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# A FIXED POINT THEOREM IN LOCALLY CONVEX SPACES 

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In this paper there is extended a fixed point theorem obtained in [1] to locally convex spaces. Therefore, let $E$ be $a$ Hausdorff locally convex space and $\mathscr{A}$ a sufficently and directed familly of seminorms that gives the topology of $E$.

Let $\varphi$ be a mapping of the family $\mathscr{A}$ satisfying the condition

$$
\begin{equation*}
\varphi[\varphi(x)]=\varphi(x) \quad(\alpha \in \mathscr{A}) \tag{I}
\end{equation*}
$$

and $H$ a closed, convex and bounded subset of $E$.
Theorem. Let f be a mapping from $H$ into $H$ such that:
(i) for all $\alpha \in \mathscr{A}$

$$
\left|f\left(x_{1}\right)-f\left(x_{2}\right)\right|_{\alpha} \leqq\left|x_{1}-x_{2}\right|_{\varphi(\alpha)} \quad\left(x_{1}, x_{2} \in H\right)
$$

(ii) there is a compact set $M \subset H$ such that for every $x \in H$ the sequence $\left\{f^{n}(x)\right\}$ has an accumulation point in $M\left(f^{(n}=f \cdot f^{n-1}, f^{1}=f\right.$.

Then $f$ has a fixed point in $H$.
Proof. We can suppose that $H$ contains the null element of $E$ such that if we put $f_{q}=q \cdot f$ with $0<q<1, f_{q}(x)=(1-q) \cdot 0+q f(x) \in H$, that is $f_{q}$ is a contraction of $H$.

For every $\alpha \in \mathscr{A}$

$$
\left|f_{q}\left(x_{1}\right)-f_{q}\left(x_{2}\right)\right|_{\alpha} \leqq q\left|x_{1}-x_{2}\right|_{\varphi(\alpha)} \quad\left(x_{1}, x_{2} \in H\right)
$$

such that, for $n=1,2, \ldots$ and $x \in H$

$$
\begin{gathered}
\left|f_{q}^{n+1}(x)-f_{q}^{n}(x)\right|_{\alpha} \leqq q \cdot\left|f_{q}^{n}(x)-f_{q}^{n-1}(x)\right|_{\varphi(\alpha)} \leqq \\
\leqq q^{2}\left|f_{q}^{n-1}(x)-f_{q}^{n-2}(x)\right|_{\psi(\alpha)} \leqq \cdots \leqq q^{n}\left|f_{q}(x)-x\right|_{\varphi^{\prime}(\alpha)}
\end{gathered}
$$

It follows that

$$
\begin{aligned}
&\left|f_{q}^{n+k}(x)-f_{q}^{n}(x)\right|_{\alpha} \leqq\left(q^{n+k-1}+\ldots+q^{n}\right)\left|f_{q}(x)-x\right|_{\varphi(\alpha)} \leqq \\
& \leqq \frac{q^{n}}{1-q}\left|f_{q}(x)-x\right|_{\varphi(\alpha)}
\end{aligned}
$$

Consequently, for each $x \in H$ the sequence $\left\{f_{q}^{n}(x)\right\}$ is a Cauchy one. It results that there exists $x_{q} \in H$ such that:

$$
\left|f\left(x_{q}\right)-x_{q}\right|_{\varphi(\alpha)} \leqq 1-q,
$$

Indeed we can take $x_{q}=f_{2}^{n}(x)$ with a sufficient large n .
On the other hand

$$
\begin{align*}
\left|f\left(x_{q}\right)-x_{q}\right|_{\psi(\alpha)} & \leqq\left|f\left(x_{q}\right)-q f\left(x_{q}\right)\right|_{\psi(\alpha)}+  \tag{3}\\
+\left|f_{q}\left(x_{q}\right)-x_{q}\right|_{\psi(\alpha)} & \leqq(1-q)\left[\left|f\left(x_{q}\right)\right|_{\psi(\alpha)}+1\right]= \\
& =(1-q) r
\end{align*}
$$

where $r$ is a positive number independent of $x_{q}$ since the set $H$ is bounded.

Hence, if $n=1,2, \ldots$

$$
\begin{equation*}
\left|f^{n+1}\left(x_{q}\right)-f^{n}\left(x_{q}\right)\right|_{\alpha} \leqq\left|f^{n}\left(x_{q}\right)-f^{n-1}\left(x_{q}\right)\right|_{q(\alpha)} \leqq \ldots \leqq(1-q) r . \tag{4}
\end{equation*}
$$

Since the sequence $\left\{f^{n}\left(x_{q}\right)\right\}$ has an accumulation point

$$
y_{q} \in M \text {, }
$$

for every $\varepsilon>o$, there exists $n$ such that:

$$
\begin{equation*}
\left|f^{n}\left(x_{q}\right)-y_{q}\right|_{B} \leqq \varepsilon \quad[\beta \geqq x, \varphi(x)] \tag{5}
\end{equation*}
$$

From (4) and (5) it follows that:

$$
\begin{gathered}
\left|f\left(y_{q}\right)-y_{q}\right|_{\alpha} \leqq\left|f\left(y_{q}\right)-f^{n+1}\left(x_{q}\right)\right|_{\alpha}+\left|f^{n+1}\left(x_{q}\right)-f^{n}\left(x_{q}\right)\right|_{\alpha}+ \\
+\left|f^{n}\left(x_{q}\right)-y_{q}\right|_{\alpha} \leqq\left|y_{q}-f^{n}\left(x_{q}\right)\right|_{q(\alpha)}+\left|f^{n+1}\left(x_{q}\right)-f^{n}\left(x_{q}\right)\right|_{\alpha}+ \\
+\left|f^{n}\left(x_{q}\right)-y_{q}\right|_{\alpha} \leqq \varepsilon+(1-q) \cdot r+\varepsilon
\end{gathered}
$$

that is

$$
\left|f\left(y_{q}\right)-y_{q}\right|_{\alpha} \leqq(1-q) . r .
$$

Let $\left\{q_{i}\right\}$ be a sequence of real numbers such that

$$
\operatorname{Lim}_{i \rightarrow \infty} q_{i}=1 \quad\left(0<q_{i}<1, i=1,2, \ldots\right) .
$$

We consider a convergent subsequence $\left\{y_{q_{i}}\right\}$ of the corresponding sequence $\left\{y_{q_{t}}\right\} \subset M$.

If $\operatorname{Lim}_{i \rightarrow \infty} y_{q_{i}^{\prime}}=y \in M$ it follows that

$$
\operatorname{Lim}_{i \rightarrow \infty}\left|f\left(y_{q_{i}^{\prime}}\right)-y_{q_{i}^{\prime}}\right|_{a} \leqq \lim \left(1-q_{i}^{\prime}\right) \cdot r=0 .
$$

Since $f$ is a continuous mapping, it holds

$$
|f(y)-y|_{\alpha}=0
$$

for all $\alpha \in \mathscr{A}$. Hence $f(y)=y, q . e . d$.
When the space $E$ is normed there is obtained theorem 5 given by D. Göhde in [1].

## REFERENCES

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