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TOPOLOGIES IN ℓ-GROUPS.

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This paper deals with the considerations of lattice-ordered groups which are topological groups and topological lattices at the same time (briefly with topological l-groups). Topology of l-groups is defined altogether by means of the complete system of neighbourhoods of zero Σ, what is possible with regard to homogeneity of l-groups.

First of all, it deals with the investigation of topological properties of l-groups in connection with their algebraic structure. A number of results about topological groups (they are not necessary topological lattices) is transferred here, as is given by Pontrjagin in [8]. Up to this time, considerations of similar kind for topological l-groups have been carried through only for special topologies, as it is e.g. the order topology (see [1]), or the interval topology (see papers [3], [4], [5], [6], [7], [11]).

Further on, some special kinds of topological l-groups are investigated. They are determined either by the algebraic character of l-groups or by the special topology of investigated l-groups. In the first case, it deals with l-groups with realization, and in the second case with topologies defined by means of a filter in the lattice of polars of l-group. The results in this part of this paper come out of the ideas of theory of disjunctivity and out of the properties of l-groups with realization which are mentioned in papers [9], [10].

PRELIMINARY NOTES AND DEFINITIONS

0.1: *A topological space* is the space in the meaning of Bourbaki (i.e. $\overline{\Phi} = \Phi$, $A \subseteqq \overline{A}$, $\overline{\overline{A}} = \overline{A}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$) — see [2]. This space is not a T_0-space which is usually defined by this condition: To arbitrary two different points of space there exists a neighbourhood at least of one of them that does not include the second point.

0.2: Lattice operations be indicated $\vee, \wedge$. Let Λ be a $\wedge$-semilattice with zero. *A filter* x in Λ is a non-empty subset of the set Λ with the following properties: 1. $0 \notin x$, 2. $a, b \in x \Rightarrow a \wedge b \in x$, 3. $a \in x$, $c \in \Lambda$, $c \geqq a \Rightarrow c \in x$. Each chain of filters (ordered by inclusion) is upper bounded and then accordingly to Zorn's Lemma, each filter is included in a maximal filter in Λ. Maximal filters are called *ultrafilters* in Λ. Further on, if Q is a lattice, then the set of all filters in Q, and the set of all ultrafilters in Q, be desiganted $\mathfrak{F}(Q)$, and $\mathfrak{UF}(Q)$, respectively. A dual object to a filter be called *antifilter*, a maximal antifilter be called *ultraantifilter*. The set of all ultraantifilters in Q be designated $\mathfrak{A}(Q)$.

0.3: *An* l-*subgroup* of an l-group G is a subgroup in G which is a sublattice of a lattice G as well. *A convex* l-*subgroup* in G is an l-subgroup H in G with the following property: $a, b \in H, g \in G, a > g > b \Rightarrow g \in H$. A convex l-subgroup which is a normal divisor in G as well is called l-*ideal* in G.

0.4: Elements $a, b \in G$ are *disjunctive* iff $|a| \wedge |b| = 0$; notation $a\delta b$. An element $g \in G$ is *a weak unit element* in G, if the only disjunctive element to g in G is zero. The needful terms as for the disjunctivity are given in introduction of [9]. Especially, *the disjunctive complement of* a set $A \subset G$ is the set $A' = \{g \in G : g\delta a$ for arbitrary $a \in A\}$. Further on, $A'' = (A')'$ and it holds $A''' = A'$. The set $A \subset G$ for which it holds $A'' = A$ is called *a polar in G*.[1]) The set a'' and a' for any element $a \in G$ is called *principal polar* and *dual principal polar*, respectively. The set of all polars, principal polars and dual principal polars, be designated Γ, Π, and Π', respectively.

0.5: The complement of a set $Q \subset G$ be indicated cQ. Further we shall indicate $g^+ = g \vee 0$, $g^- = g \wedge 0$, $|g| = g \vee -g$.If the elements a, b are incomparable, we write $a \| b$. If a set has no incomparable elements, then it is a *fully ordered set*.

0.6: We say that *an* l-*group G has a realization* if it is isomorphic with a subdirect sum of fully ordered groups G_x, $x \in M$ (notation $(G_x : x \in M)$). We shall assume that $G_x \neq 0$, for any $x \in M$. An l-group with realization is briefly called an *r-group*. Further let us introduce this indication for $f \in G, P \subset G, A \subset M : Z(f) = \{x \in M : f(x) = 0\}$, $cZ(f) = M \setminus Z(f) = \{x \in M : f(x) \neq 0\}$, $Z(P) = \{x \in M : f(x) = 0$ for arbitrary $f \in P\}$, $\Psi(A) = \{f \in G : f(x) = 0$ for arbitrary $x \in A\}$. A realization $(G_x : x \in M)$ of an l-group G be called *completely regular* if for any element $f \in G$ and any element $x \in Z(f)$ there exists an element $g \in G$ so that $x \in cZ(g) \subseteqq Z(f)$. Every r-group always has a completely regular realization, see [10].

0.7: Let G be an l-group and x an arbitrary ultrafilter in the set Γ of all polars in G; let us indicate Ux the union of all polars in x. Then the system $\mathfrak{R} = \{G/Ux : x \in \mathfrak{A}(\Gamma), Ux \neq G\}$ defines a realization of the l-group G, which is called the *Γ-realization of G*. Similarly the system $\mathfrak{S} = \{G/Ux : x \in \mathfrak{A}(\Pi')\}$ defines a realization of G, which is called the *Π'-realization of G*. The introduction of these terms is given in [10].

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[1]) This set A is sometimes called *a component in G*.

1.

Definition: A topological space G is called *a topological* 1-*group* if G has the following properties:

1. G is an 1-group.
2. The group operation (notation $+$) and lattice operations are continuous in G, it means to hold:

Let the symbol $\circ$ indicates any of the following symbols: $+$, $-$, $\vee$, $\wedge$ and let a, $b \in G$. Then for each neighbourhood W for which it holds $a \circ b \in W$ there exist neighbourhoods U, V such that $a \in U$, $b \in V$ and for arbitrary elements $x \in U$, $y \in V$ it holds $x \circ y \in W$.

Let G be a topological 1-group. Everywhere on, we shall indicate with Σ_G^* the complete system of neighbourhoods in G and with Σ_G the complete system of neighbourhoods of zero in G. In the case it does not come to misunderstanding, we shall write briefly Σ^* and Σ. With regard to homogeneity of 1-group G, Σ determines fully Σ^*. A topological 1-group G with the complete system of neighbourhoods of zero Σ be indicated (G, Σ).

1.1: Let G be an 1-group which is a topological group and one of operations $\vee$, $\wedge$ is continuous. Then the second operation is also continuous.

Proof: Let the operation $\wedge$ be continuous and let $W \in \Sigma^*$ be a neighbourhood of the point $a \vee b$. Then $-(a \vee b) = -a \wedge -b \in -W$. Accordingly to [8], p. 104, def. 22 b) there exists a neighbourhood $W_0 \in \Sigma^*$ such that $-a \wedge -b \in W_0 \subset -W$. According to supposition there exist neighbourhoods U, U_0, V. $V_0 \in \Sigma^*$ such that $a \in U_0 \subset -U$, $b \in V_0 \subset -V$ and $U \wedge V \subset W_0$. Hence $-U_0 \wedge -V_0 \subset U \wedge V \subset W_0 \subset \subset -W$. Together $U_0 \vee V_0 \subset W$, such that the operation $\vee$ is continuous. The second part is proved dually (with the change $\vee$ and $\wedge$).

The following theorems are principal for further considerations:

1.2. Theorem: Let (G, Σ) be a topological 1-group. Then Σ fulfils the following conditions:

1. The intersection of two arbitrary sets of Σ contains a set of Σ.
2. For any set $U \in \Sigma$ there exists a set $V \in \Sigma$ such that $V - V \subset U$.
3. For any set $U \in \Sigma$ and any element $u \in U$ there exists a set $V \in \Sigma$ such that $V + u \subset U$.
4. For any set $U \in \Sigma$ and any element $g \in G$ there exists a set $V \in \Sigma$ such that $-g + V + g \subset U$.
5. For any set $U \in \Sigma$ and any element $g \in G$ there exists a set $V \in \Sigma$ such that $(V - g^+) \vee (V + g^-) \subset U$.

Note: In the condition 5., it is possible to write equivalently $(V - g^-) \wedge (V + g^+) \subset U$.

Proof: The validity of conditions 1.—4. is proved in [8], p. 107, B.

We shall prove the condition 5: Let us choose $U \in \Sigma$, $g \in G$. The set $U + (g \vee 0)$ is a neighbourhood of the point $g \vee 0$ and as the operation $\vee$ is continuous, there exist neighbourhoods V' $V'' \in \Sigma^*$, such that $g \in V'$, $0 \in V''$ and $V' \vee V'' \subset U + (g \vee 0)$. According to the condition 4., there exists a neighbourhood $V_0 \in \Sigma$ such that $V_0 + g \subset V'$ and consequently $(V_0 + g) \vee V'' \subset U + (g \vee 0)$. Hence $(V_0 + g - g^+) \vee (V'' - - g^+) = [(V_0 + g) \vee V''] - g^+ \subset U$, i.e. $(V_0 + g^-) \vee (V'' - g^+) \subset U$.

Finally in accordance with condition 1., there exists a neighbourhood $V \in \Sigma$ such that $V \subset V_0 \cap V''$ and therefore it holds $(V + g^-) \vee \vee (V - g^+) \subset U$.

1.3. Theorem: Let G be an l-group. Let Σ be a system of subsets of G fulfilling the conditions 1.—5. of Theorem 1.2. Then (G, Σ) is a topological l-group. Topology in G defined by means of Σ is determined uniquely.

Proof: According to [8], p. 108, Th. 9, (G, Σ) is a topological group. Topology in G defined by means of Σ is uniquely determined. The continuity of lattice operations with regard to this topology is remaining to be proved: Let $g, h \in G$, U be a neighbourhood of the point $g \vee h$. It holds $g \vee h = [(g - h) \vee O] + h = (g - h)^+ + h$. Let us indicate $f = g - h$. Then in accordance with [8], p. 104, def. 22 a), there exist neighbourhoods U_1, U_2 such that $f^+ \in U_1$, $h \in U_2$ and $U_1 + + U_2 \subset U$. Further there exists a neighbourhood $U_3 \in \Sigma$ such that $U_3 + f^+ \subset U_1$. Regarding the condition 5, of 1.2. there exists a neighbourhood $V \in \Sigma$ such that $(V + f^-) \vee (V - f^+) \subset U_3$. Hence $(V + + f^- + f^+ + h) \vee (V - f^+ + f^+ + h) = [(V + f^-) \vee (V - f^+)] + f^+ + + h \subset U_3 + f^+ + h \subset U_1 + h \subset U_1 + U_2 \subset U$ and consequently $(V + f + h) \vee (V + h) \subset U$. Together $(V + g) \vee (V + h) \subset U$, what means the operation $\vee$ in G to be continuous. The continuity of operation $\wedge$ follows from 1.1. and the proof of the Theorem is finished.

1.4: Let (G, Σ) be a topological l-group. Then it holds:

1. The topological space G is discrete if, and only if, $\{O\} \in \Sigma$.
2. The topological space G is a Kuratowski space if, and only if, the intersection of all neighbourhoods of Σ is zero in G.
3. The topological space G is regular.
4. If G is a Kuratowski space, then G is completely regular.

Proof: 1. It results from [8], p. 106, A. 2. With regard to the homogeneity of G it holds, the intersection of all neighbourhoods of Σ is zero in G if, and only if, all points of G are closed. 3. Let $U \in \Sigma$. Then in accordance with the condition 2. of 1.2. there exists $V \in \Sigma$ such that $V - V \subset U$. We shall prove that $\overline{V} \subset U$. In fact if it is $p \in \overline{V}$, then an arbitrary neighbourhood of the point p has a common point with V. Consequently for the neighbourhood $p + V$ of the point p it holds $(p + V) \cap V \neq \Phi$, i.e. there exist elements $a, b \in V$ such that $p + a = b$. Hence $p = b - a \in V - V \subset U$. In conclusion, for arbitrary $U \in \Sigma$

there exists $V \in \Sigma$ such that $\bar{V} \subset U$ and consequently with regard to the homogeneity of 1-group G, G is a regular topological space. 4. It follows from [8], p. 114, Th. 10.

2.

Let us give now some examples of topological 1-groups.

2.1: Let G be a complete 1-group. Then G is a topological 1-group with respect to the order topology. The space G is, moreover, Hausdorff and completely regular.

Proof: It follows from the corollary of Th. 18, [1], p. 320.

2.2: A fully ordered additive group of real numbers R with the order topology is a topolotical 1-group.

2.3: Let R^n be an n-dimensional Euclidean space. Let a partial order on R^n be introduced in the following way:
$(x_1, x_2, \ldots, x_n) \geqq (y_1, y_2, \ldots, y_n)$, where $x_i, y_i \in R$, $i = 1,2, \ldots, n$ $\Leftrightarrow x_i \geqq y_i$, for $i = 1,2, \ldots, n$.

Then R^n is a topological 1-group with respect to the order topology.

The 1-groups in the examples 2.2 and 2.3 are complete 1-groups and so they have the properties described in 2.1.

2.4: *The interval topology* in an 1-group G is usually defined by means of the subbasis of closed sets. This subbasis is formed by closed intervals $\{g \in G : g \geqq a\}$, $\{g \in G : g \leqq a\}$, for all $a \in G$. The interval topology in an 1-group G is a Kuratowski one. If G is a topological group with respect to the interval topology, then its space is regular and also Hausdorff.

Northam has given in [7] an example of the additive group of continuous functions on the closed interval [0, 1], the space of which is not Hausdorff with respect to the interval topology. Choe in [5], Conrad in [3], Jakubík in [6], Wolk in [11] have been investigating the classes of 1-groups with the request so that these groups may form topological groups with respect to the interval topology. In all cases, these 1-groups had to be fully ordered and it arose a presumption that an 1-group which is a topological group with respect to the interval topology is fully ordered. This presumption has been disproved by Holland in [4] with the case of non-fully ordered topological 1-group:

Let G be the set of all o-automorphisms f of the fully ordered set R of real numbers that fulfil this condition:

$$(x - 1) f = xf - 1, \text{ for all } x \in R.$$

The group operation in G is defined by composition of o-automorphisms and the partial order in this way:

$$f \geqq g \Leftrightarrow fx \geqq gx, \text{ for all } x \in R.$$

2.5: Let R be the fully ordered additive group of real numbers. The topology in R is defined by means of the system Σ, which consists of the sets $U_r = \bigcup_m \{x \in R : m - r < x < m + r\}$, $m = 0, \pm 1, \pm 2, \ldots,$ for all rational numbers r, $0 < r < 1$.

Then R is a topological group with the complete system of neighbourhoods of zero Σ. It is neither a topological l-group nor a Kuratowski space.

Note: If we define the system Σ in the example 2.7 with the sets $U_r = \{x \in R : -r < x < r\}$, for all rational numbers r, $0 < r < 1$, then it holds for R the same as in 2.7 and R is a Kuratowski space.

3.

Let us go on in investigation of topological l-groups, whose complete system of neighbourhoods of zero is formed with subgroups.

3.1: Let (G, Σ) be a topological l-group and let $U \in \Sigma$ be a subgroup in G. Then U is a closed set.

Proof: Let $g \in \overline{U}$. Since $U + g$ is a neighbourhood of the point g, the sets $U + g$ and U have a common point. Then there exist elements f, $\in U$ such that $f + g = h$. Hence $g = -f + h \in U$ i.e. $g \in U$. Thus $\overline{U} \subset U$ and evidently $U \subset \overline{U}$, so that $\overline{U} = U$.

Corollary: Let (G, Σ) be a topological l-group and let any neighbourhood $U \in \Sigma$ be a subgroup in G. Then it holds:

1. Each open set in G is the union of closed sets.
2. Each closed set in G is the intersection of open sets.

Proof: Results follow immediately from 3.1.

3.2: Let (G, Σ) be a topological l-group, let any neighbourhood $U \in \Sigma$ be a closed set in G. Further on, let the topological space G be a T_0-space. Then G is a Kuratowski space.

Proof: Let G be a T_0-space and let us assume that $\bigcap_{U \in \Sigma} U \neq 0$. It means, there exists an element $0 \neq g \in G$ such that $g \in U$ for every neighbourhood $U \in \Sigma$ and g is also element of every neighbourhood of zero (which is not necessary an element of Σ).

Then there exists a neighbourhood W of the point g such that $0 \notin W$. Further there exists a neighbourhood $V \in \Sigma$ such that $V + g \subset W$. The neighbourhood $V + g$ is a closed set and thus $P = G \setminus (V + g)$ is an open set, $0 \in P$, $g \notin P$. There exists evidently a neighbourhood $Q \in \Sigma$ such that $0 \in Q$, $g \notin Q$, what is a contradiction. Thus, $\bigcap_{U \in \Sigma} U = 0$ and G is according to 1.4 a Kuratowski space.

Corollary: Let (G, Σ) be a topological l-group and let any neighbourhood $U \in \Sigma$ be a subgroup in G. Let further G be a T_0-space. Then G is a Kuratowski space.

Proof: It follows from 3.1 and 3.2.

3.3: Let $(G, \Sigma) \neq 0$ be a topological l-group and let a neighbourhood $U \in \Sigma$ exists, which is a proper subgroup in G. Then G is a disconnected topological space.

Proof: Let us assume G be a connected topological space and let U be a subgroup in G. According to 3.1 U is a closed set in G. And so is the set $G \setminus U$ closed in G and it holds $U \cup (G \setminus U) = G$. That is possible if, and only if, $G \setminus U = \Phi$ i.e. $G = U$ and this is a contradiction.

3.4: Let (G, Σ) be a topological l-group and let any neighbourhood $U \in \Sigma$ be a subgroup in G. Further let G be a T_0-space. Then G is a totally disconnected topological space.

Proof: Let E be a maximal connected set containing zero in G. Then $E \cap U \neq \Phi$ for any neighbourhood $U \in \Sigma$. Both sets E and U are closed and U is open, too. Therefore $E \cap U$ and $(G \setminus U) \cap E$ are closed sets in G. Thus, $(G \setminus U) \cap E = \Phi$ and hence $E \subset U$ for any neighbourhood $U \in \Sigma$. Further $E \subset \bigcap_{U \in \Sigma} U = \{0\}$ and in accordance with 3.2 and [8], p. 137, B) the space G is totally disconnected.

Note: All previous statements of this paragraph and the statement 1.4 hold in topological group, too.

3.5: Theorem: Let (G, Σ) be a topological l-group. Then the intersection of all neighbourhoods from Σ is a normal l-subgroup in G.

Proof: In the proof, the theorem 1.2 is used (conditions 2.—5.). Let $I = \bigcap_{U \in \Sigma} U \supseteqq \{0\}$. To each $U \in \Sigma$ and to each $a \in U$ there exists a neighbourhood $V \in \Sigma$ such that $V + a \subset U$. Particularly for any neighbourhood $U \in \Sigma$ and for any element $j \in I$ is $I + j \subset U$ i.e. $I + j \subset \bigcap_{U \in \Sigma} U = I$. Hence $I + I \subset I$. Further, to any neighbourhood $U \in \Sigma$ there exists a neighbourhood $V \in \Sigma$ such that $-V \subset U$. But $I \subset V$ i.e. $-I \subset -V \subset U$ and hence $-I \subset \bigcap_{U \in \Sigma} U = I$. Thus I is a subgroup in G.

For any neighbourhood $U \in \Sigma$ and any element $g \in G$ there exists a neighbourhood $V \in \Sigma$ such that $-g + V + g \subset U$. Thus, $-g + I + g \subset -g + V + g \subset U$ for any $U \in \Sigma$. It means to be $-g + I + g \subset I$.

To the end, it follows from the continuity of lattice operations, neighbourhoods V, $W \in \Sigma$ exist to any neighbourhood $U \in \Sigma$ such that $V \vee V \subset U$, $W \wedge W \subset U$. Then $I \vee I \subset \bigcap_{U \in \Sigma} U = I$ and $I \wedge I \subset I$ so that I is a normal l-subgroup in G.

Corollary: Let (G, Σ) be a topological 1-group and let any neighbourhood in Σ be a subgroup in G. Then the intersection of all neighbourhoods from Σ is a closed normal 1-subgroup in G.

If it is, moreover, any neighbourhood from Σ a convex subset in G, then the intersection of all neighbourhoods from Σ is an 1-ideal in G.

Proof: Results follow from 3.1, 3.5 and from this fact, that the intersection of convex sets is a convex set.

4.

In this paragraph, topological 1-groups are investigated, whose complete system of neighbourhoods of zero is formed by convex 1-subgroups.

Definition: A system $\mathfrak{C}$ of convex 1-subgroups of an 1-group G be called *an 0-system*, if following properties are fulfilled:

1. The intersection of arbitrary two elements of $\mathfrak{C}$ contains an element of $\mathfrak{C}$.

2. The system $\mathfrak{C}$ contains with each element both all elements conjugated with it.

An 0-system be called *a zero 0-system*, if the intersection of all its elements is zero in G.

4.1: An 0-system $\mathfrak{C}$ in an 1-group G is a zero one, iff $(G, \mathfrak{C})$ is a Kuratowski space.

Proof: Evident from 1.4.

4.2: A right (left) decomposition of an 1-group G modulo a convex 1-subgroup H is a congruence in the lattice G.

Proof: We carry out only for the right decomposition G/H.

It is sufficient to prove, for arbitrary elements $a, b \in G$, $h_1, h_2 \in H$ it holds $(h_1 + a) \vee (h_2 + b) — (a \vee b) \in H$ and $(h_1 + a) \wedge (h_2 + b) — — (a \wedge b) \in H$. It holds $(h_1 + a) \vee (h_2 + b) \leqq [(h_1 \vee h_2) + a] \vee [(h_1 \vee h_2) + + b] = (h_1 \vee h_2) + (a \vee b)$ and hence $(h_1 + a) \vee (h_2 + b) — (a \vee b \leqq \leqq h_1 \vee h_2$. In an analogical way, it holds $(h_1 + a) \vee (h_2 + b) \geqq \geqq [(h_1 \wedge h_2) + a] \vee [(h_1 \wedge h_2) + b] = (h_1 \wedge h_2) + (a \vee b)$ and hence $(h_1 + a) \vee (h_2 + b) — (a \vee b) \geqq h_1 \wedge h_2$. Together $h_1 \vee h_2 \geqq (h_1 + a) \vee \vee (h_2 + b) — (a \vee b) \geqq h_1 \wedge h_2$ and with regard to a convexity of H it is $(h_1 + a) \vee (h_2 + b) — (a \vee b) \in H$. Similarly we can prove $(h_1 + a) \wedge \wedge (h_2 + b) — (a \wedge b) \in H$.

4.3: Let K be a convex 1-subgroup in an 1-group G. Then for any element $g \in G$ it holds $(K + g^-) \vee (K — g^+) \subset K$.

Proof: According to 4.2 there is $(K + g^-) \vee (K — g^+) \subset K + + (g^- \vee —g^+) = K — (g^+ \wedge —g^-) = K$, because $g^+ \wedge —g^- = 0$.

Corollary: Any set of convex 1-subgroups of an 1-group G fulfils the condition 5. of the Theorem 1.2.

4.4: Theorem: Each 0-system of an l-group G is a complete system of neighbourhoods of zero.

Proof: Let L be an 0-system of an l-group G. We are going to prove, L fulfils the conditions of the Theorem 1.2. The condition 1. is fulfilled evidently. In the conditions 2. and 3., it is sufficient to put $V = U$ and their validity is guaranteed by the fact, it is dealt with l-subgroups in G. The condition 4. is fulfilled for $V = g + U - g$ (it follows from the definition of an 0-system). At the end, the condition 5. is fulfilled for $V = U$ with regard to the corollary 4.3. Thus, according to the theorem 1.3, (G, L) is a topological l-group.

5.

Each filter x in a lattice of l-ideals of an l-group G forms an 0-system. A topology defined in G in this way is not a discrete one, because the filter x does not contain zero in G. If, moreover, the intersection of l-ideals belonging to the filter x is zero, then it is dealt with a Kuratowski space. Let us investigate similar questions for an ultrafilter in the lattive of convex l-subgroups of an l-group G. Everywhere in this and in the next paragraph, let us denote K the lattice of convex l-subgroups of an l-group G.

Definition: A filter $x \in \mathfrak{F}(K)$ be called a *normal filter* if x contains with each element of x all with it conjugated elements. An ultrafilter $x \in \mathfrak{A}\mathfrak{F}(K)$, which is a normal filter, be called *a normal ultrafilter.*

Each normal filter in K forms an 0-system and its corresponding topology in G is evidently not a discrete one.

5.1: An ultrafilter $x \in \mathfrak{A}\mathfrak{F}(K)$ is normal, iff it holds:

To arbitrary elements $g \in G$ and $\varkappa \in x$ there exists an element $c \in \varkappa$,

(N) $c \neq 0$ such that $-g + c + g \in \varkappa$.

Proof: Let x be a normal ultrafilter. Then for arbitrary elements $\varkappa \in x$ and $g \in G$ there exists an element $\lambda \in x$ such that $-g + \varkappa + g = \lambda$. Let us choose an element $d \in \varkappa \cap \lambda$. Then it holds $g + d - g = c$, for some element $c \in \varkappa$. Thus, $-g + c + g \in \varkappa$ and the condition of the statement is fulfilled.

Conversely, let $x \in \mathfrak{A}\mathfrak{F}(K)$ be not a normal filter. Then there exists $\varkappa \in x$ such that $-g + \varkappa + g \notin x$ for suitable element $g \in G$. As x is an ultrafilter, there exists $\lambda \in x$ such that $\lambda \cap (-g + \varkappa + g) = 0$. Hence $(\varkappa \cap \lambda) \cap [-g + (\varkappa \cap \lambda) + g] = 0$ and x does not fulfil (N).

5.2: Let it be $x \in \mathfrak{A}\mathfrak{F}(K)$. Then the ultrafilter x forms an 0-system in G, iff the condition (N) is fulfilled.

Proof: 1. If (N) is fulfilled, then x is a normal ultrafilter according to 5.1, and so is an 0-system.

2. An ultrafilter $x \in \mathfrak{A}\mathfrak{F}(K)$ forms an 0-system. Then it contains

with each element all elements conjugated with it, thus it is a normal ultrafilter and according to 5.1 the condition (N) is fulfilled.

Corollary: Let $x \in \mathfrak{A}\mathfrak{F}(K)$ fulfils the condition (N). Then (G, x) is. a topological 1-group and the topological space G is not a discrete one.

Proof follows from 4.4.

6.

Let us state further the conditions guaranteeing the topological space of a topological 1-group, for which the system Σ is a normal ultrafilter in K, to be a Kuratowski space.

6.1: 1. Let Q be any lattice with zero. Then the infimum of elements of an ultrafilter x in Q is zero if, and only if, x is not principal filter.

2. Let Q be any lattice. Then an ultrafilter x in Q is principal filter if, and only if, x is generated by a minimal element in $Q \setminus \{0\}$.

Proof: Let $x \in \mathfrak{A}\mathfrak{F}(Q)$, which is not principal. If it is $\varkappa \in Q$, $\varkappa \leqq \lambda$ for all elements $\lambda \in x$, then it is $\varkappa \notin x$. Thus, there exists an element $\tau \in x$ such that $\tau \cap \varkappa = 0$. As $\varkappa \leqq \tau$, it is $\varkappa = 0$ i.e. $\bigwedge\limits_{\delta \in x} \delta = 0$.

Conversely, let $\bigwedge\limits_{\delta \in x} \delta = 0$ be for some ultrafilter x in Q. Then x is not principal, because for each principal filter y it holds $\bigcap\limits_{\delta \in y} \delta = \mu \neq \{0\}$, where μ is the least element in y.

The proof of the second part is evident.

6.2: In the lattice K of convex 1-subgroups of an 1-group G atoms are exactly these convex 1-subgroups that are isomorphic with a subgroup of a fully ordered additive group R of real numbers.

Proof: If $J \in K$ is isomorphic with a group $R_1 \subset R$, φ is the corresponding isomorphism and $J_0 \subseteqq J$, $J_0 \in K$, then $J_0\varphi$ is a convex 1-subgroup in R_1. Thus, $J_0\varphi = 0$ or $J_0\varphi = R_1$, and hence $J_0 = 0$ or $J_0 = J$. Hence J is an atom in K.

If J is an atom in K, then it is a fully ordered group. If it is not namely a fully ordered group, then there exist elements a, $b \in J$, $a \neq 0 \neq b$, $a\delta b$. Thus, $b \in a' \cap J$, $a \in a'' \cap J$, i.e. $a' \cap J \in K$, $0 \neq a' \cap J \subseteqq J$ and J is not an atom in K, what is a contradiction. Further J is an Archimedean group. If namely elements a, $b \in J$, $0 < a < b$ existed, such that it holds $na < b$ for all positive integers n, then the set of all elements $c \in J$ such that $|c| \leqq na$ for some positive integers n, would be a convex 1-subgroup different from zero and J, what is again a contradiction. Thus J is according to Hölder's Theorem isomorphic with a subgroup in R.

6.3: Theorem: Let $x \in \mathfrak{A}\mathfrak{F}(K)$. Then the ultrafilter x is a zero 0-system, iff the following properties are fulfilled:

1. To arbitrary elements $g \in G$, $\varkappa \in x$ there exists an element $c \in \varkappa$ such that $0 \neq c$ and $-g + c + g \in \varkappa$.

2. The ultrafilter x does not contain any convex l-subgroup in G, which is isomorphic with a subgroup of the fully ordered additive group of real numbers.

Proof: It follows from 4.1, 5.2, 6.1 and 6.2.

7.

Let us specialize the considerations from topological l-groups to r-groups, whose complete system of neighbourhoods of zero is formed by a filter in the set Γ, respectively Π'.

7.1: Let G be an r-group, $x \in \mathfrak{F}(\Gamma)$. Then x is an 0-system.

Proof: According to [9], Lemma 3 or [10], I., Theorem 2.2, all polars in G are l-ideals and then x is an 0-system.

7.2: Let G be an r-group. Then (G, Π') is a topological l-group for which it holds:

1. Π' is a zero 0-system in G.

2. G is a discrete topological space, iff G contains a weak unit element.

Proof: For an arbitrary element $g \in G$, g' is a convex l-subgroup in G. Further for arbitrary elements $g_1, g_2 \in G$ it holds $g_1' \cap g_2' = (|g_1| \vee |g_2|)'$. Elements of Π' are according to [10], I., Theorem 2.2. normal l-subgroups and by that it is proved Π' to be an 0-system in G. Further for $0 \neq g \in G$ it holds $g \notin g'$ and thus, $\bigcap_{g' \in \Pi'} g' = 0$ and Π' is a zero 0-system in G. Finally, let G be a discrete space. Then $\{0\} \in \Sigma = \Pi'$ i.e. $\{0\} = a'$, for a suitable element $a \in G$ and this is possible if, and only if, a is a weak unit element in G.

Note: If it is $\Sigma = \Gamma$ or $\Sigma = \Pi$, then a topology in G is discrete as $\{0\} = 0'' \in \Pi \subset \Gamma$.

Let (G, Σ) be a topological l-group and let $\Sigma \in \mathfrak{F}(\Pi')$. Then we denote $G(\Sigma) = \{g \in G^+ : g' \in \Sigma\}$.

7.3: Let (G, Σ) be a topological l-group, $\Sigma \in \mathfrak{F}(\Pi')$. Then it holds:

1. $G(\Sigma)$ does not contain any weak unit element in G.

2. $G(\Sigma)$ is a convex $\wedge$-subsemilattice of the lattice G.

Proof: As Σ is a filter in Π', it is $\{0\} \notin \Sigma$. Let $a \in G(\Sigma)$ be a weak unit element in G. Then $\{0\} = a' \in \Sigma$, what is a contradiction. Further let $r, s \in G(\Sigma)$, $g \in G$, $0 \leqq r < g < s$. Then $s' \subset g'$ and thus, $g' \in \Sigma$ i.e. $g \in G(\Sigma)$. Finally for $r, s \in G(\Sigma)$ it holds $(r \wedge s)' \supseteqq r'$ and then $r \wedge s \in G(\Sigma)$.

7.4: Let (G, Σ) be a topological r-group, $\Sigma \in \mathfrak{F}(\Pi')$. If we identify the r-group G with some of its complete regular realization, then for arbitrary elements $r, s \in G(\Sigma)$ there exists an element $t \in G(\Sigma)$ such that it holds $\Phi \neq Z(t) \subset Z(r) \cap Z(s)$.

Proof: Let r, $s \in G(\Sigma)$. Then there exists an element $t \in G(\Sigma)$ such that $t' = r' \cap s'$. Then it holds $Z(t') = Z(r' \cap s') \supseteqq Z(r') \cup Z(s') = cZ(r) \cup cZ(s) = c[Z(r) \cap Z(s)]$ according to (10], IV., 8.10. Moreover, t is not a weak unit element in G and thus according to [10], V., 12.11 it is $\Phi \neq Z(t) = cZ(t') \subset Z(r) \cap Z(s)$.

7.5: Let (G, Σ) be a topological r-group, $\Sigma \in \mathfrak{F}(\Pi')$. Then Σ is a zero 0-system, iff to any element $g \in G$, $g \neq 0$ there exists an element $s \in G(\Sigma)$ such that $|g| \wedge s \neq 0$.

Proof: Σ is a zero 0-system if, and only if, $\bigcap_{U \in \Sigma} U = 0$, what is equivalent with the fact, that to any element $0 \neq g \in G$ there exists an element $s \in G(\Sigma)$ such that $g \notin s'$.

Example: Let G be an r-group, which has a weak unit element. Let $\Sigma \in \mathfrak{A}\mathfrak{F}(\Pi')$ be a principal filter. Then (G, Σ) is a topological r-group according to 7.1 and accordintg to 6.1 it contains an atom in Π'. Further it holds $\bigcap_{g' \in \Sigma} g' = m' \neq \{0\}$, what means G not to be a Kuratowski space.

7.6. Theorem: Let G be a complete regular realization of an r-group and let $\Sigma \in \mathfrak{F}(\Pi')$. Then Σ is a zero 0-system in G, iff to any element $g \in G$, $g \neq 0$ there exists an element $s \in G(\Sigma)$ such that $Z(g)$ non $\supseteqq cZ(s)$.

Proof: Let $g \in G$, $s \in G(\Sigma)$, $g \neq 0$. According to [10], IV., 8.10, $g \in s'$ is equivalent with $Z(g) \supseteqq Z(s') = cZ(s)$. Hence it follows $|g| \wedge s \neq 0$ to be equivalent with $Z(g)$ non $\supseteqq cZ(s)$. It occurs according to 7.5 if, and only if, Σ is a zero 0-system in G.

Note: A realization of an r-group G, which is complete regular, is e.g. a Π'-realization, see [10], I.

Corollary: Let G be a complete regular realization of an r-group and let $\Sigma \in \mathfrak{F}(\Pi')$. Then (G, Σ) is a topological r-group, the space of which is a Kuratowski one, iff to any element $g \neq 0$, $g \in G$ there exists an element $s \in G(\Sigma)$ such that $Z(g)$ non $\supseteqq cZ(s)$.

Proof: Results follow from 4.1, 4.4 and 7.6.

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