \geqslant 1$, we can update M. The matrix M should be chosen so that the condition number κ_k of $F'(x^k)$ gets small. This helps in general in speeding up the inner iterations but also in speeding up the outer iteration method (3.3)–(3.5), as we shall see.

We present now the global convergence result.

4. Global convergence of the approximate Newton direction iteration method

Let F be a mapping $\mathbf{R}^n \to \mathbf{R}^n$ and let the vector sequence $\{x^k\}$ be computed by the algorithm in Section 3. Assume that $F'(x^k)$ is nonsingular (but $F'(x)$ may be singular at the limit point) and that

$$\delta = \sup_k \|F'(x^k)^{-1} F(x^k)\| / \|F(x^k)\|^{\nu} \tag{4.1}$$

exists for some ν, $0 < \nu \leqslant 1$. In particular, this can be seen to hold if F is a strongly monotone operator. Assume in addition that F' is Hölder continuous, i.e. there exists a γ, $0 < \gamma \leqslant 1$ such that

$$K_\gamma = (1+\gamma) \sup_{k, x \neq x^k} \left\| \int_0^1 [F'(x^k + t(x - x^k)) - F'(x^k)] \, \mathrm{d}t \right\| \Big/ \|x - x^k\|^{\gamma} \tag{4.2}$$

exists. Note that in practice it suffices to take the supremum in some balls about the points x^k. Note also that such a δ exists (with $\nu = 1$) if $F'(x^k)^{-1}$ is uniformly bounded and also if F has a multiple zero.

Theorem 4.1. *Under the above assumptions the sequence $\{x^k\}$ defined in the algorithm (3.3)–(3.5) where $\varrho_k \leqslant \min(\frac{1}{2} c_0 \kappa_k^{-1})$, and c_0 is a constant, converges for any x^0 and there exists an ε independent on k, $0 < \varepsilon < 1$, such that*

$$\|F(x^k)\| / \|F(x^0)\| \leqslant (1 - \varepsilon)^k.$$

Proof. Note first that by the assumption made in the algorithm,

$$\|F(x^{k+1})\| \leqslant \min \|F(\tilde{x})\|, \quad \tilde{x} = x^k + \tau_k p^{k+1}, \tag{4.3}$$

for any τ_k, $0 < \tau_k \leqslant 1$. Next

$$\begin{aligned} F(\tilde{x}) &= F(x^k) + F'(x^k)(\tilde{x} - x^k) \\ &\quad + \int_0^1 [F'(x^k + t(\tilde{x} - x^k)) - F'(x^k)](\tilde{x} - x^k)\mathrm{d}t \\ &= (1 - \tau_k)F(x^k) + \tau_k\big(F(x^k) + F'(x^k)p^{k+1}\big) \\ &\quad + \tau_k \int_0^1 \big[F'(x^k + t(\tilde{x} - x^k)) - F'(x^k)\big]p^{k+1}\mathrm{d}t. \end{aligned} \tag{4.4}$$

In order to estimate the norm of p^{k+1}, we use (3.3), to find

$$\begin{aligned} \|F'(x^k)^{-1}F(x^k) + p^{k+1}\| &\leqslant \|F'(x^k)^{-1}\| \, \|F(x^k) + F'(x^k)^{-1}p^{k+1}\| \\ &\leqslant \varrho_k \|F'(x^k)^{-1}\| \, \|F(x^k)\| \leqslant \varrho_k \|F'(x^k)^{-1}F(x^k)\|, \end{aligned}$$

where

$$\kappa_k = \|F'(x^k)^{-1}\| \, \|F'(x^k)\|.$$

Using the triangle inequality, this shows

$$\begin{aligned} \|p^{k+1}\| &\leqslant \|F'(x^k)^{-1}F(x^k)\| + \|F'(x^k)^{-1}F(x^k) + p^{k+1}\| \\ &\leqslant (1 + \varrho_k\kappa_k)\|F'(x^k)^{-1}F(x^k)\| \leqslant (1 + \varrho_k\kappa_k)\delta\|F(x^k)\|^{\nu}. \end{aligned} \tag{4.5}$$

This and (4.4) shows that

$$\begin{aligned} \frac{\|F(\tilde{x})\|}{\|F(x^k)\|} &\leqslant 1 - \tau_k + \tau_k\varrho_k \\ &\quad + \tau_k(1 + \varrho_k\kappa_k)\delta \, \frac{\| \int_0^1 [F'(x^k + t(\tilde{x} - x^k)) - F'(x^k)]\mathrm{d}t\|}{\|\tilde{x} - x^k\|^{\gamma}} \cdot \|\tilde{x} - x^k\|^{\gamma}, \end{aligned}$$

so, again using (4.5),

$$\frac{\|F(\tilde{x})\|}{\|F(x^k)\|} \leqslant 1 - \tau_k + \tau_k\varrho_k + \tau_k^{1+\gamma}[\delta(1 + \varrho_k\kappa_k)]^{1+\gamma}\frac{K_\gamma}{1+\gamma}\|F(x^k)\|^{\nu\gamma} \tag{4.6}$$

We now let $\tau_k = \min(\hat{\tau}_k, 1)$, where $\hat{\tau}_k$ minimizes the right hand side expression in (4.6). We find

$$\hat{\tau}_k = \left[\frac{1 - \varrho_k}{[\delta(1 + \varrho_k\kappa_k)]^{1+\gamma}} K_\gamma^{-1}\right]^{1/\gamma} \|F(x^k)\|^{-\nu}, \tag{4.7}$$

and

(4.8)

$$\frac{\|F(\tilde{x})\|}{\|F(x^k)\|} \leqslant \begin{cases} \varrho_k + [\delta(1+\varrho_k\kappa_k)]^{1+\gamma}\dfrac{K_\gamma^{-1}}{1+\gamma}\|F(x^k)\|^{\nu\gamma}, & \text{if } \hat{\tau}_k \geqslant 1 \text{ (i.e. } \tau_k = 1) \\ 1 - \dfrac{\gamma}{1+\gamma}\hat{\tau}_k(1-\varrho_k), & \text{if } \hat{\tau}_k < 1 \text{ (i.e. } \tau_k = \hat{\tau}_k < 1). \end{cases}$$

Initially, $\hat{\tau}_0 = [\frac{1-\varrho_0}{[\delta(1+\varrho_0\kappa_0)]^{1+\gamma}} K_\gamma^{-1}]^{1/\gamma}\|F(x^0)\|^{-\nu} > 0$ so (4.8) shows that

$$\frac{\|F(x^1)\|}{\|F(x^0)\|} \leqslant 1 - \varepsilon_0,$$

for some ε_0, $0 < \varepsilon_0 < 1$.

Further, since $\varrho_k\kappa_k$ is bounded, $k = 0, 1, \ldots$ all values $\hat{\tau}_k$ are bounded from below and there exists an ε, $0 < \varepsilon < 1$, such that

(4.9)
$$\frac{\|F(\tilde{x})\|}{\|F(x^k)\|} \leqslant 1 - \varepsilon, \quad k = 0, 1, \ldots$$

Further, since $\tilde{x} = x^k + \tau_k p^{k+1}$, then by assumption,

$$\|F(x^{k+1})\| \leqslant \min_\tau \|F(x^k + \tau p^{k+1})\| \leqslant \|F(\tilde{x})\|,$$

so

$$\frac{\|F(x^{k+1})\|}{\|F(x^k)\|} \leqslant 1 - \varepsilon, \quad k = 0, 1, \ldots$$

which shows the global convergence. □

Note that as $\|F(x^k)\|$ decreases monotonically, at least with the factor $1 - \varepsilon$, eventually for some $k = k_0$, $\hat{\tau}_k$ becomes bigger than one and (4.8) shows that

$$\frac{\|F(x^{k+1})\|}{\|F(x^k)\|} \leqslant \varrho_k + \text{const}\, \|F(x^k)\|^{\nu\gamma}.$$

Hence, if we let $\varrho_k = \min\{\|F(x^k)\|^{\nu\gamma}, \kappa_k^{-1}\}$ then this shows that *superlinear rate of convergence*,

$$\|F(x^{k+1})\| \leqslant O(\|F(x^k)\|^{1+\nu\gamma}), \quad k = k_0, k_0+1, \ldots$$

In particular, if F' is Lipschitz-continuous (i.e. $\gamma = 1$) and $\|F'(x^k)^{-1}\|$ is uniformly bounded, (so $\nu = 1$), then

$$\|F(x^{k+1})\| \leqslant O(\|F(x^k)\|^2), \quad k = k_0, k_0+1, \ldots$$

i.e. it shows the quadratic rate of convergence of the inexact Newton method. If F has a root of multiplicity m, then it can be seen that $\nu = \frac{1}{m}$ and the superlinear rate of convergence is correspondingly smaller.

Since the condition number κ_k increases near the solution, if F' is singular there, we must spend an increasing amount of work in solving the inexact Newton equations because $\varrho_k \kappa_k$ is bounded. However, it is well known that the number of iterations typically depends of the iteration error accuracy as $\log(\varrho_k^{-1})$. Hence, the dependence of the amount of work on ϱ_k is relatively minor.

Finally we state the following mesh independence result. Let $\mathbf{R}^n$ correspond to the finite dimensional space V_H and assume that upper bounds $\hat{\delta}$ and $\hat{K}_\gamma$ of the constant δ in (4.1) and of the Hölder constant in (4.2), respectively hold independent of the dimension n. Then it can be seen that $\hat{\tau}_k$ in (4.7) is bounded below by a number which does not depend on the dimension and (4.9) holds with an ε independent of the dimension. Hence (4.7) shows that

$$\hat{\tau}_k \geqslant \left[\frac{1-\varrho_k}{[\delta(1+\varrho_k\kappa_k)]^{1+\gamma}} K_\gamma^{-1}\right]^{\frac{1}{\gamma}} (1-\varepsilon)^{-\nu k} \geqslant \frac{\frac{1}{2}\hat{K}_\gamma^{-1}}{[\hat{\delta}(1+c_0)]^{1+\gamma}}(1-\varepsilon)^{-\nu k} \geqslant 1,$$

for $k \geqslant k_0$, where k_0 does not depend on n. We collect this result in the final Theorem.

Theorem 4.2. *Consider the finite element approximation of the nonlinear mapping $F(\cdot)\colon V \to V'$, on V_H, where V_H has dimension n. Assume that the constants δ and K_γ are bounded above uniformly with respect to n. Then the approximate Newton direction iteration method needs a bounded number of steps, where the bound holds uniformly in n, before the superlinear rate of convergence sets in.*

Conclusions

Under certain assumptions, we have shown two mesh-independence principles of (approximate) Newton iteration methods when solving strongly monotone operator problems. The first result shows that for a proper relation between the mesh parameters it suffices with solving the nonlinear problem on a coarse mesh and subsequently correct it once on the fine mesh, where the solution is wanted, to get the full discretization order of the error of the computed approximation.

The second result shows that when solving the nonlinear problem on the coarse mesh there is at most a finite number of steps independent on H, before the superlinear rate of convergence can be seen. Furthermore, this holds irrespective of the initial vector.

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