C. Tudor A fixed point theorem in locally convex spaces

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CONVEX SPACES

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In this paper there is extended a fixed point theorem obtained in [1] to locally convex spaces. Therefore, let E be a Hausdorff locally convex space and \mathscr{A} a sufficiently and directed familly of seminorms that gives the topology of E.

Let φ be a mapping of the family \mathscr{A} satisfying the condition

(1)
$$\varphi[\varphi(x)] = \varphi(x) \quad (\alpha \in \mathscr{A})$$

and H a closed, convex and bounded subset of E.

Theorem. Let f be a mapping from H into H such that:

(i) for all $\alpha \in \mathscr{A}$

2)
$$|f(x_1) - f(x_2)|_{\alpha} \leq |x_1 - x_2|_{\varphi(\alpha)} \quad (x_1, x_2 \in H)$$

(ii) there is a compact set $M \subseteq H$ such that for every $x \in H$ the sequence $\{f^n(x)\}$ has an accumulation point in M $(f(^n = f \cdot f^{n-1}, f^1 = f \cdot f^{n-1})$.

Then f has a fixed point in H.

Proof. We can suppose that H contains the null element of E such that if we put $f_q = q \cdot f$ with 0 < q < 1, $f_q(x) = (1 - q) \cdot 0 + qf(x) \in H$, that is f_q is a contraction of H.

For every $\alpha \in \mathscr{A}$

$$|f_q(x_1) - f_q(x_2)|_{\alpha} \leq q |x_1 - x_2|_{\varphi(\alpha)} \quad (x_1, x_2 \in H)$$

such that, for n = 1, 2, ... and $x \in H$

$$egin{aligned} &|f_q^{n+1}(x)-f_q^n(x)\mid_{lpha}\leq q\,.\;|f_q^n(x)-f_q^{n-1}(x)\mid_{arphi(lpha)}\leq & \ &\leq q^2\,|f_q^{n-1}(x)-f_q^{n-2}(x)\mid_{arphi(lpha)}\leq \ldots \leq q^n\,|f_q(x)-x\mid_{arphi'(lpha)} \end{aligned}$$

It follows that

$$egin{aligned} &|f_q^{n+k}(x)-f_q^n(x)|_{lpha} \leq (q^{n+k-1}+\ldots+q^n) |f_q(x)-x|_{arphi(lpha)} \leq \ &\leq rac{q^n}{1-q} \left| f_q(x)-x
ight|_{arphi(lpha)}. \end{aligned}$$

Consequently, for each $x \in H$ the sequence $\{f_q^n(x)\}$ is a Cauchy one. It results that there exists $x_q \in H$ such that:

$$|f(x_q) - x_q|_{\varphi(\alpha)} \leq 1 - q,$$

Indeed we can take $x_q = f_2^n(x)$ with a sufficient large n. On the other hand

(3)
$$|f(x_q) - x_q|_{\varphi(\alpha)} \leq |f(x_q) - qf(x_q)|_{\varphi(\alpha)} + + |f_q(x_q) - x_q|_{\varphi(\alpha)} \leq (1 - q) [|f(x_q)|_{\varphi(\alpha)} + 1] = = (1 - q) r$$

where r is a positive number independent of x_q since the set H is bounded.

Hence, if n = 1, 2, ...

(4)
$$|f^{n+1}(x_q) - f^n(x_q)|_{\alpha} \leq |f^n(x_q) - f^{n-1}(x_q)|_{q(\alpha)} \leq \dots \leq (1-q) r.$$

Since the sequence $\{f^n(x_q)\}$ has an accumulation point

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$$y_a \in M$$

for every $\varepsilon > o$, there exists n such that:

(5)
$$|f^n(x_q) - y_q|_{\beta} \leq \varepsilon \quad [\beta \geq \alpha, \varphi(\alpha)]$$

From (4) and (5) it follows that:

$$\begin{split} |f(y_q) - y_q|_{a} &\leq |f(y_q) - f^{n+1}(x_q)|_{a} + |f^{n+1}(x_q) - f^{n}(x_q)|_{a} + \\ + |f^{n}(x_q) - y_q|_{a} &\leq |y_q - f^{n}(x_q)|_{\varphi(a)} + |f^{n+1}(x_q) - f^{n}(x_q)|_{a} + \\ + |f^{n}(x_q) - y_q|_{a} &\leq \varepsilon + (1 - q) \cdot r + \varepsilon \end{split}$$

that is

$$|f(y_q) - y_q|_a \leq (1 - q) \cdot r.$$

Let $\{q_i\}$ be a sequence of real numbers such that

$$\lim_{i \to \infty} q_i = 1 \qquad (0 < q_i < 1, \ i = 1, 2, ...).$$

We consider a convergent subsequence $\{y_{q_i}\}$ of the corresponding sequence $\{y_{q_i}\} \subset M$. If $\lim y_{q'_i} = y \in M$ it follows that

 $i \rightarrow \infty$

$$\lim_{i\to\infty} |f(y_{q'_i}) - y_{q'_i}|_{\alpha} \leq \lim (1 - q'_i) \cdot r = 0.$$

Since f is a continuous mapping, it holds

$$|f(y) - y|_{a} = 0$$

for all $\alpha \in \mathscr{A}$. Hence $f(y) = y, q \cdot e \cdot d$.

When the space E is normed there is obtained theorem 5 given by D. Göhde in [1].

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