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A NOTE TO D. GALE'S PRODUCTIVITY

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The sufficient condition for $n + 1$ points on the n-sphere in euclidean space E^n to be „global" (reformulated as a covering theorem) and its application to the theory of non-negative matrices and to the economy is given. I am indebted to K. Kříž for consultations in economics. Other economic applications of presented theorems are given in [6].

Let $\gamma = \{R_1, \ldots, R_{n+1}\}$ be a covering of a set $E^n - \{o\}$[1]) by open half-spaces R_i' s the boundaries of which H_i' s pass through o. Then every R_i is *indespensable* (i.e. $\gamma - \{R_i\}$ is not a covering) and the sets $S_i = \; = : \bigcap_{j \neq i} (-R_j)$ are mutually disjoint n-dimensional sharp cones satisfying $H_i \cap \bar{S}_i = \{o\}$[2]) (Since γ is the covering of $E^n - \{o\}$, $\bigcap_{j \neq i} H_j = \{o\}$ is valid for any i. Consequently every R_i is indispensable and $\bar{S}_i - \{o\} \subset \; \subset R_i$). Hence it follows that the sets $r_j^{(i)} = : R_j \cap H_i \; (j \neq i)$ are open $(n - 1)$ - half - spaces in H_i and $\sigma^{(i)} = : \{r_j^{(i)}\}_{j \neq i}$ forms the covering of the set $H_i - \{o\}$. For an arbitrary $x \in S_i$, $-r_j^{(i)}$ is a projection of the set $(-\bar{R}_j) \cap (-R_i)(j \neq i)$ in the direction $\overrightarrow{ox}$ into H_i (as the open projecting ray lies always in $-R_j$) and consequently the projection of the set $\bar{S}_j - H_i$ is $s_j^{(i)} = : \bigcap_{i \neq k \neq j} (-r_k^{(i)})$ for any $j \neq i$.

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[1]) A point x of the euclidean n-dimensional space E^n is a column (i.e. an n-by-1 matrix with x^i as its i-th component). o means the origin, $d(x, y) = \sqrt{{}^T(x - y)(x - y)}$ the distance (for matrices A, B ${}^TA$ means the transpose of A, AB the matrix multiplication row by column and for a nonsingular C, C^{-1} denotes its invers, i.e. $CC^{-1} = \; = I = (\delta_{ij})$), $\mathfrak{S}_{n-1}(x_0, r) = \{x : d(x, x_0) = r\}$ the spherical $(n - 1)$ - space, and for U, V, $X = \{x_1, \ldots, x_k\} \subset E^n$ we denote by $\bar{U}$ or IU the closure or the interior of U (in the usual metric topology), $U + V = \{x : x = x_1 + x_2, \; x_1 \in U, \; x_2 \in V\}$, $-V = \{x : x = -y, \; y \in V\}$, $CX = \{x : x = \sum_{i=1}^{k} \lambda_i x_i, \; \lambda_i \geqq 0, \; \sum_{i=1}^{k} \lambda_i = 1\}$ (called a *convex polyhedron*) and $KX = \{x : x = \sum_{i=1}^{k} \lambda_i x_i, \; \lambda_i \geqq 0\}$ (called a *cone*). KX is called a *sharp cone* if $C(X + \{o\})$ has o as its vertex. For $K = \{i_1, \ldots, i_k\} \subset \; \subset \{1, \ldots, n\} = : N$ x^K is a point in E^k such that ${}^Tx^K = (x^{i_1}, \ldots, x^{i_k})$ if $x \in E^n$ is given. For a matrix A we denote A^i or A_j its i-th row of j-th column.

[2]) $\bar{R}_i = : R_i \cup H_i$, $-\bar{S}_i = \bigcap_{j \neq i} \bar{R}_j$, hence $\bar{R}_i$' s are closed half-spaces and S_i' s are interiors of $\bar{S}_i$' s.

Theorem 1. *For a set* $X = \{x_i\}_{i=1}^{n+1}(n \geqq 2,\ o \notin X$ *and always* $x_i \in \overline{S}_i)$ *we have* $o \in ICX$ *and* $-\overline{S}_i \subset K(X - \{x_i\})$ *for all* i, *if it is*

(α) $x_{n+1} \in S_{n+1}$

and

(β) $x_n \notin H_{n+1},\ x_{n-1} \notin H_n \cap H_{n+1}, \ldots, x_2 \notin \bigcap_{k=3}^{n+1} H_k$.

Proof: Since $-\overline{S}_i \subset K(X - \{x_i\}$ follows from $o \in ICX$ (For $o \in ICX$, $K_i = : K(X - \{x_i\})$ it is $\bigcup_{k=1}^{n+1} K_k = \mathrm{E}^n$. Since $(\bigcup_{k \neq i} K_k) \cap (-S_i) = 0$ (cones $-S_i$, $K_k (k \neq i)$ are separated by the hyperplane H_k) it is $-S_i \subset K_i$) it is sufficient to prove $o \in ICX$. The proof will be done by means of the mathematical induction. If we project the set $X - \{x_{n+1}\}$ into H_{n+1} in the direction $\overrightarrow{ox_{n+1}}$ (the corresponding projections will be denoted by an asterisk), it is (for $j = 1, 2, \ldots, n$) $x_j^* \in \bar{s}_j^{(n+1)}$. Furthermore, there is $x_n^* \in s_n^{(n+1)}$ (because $x_n \notin H_{n+1}$), $x_{n-1}^* \notin h_n^{(n+1)} = : H_n \cap H_{n+1}$ (since it is $x_{n-1} \notin h_n^{(n+1)}$, we have in the case $x_{n-1}^* \in h_n^{(n+1)}$ $x_{n-1} \neq x_{n-1}^*$ and therefore $x_{n-1} \in R_n$ (because the open back projecting ray $\overrightarrow{x_{n-1}^* x_{n-1}}$ is disjoint with $(-\overline{R}_n)$ if it starts on H_n) — a contradiction), $x_{n-2}^* \notin h_{n-1}^{(n+1)} \cap h_n^{(n+1)}$ (in the case $x_{n-2}^* \in h_{n-1}^{(n+1)} \cap h_n^{(n+1)}$ it would be analogically $x_{n-2} \in R_{n-1}$ — a contradiction as before), $\ldots$, $x_2^* \notin \bigcap_{k=3}^{n} h_k^{(n+1)}$. Thus conditions ($\alpha$), ($\beta$) are fulfilled for the set $X_{n+1}^* = : \{x_1^*, \ldots, x_n^*\}$ in H_{n+1} and consequently we have $o \in ICX_{n+1}^*$ in the case that our theorem holds for the dimension $n - 1 (n > 2)$. Consequently there exist nonnegative λ_i' s not all zero such that $o = \sum_{i=1}^{n} \lambda_i x_i^*$. All λ_i' s are positive, because each of points x_i^* is indispensable for the condition $o \in ICX_{n+1}^*$. For $x = : \sum_{i=1}^{n} \lambda_i x_i$ it is $x^* = o((\Sigma\lambda_i x_i)^* = \Sigma\lambda_i x_i^* = o)$ and $x \neq o(x_n \notin H_n,\ \lambda_n > 0)$. Consequently $K\{x_{n+1}, - x_{n+1}\} \subset KX$. Let $y \in \mathrm{E}^n$ and $y^* = \sum_{i=1}^{n} \mu_i x_i^*,\ \mu_i \geqq 0$. Since $y_1 = \Sigma\mu_i x_i (\in KX)$ is situated on the projecting ray $p (\subset KX)$ of the point y, we have $y \in KX$. Thus $KX = \mathrm{E}^n$, which means $o \in ICX$. Since our theorem is true for the plane (it can be seen easily) $o \in ICX$ follows from (α), (β) for any $n \geqq 2$, q.e.d.

Remark. 1. λ_i's in $x = \sum_{j \neq i} \lambda_j x_j$ exist, are uniquely determined and all positive for all $x \in -S_i$ (it follows from $-S_i \subset K(X - \{x_i\})$).

Remark 2. Namely we have $KX = \mathrm{E}^n$ in the case $x_i \in S_i$ $(i = 1, 2, \ldots, n + 1)$.

Remark 3. Theorem 1 also expresses the sufficient condition for the set $X = \{x_1, \ldots, x_{n+1}\}$ of points on the spherical $(n-1)$ - space $\mathfrak{S} = \mathfrak{S}_{n-1}(o, r)$ to be *global* (which should mean that $o \in ICX$), if we consider spherical simplices S_i constructed by open hemispheres R_i forming the covering $\gamma = \{R_i\}_{i=1}^{n+1}$ of the set $\mathfrak{S}$: X is global, if it holds $x_i \in \overline{S}_i$ for all i and (α), (β) are true. The property of the system γ to be a covering is essential, 'because the set $X = \{x_1, x_2, x_3, x_4\}$, ${}^Tx_1 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, ${}^Tx_2 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, ${}^Tx_3 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, ${}^Tx_4 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ lies on the unit sphere $\mathfrak{S}$ with its center in the origine, a system $\gamma = \{R_i\}$, $R_1 = \left\{x^3 < \frac{3}{2}x^2\right\} \cap \mathfrak{S}$, $R_2 = \left\{x^3 < -\frac{3}{2}x^2\right\} \cap \mathfrak{S}$, $R_3 = \left\{x^3 < \frac{3}{2}x^1\right\} \cap \mathfrak{S}$ and $R_4 = \left\{x^3 < -\frac{3}{2}x^1\right\} \cap \mathfrak{S}$ defines spherical triangles $-S_i = \bigcap_{j \neq i} R_j$, $x_i \in S_i$ for all i, γ does not cover $\mathfrak{S}$ and it is $KX \neq E^3$.

Theorem 1 has an interesting application in the economics: Let us assume the production fulfils the following conditions:

(P_1) There exist $n(\geqq 2)$ different kinds of goods.

(P_2) Each of goods is measured in the fixed units.

(P_3) A square matrix $A = (a_{ij})$ of the type n-by-n is given where a_{ij} is the quantity of the j^{th} good which vanishes during the production (any production process is of the unit time duration) of the unit of the i^{th} one.

(P_4) It is $A^i \geq {}^To$ for all i.[3])

(P_5) The production takes place in production branches.

(P_6) Each of goods is produced in one production branche only (nameny, the i-th good in the branche i).

(P_7) Each branche produces one kind of goods only.

Thus, A can be considered as an input matrix (A^i is an input for the i-th branche, if it produces the unit of the good i) for the production with the output unit matrix $I = (\delta_{ij})$. The intensity vector of production $x \in E^n$ is called *reproductive*, if $x > o$ and ${}^Tx(I - A) \geqq {}^To$. We say S, $\emptyset \neq S \subset N = : \{1, 2, \ldots, n\}$ is the *fundament* for the vector x if it is $[{}^Tx(I - A)]^S = {}^To^S$ and $a_{ij} = 0$ for $i \notin S$, $j \in S$.

(P_8) There exists a reproducting intensity vector with no fundament.

Theorem 2. *For the production of type* $P_1 - P_8$ *the matrix* $I - A$ *is*

[3]) For $x, y \in E^n$ we write $x > y$, or $y \geq y$, or $x \geqq y$ if for all i $x^i > y^i$, or $x^i \geqq y^i$ and $x \neq y$, or $x^i \geqq y^i$, respectively.

regular and we have $(I - A)^{-1}\, l \gg o$ *for* $l \gg o$ *(with* $(I - A)^{-1}\, l > o$ *for* $l > o$*).*

Proof. Let y be a reproductive intensity vector of the production given by P_8. Put $x_{n+1} =: -y$, $R_{n+1} = \{x : {}^T yx < 0\}$, $x_j =: Z_j (Z =: (I - A))$ and $R_j = \{x : x^j > 0\}$ for $1, 2, \ldots, n$, $\gamma =: \{R_1, \ldots, R_{n+1}\}$. γ covers $E^n - \{o\}$ ($E^n - \bigcup_{j=1}^{n} R$ is non-positive cone (i.e. $\{x : x \leqq o\}$) which is (except for its vertex) contained in R_{n+1}), $o \notin X =: \{x_1, \ldots, \ldots, x_{n+1}\}$ ($x_{n+1} \neq o$ because $y > o$; but it as also $Z_j \neq o$ because in the case of any $Z_j = o$ it would be $a_{ij} = 0$ for $i \neq j$ and $a_{jj} = 1$, and $S =: \{j\}$ would be a fundament), $x_{n+1} \in S_{n+1}$ (because S_{n+1} is a negative cone) and the others $x_j \in \bar{S}_i$ (because ${}^T Z_j = (-a_{1,j}, \ldots, -a_{j-1,j}, 1 - a_{jj}, -a_{j+1,j}, \ldots, -a_{n,j})$, $a_{ij} \geqq 0$ (and $1 - a_{jj} \geqq 0$ ($\{j\}$ is not fundament, we have even $a_{jj} < 1$))). By means of the suitable change of the denotatiton of points from X (and the corresponding change of the denotation of R_i's and S_i's) we can achieve thruthfullness of (α) (β) (Let us put $x'_{n+1} =: x_{n+1}$, $R'_{n+1} =: R_{n+1}$. Since ${}^T yZ \geq {}^T o$, j_1 exists such that ${}^T yZ_{j_1} > 0$. Let us put $x'_n =: x_{j_1}$, $R'_n =: R_{j_1}$, and consequently $x'_n \notin H'_{n+1}$. j_2 exists in the set $N - \{j_1\}$ such that ${}^T yZ_{j_2} = 0$ and $a_{j_1 j_2} = 0$ are not true simultaneously (otherwise the set $N - \{j_1\}$ would be fundament for y) and for that reason we have $x'_{n-1} \notin H'_n \cap H'_{n+1}$ for $x'_{n-1} =: x_{j_2}$, $R'_{n-1} =: R_{j_2}$. In a similar way we proceed up to the case $x'_2 =: x_{j_{n-1}}$, $R'_2 =: R_{j_{n-1}}$, $x'_2 \notin \bigcap_{k=3}^{n+1} H'_k$. Finally let us denote by x'_1 the remaining element of X and by R'_1 the remaining element of γ. The sets X', γ' and S'_i ($i = 1, 2, \ldots, n + 1$) fulfil the assumptions (α), (β).). Theorem 2 is, now, the immediate consequence of Theorem 1 and Remark 1.

Remark 4: Notice that we used only $A \geqq O$ in the preceding proof (hence theorem 2 is true when we exchange (P_4) for this weaker property).

Remark 5: Since there is a $w > o$ such that $(I - A)\, w = l > o$, it follows that $(I - A)^{-1} \geqq O$, each principal minor of $I - A$ is positive and $0 <$ determinant $(I - A) \leqq 1$. (These results follow from the theory of non-negative matrices — see [1], [2] and [4].)

Remark 6: A real square matrix Z is called of type Z (see [2]) if all its off-diagonal elements are nonpositive. In [2] the equivalence of the following three statements is proved:

(1) There exists a point $x \geqq o$ such that $Zx > o$.
(2) There exists a point $x > o$ such that $Zx > o$.
(3) The invers Z^{-1} exists and $Z^{-1} \geqq O$.

We call a square matrix $A \geqq O$ *productive* (see [3]), if there is a $s \geqq o$ such that ${}^T s(I - A) > {}^T o$.

If A is productive, $Z =: {}^T(I - A)$ is of type Z and (1) is true. Hence according to (2) there is a $s^* > o$ such that ${}^Ts^*(I - A) > {}^To$. Thus (P_8) is fulfilled.

Let a matrix $A \geqq O$ have the property (P_8). From theorem 2 and Remark 4 it follows (3) for $Z =: I - A$ and hence (3) is true also for ${}^TZ$. Hence we have (1) for this matrix. Consequently A must be productive.

Hence we have this theorem: *A square matrix $A \geqq O$ is productive iff (P_8) is fulfilled.*

Remark 7: Desired properties of $I - A$ follow also from the existence of prices $p > o$ such that $(I - A)p > o$.

Assume that the production has the following properties: (L_1) There exist n different kinds of the labour. (L_2) Any labour is measured in fixed units. (L_3) For the production of a unit of the good i one needs the quantity $l_i > 0$ of labour i and no quantity of labour j $(j \neq i)$. (L_4) An abstract labour is given and any labour can be transferred on abstract one. (We say an *abstract labour* is given if any quantity of labour of any kind corresponds to a real number (called an abstract labour) in such a way that for each i it is settled how much units of abstract labour is one unit of labour i. Hence for each i such a linear function $\lambda_i(u)$ is given that $\lambda_i(0) = 0$ and $\lambda_i(u) > 0$ for $u > 0$. One can now add the quantities of different kinds of labour (each kind of labour is transferred on abtract one and these numbers are summed up).

K. Marx has defined (see [5]) the *labour value w^i of the good i* by this rule: The labour value of any good is the quantity of abstract „live" labour (i.e. one really exerted in the course of the production of this good) plus the quantity of the abstract labour which is „objectified" (i.e. the quantity of abstract labour contained as labour value in the goods, exhausted during the production of our good) thus, $w^i = \lambda_i(l_i) + \sum_{j=1}^{n} a_{ij}w^i$ i.e. $(I - A)w = l$ where ${}^Tw =: (w^1, \ldots, w^n)$, ${}^Tl =: (l^1, \ldots, l^n)$ and $l^i =: \lambda_i(l_i)$.

Theorem 3. *If the production fulfils* $P_1 - P_8$ *and* $L_1 - L_4$, *then the labour value w^i of the i-th good (for any i) exists, it is positive and even defined uniquely:* $w = (I - A)^{-1}\,l$. (This fact follows immediately from the Theorem 2.)

REFERENCES

[1] G. Debreu and I. N. Herstein: *Nonnegative square matrices.* Econometrica 21 (1953), 597—607.

[2] M. Fiedler and Vl. Pták: *On matrices with non-positive off-diagonal elements and positive principal minors.* Czechoslovak Math. Journal 12 (1962), 382—400.

[3] D. Gale: *The theory of linear economic models.* McGraw-Hill Comp., New York 1960 (Russian translation in 1963).
[4] S. Karlin: *Mathematical methods and theory in games, programming and economics,* 1959 (Russian translation in 1964).
[5] K. Marx: *Capital.* Chicago, Charles Kerr and Company, 1933.
[6] V. Polák: *A note to mathematical aspects of the political decision theory.* Archivum mathematicum (in print).