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CENTROAFFINE INVARIANTS OF PLANE CURVES IN CONNECTION WITH THE THEORY OF THE SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

In honour of the 70th birthday anniversary in 1969 of PROF. O. BORŮVKA.

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1. INTRODUCTION

Let $\boldsymbol{u}[u_1(t), u_2(t)]$ denote a curve of class C^2_{I} given in the plane (u_1, u_2) by its parametric expression

$$\begin{matrix} u_1 = u_1(t), & u_1(t) \in C^2_{\mathrm{I}}, \\ u_2 = u_2(t), & u_2(t) \in C^2_{\mathrm{I}}, \end{matrix} \quad \begin{vmatrix} u_1(t), & u_2(t) \\ u_1(t), & u_2(t) \end{vmatrix} \neq 0, \quad t \in I.$$

For $n \geqq 0$, C^n_{I} is the set of all real continuous functions defined on the interval I and being of continuous derivatives up to and including the order n; $C^n_{(-\infty, \infty)} \equiv C^n$. For brevity, let $(\boldsymbol{u}; \boldsymbol{u}')$ denote a determinant

$$\begin{vmatrix} u_1(t), & u_1'(t) \\ u_2(t), & u_2'(t) \end{vmatrix}$$

and symbols $(\boldsymbol{u}; \boldsymbol{u}'')$ etc. are defined in the same manner. A centroaffine invariant of a curve $\boldsymbol{u}(u_1, u_2)$ is an operator $F[t;\, u_1(t), u_2(t)]\,(x)$ if the two following properties are satisfied:

(*) for every transformation $\tau = \tau(t)$, $\tau \in C^2_{\mathrm{I}}$, $\mathrm{d}\tau/\mathrm{d}t \neq 0$, $u^*(\tau) = u(t)$ there holds

$$F[\tau, u_1^*(\tau), u_2^*(\tau)]\,(\tau(x)) = F[t, u_1(t), u_2(t)]\,(x)$$

for all $x \in I$

(**) for

$$c = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0$$

and all $x \in I$ it holds

$F[t, c_{11}u_1(t) + c_{12}u_2(t), c_{21}u_1(t) + c_{22}u_2(t)]\,(x) = h(c)\, F[t, u_1(t), u_2(t)]\,(x)$ where $h(c)$ is a real function of a real parameter.

Remark. Usually the term „invariant“ is reserved for the case $h(c) \equiv 1$, and the term „semiinvariant“ will be more convenient for our purpose. Our considerations, however, will mainly deal with $h(c) \not\equiv 1$, and we shall use „invariant“ only. If the case $h(c) \equiv 1$ should be stressed, the notion „invariant in the strict sense“ will be used.

Denote by $U(t)$ a point $[u_1(t), u_2(t)]$ of a curve $\boldsymbol{u}$ and $|U(t)| = \sqrt{[u_1^2(t) + u_2^2(t)]}$ its distance from the origin.

Functions $u_1(t)$, $u_2(t)$ satisfy the differential equation

$$\begin{vmatrix} u_1 & u_2 & y \\ u_1' & u_2' & y' \\ u_1'' & u_2'' & y'' \end{vmatrix} = 0,$$

which can be written in the form $y'' + a(t)\,y' + b(t)\,y = 0$. By the transformation $\tau = \tau_0 + \tau_0' \int_{t_0}^{t} \exp\left[-\int_{t_0}^{\sigma} a(s)\,\mathrm{d}s\right] \mathrm{d}\sigma$, $t_0 \in I$, $\tau_0' \neq 0$ of parameter t, the interval I is transformed into the interval i and the above differential equation is in the form (again the independent variable is denoted by t)

(q) $$y'' = q(t)\,y,$$

where $q(t) \in C_i$ is given by the relation

$$q(t) = \frac{1}{\tau_0'^2}\,b(t)\,.\,\exp 2 \int_{t_0}^{t} a(s)\,\mathrm{d}s.$$

In other words: For every curve $\boldsymbol{u}$ there exists a transformation of parameter t such that the components u_1 and u_2 satisfy a differential equation (q). If a curve $\boldsymbol{u}$ has this parametric expression, we shall denote the curve by $\boldsymbol{u}_q$.

Now, let us consider a differential equation (q) and its two solutions u_1, u_2. Let $W = W(u_1, u_2) = (\boldsymbol{u}; \boldsymbol{u}')$ stand for the Wronskian of these solutions. Denote by $\varphi(t)$ the basic central dispersion of the 1^{st} kind (further only dispersion) of the differential equation (q), [3], i.e. $\varphi(t_0)$ is the first zero of a solution $y(t)$, $y(t_0) = 0$, of the equation (q) lying on the right of t_0.

If $U(t_0)$ is a point of a curve $\boldsymbol{u}_q$ and if $\varphi(t_0)$ exists then the point $U[\varphi(t_0)]$ is again a point of $\boldsymbol{u}_q$ and these two points lie on the straight line passing the origin; moreover, $U[\varphi(t_0)]$ is the first point of intersection of the curve $\boldsymbol{u}_q$ with the straight line passing the origin and the point $U(t_0)$ when the curve $\boldsymbol{u}_q$ is observed from the point $U(t_0)$ in the direction of an increasing parameter t. This fact is obvious from the following consideration: There exist c_1, c_2, $c_1^2 + c_2^2 \neq 0$ such that $c_1u_1(t_0) + c_2u_2(t_0) = 0$. Then $c_1u_1(t) + c_2u_2(t)$ is a solution of (q) which vanishes at t_0 and the first number on the right of t_0 where the solution again vanishes being $\varphi(t_0)$. Therefore $U[\varphi(t_0)]$ is the first point of the curve $\boldsymbol{u}_q$ from $U(t_0)$ in direction of increasing parameter which lies on the straight line $c_1u_1 + c_2u_2 = 0$.

Let $\alpha(t)$ denote some of the 1^{st} phases corresponding to the pair of independent solutions u_1, u_2 of the differential equation (q), [3], i.e. a continuous function $\alpha(t)$ for all $t \in i$ complying with the relation

$$\operatorname{tg} \alpha(t) = u_1(t)/u_2(t)$$

for all $t \in i$, for which $u_2(t) \neq 0$.

It is obvious that

$$\alpha'(t) = -W(u_1, u_2)/[u_1^2(t) + u_2^2(t)]$$

and then $\alpha \in C_i^3$ and α' does not change its sign on i. For a point $\boldsymbol{U}(t)$ of a curve $\boldsymbol{u}_q$ one can write

$$| \boldsymbol{U}(t) | = \sqrt{\frac{-W}{\alpha'(t)}},$$

where $W = W(u_1, u_2)$ and α is a 1^{st} phase corresponding to the components of the curve $\boldsymbol{u}_q$.

A curve $\boldsymbol{u}_q$ is simple and centrosymmetric with respect to the origin iff the dispersion $\varphi(t)$ of (q) is defined at least on an interval $i \supset [t_0, \varphi(t_0)]$ and the relation $| \boldsymbol{U}(t) | = | \boldsymbol{U}[\varphi(t)] |$ is satisfied on it. As $W =$ const., the last relation can be written in the form

$$\sqrt{| \alpha'(t) |} = \sqrt{| \alpha'[\varphi(t)] |}.$$

And, as α' does not change its sign, the last condition is equivalent to the condition

$$\alpha'(t) = \alpha'[\varphi(t)], \quad \text{for} \quad t \in i \supset [t_0, \varphi(t_0)].$$

The dispersion $\varphi(t)$ and every 1^{st} phase of the same differential equation (q) satisfy the relation $\alpha[\varphi(t)] = \alpha(t) + \pi$, [3], or $\alpha'(t) = \alpha'[\varphi(t)] \cdot \varphi'(t)$. Therefore $\varphi'(t) \equiv 1$ and $\varphi(t) \equiv t + d$ ($d > 0$ as $\varphi(t)$ is increasing) everywhere, where $\varphi(t)$ is defined, i.e. at least on the interval $\langle t_0, t_0 + d)$.

The curve $\boldsymbol{u}_q$ being centrosymmetric and then closed, it is possible to extend its parametric expression $\boldsymbol{u}_q$. At the same time the dispersion $\varphi(t)$ must be of the form $\varphi(t) = t + d$. With respect to this, $q(t)$ must be periodic with period d; it is defined at least on the interval $\langle t_0, t_0 + d)$, hence the extending of parametric expression of the curve $\boldsymbol{u}_q$ on the interval $(-\infty, \infty)$ is unique. In what follows we shall suppose $i =$ $= (-\infty, \infty)$ for centrosymmetric curves $\boldsymbol{u}_q$ without lost of generality.

As the properties of "simplicity and centrosymmetricity" are not changed by any change of parameter or any centroaffine transformation

$$\begin{aligned} u_1 &= c_{11}u_1 + c_{12}u_2, \\ u_2 &= c_{21}u_1 + c_{22}u_2, \end{aligned} \qquad \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0,$$

the form $\varphi(t) = t + d$ of the dispersion of a differential equation (q) is the sufficient and necessary condition for the curve $\boldsymbol{u}$ (with a parametric expression $\boldsymbol{u}_q$) to be simple and centrosymmetric.

By the change of parameter: $t = \pi t/d$ we can restrict our study only on the dispersion $\varphi(t) = t + \pi$.

Let us summarize our considerations:

In every simple centrosymmetric curve $\boldsymbol{u}$ $\langle$*of class* $C^2\rangle$ *such a parametrisation can be introduced that its components* $u_1(t)$, $u_2(t)$ *are independent solutions of a differential equation* (q) *with the dispersion* $\varphi(t) = t + \pi$, and vice versa:

Every two independent solutions $u_1(t)$, $u_2(t)$ *of a differential equation* (q) *with the dispersion* $\varphi(t) = t + \pi$ *are components of a simple centrosymmetric curve* $\langle$*of class* $C^2\rangle$.

Analogously one can derive that a curve $\boldsymbol{u}$ (with a parametric expression $\boldsymbol{u}_q$) is simply closed iff

$$| \boldsymbol{U}\{\varphi[\varphi(t)]\} | = | \boldsymbol{U}(t) |, \quad \text{or} \quad \sqrt{| \alpha'\{\varphi[\varphi(t)]\} |} = \sqrt{| \alpha'(t) |}.$$

Thus, with respect to the relation

$$\alpha\{\varphi[\varphi(t)]\} = \alpha[\varphi(t)] + \pi = \alpha(t) + 2\pi,$$

the last condition is equivalent to

$$\frac{\mathrm{d}\varphi[\varphi(t)]}{\mathrm{d}t} = 0, \quad \text{or} \quad \varphi[\varphi(t)] = t + d \quad (d > 0, \text{const.}).$$

where $\varphi(t)$ is the dispersion of the differential equation (q).

Generally, *a curve* $\boldsymbol{u}$ $\langle$*with the parametric expression* $\boldsymbol{u}_q\rangle$ *is closed iff*

$$\overbrace{\varphi(\varphi\{\cdots[\varphi}^{n}(t)]\ldots\}) = \varphi^{[n]}(t) = t + d$$

for an integer n *and a positive constant* $d > 0$. *If* n *is odd, then the curve* $\boldsymbol{u}$ *is also centrosymmetric.*

When a curve $\boldsymbol{u}_q$ is closed we can suppose $\boldsymbol{i} = (-\infty, \infty)$ without lost of generality, because the periodicity of $\alpha'(t)$ follows from relations $\varphi^{[n]}(t) = t + d$ and $\alpha[\varphi^{[n]}(t)] = \alpha(t) + n\pi$, and with respect to the relation

$$q(t) = -\frac{1}{2}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)' + \frac{1}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 - \alpha'^2(t), \qquad [3, \text{p. } 35]$$

the function $q(t)$ is periodic with period d. At the same time $q(t)$ is defined at least on the interval of the length d and, therefore, its extension on the interval $(-\infty, \infty)$ is unique.

2. CONSTRUCTION OF ALL SIMPLE CENTROSYMMETRIC CURVES

In paper [5] (or in [6], too), all the differential equations (q) with the dispersion $\varphi(t) = t + \pi$ have been constructed. These functions $q(t)$ satisfy

$$q(t) = f''(t) + f'^2(t) + 2f' \operatorname{cotg} t - 1$$

where $f \in C^2$, $f(0) = f'(0) = 0$, $f(t + \pi) = f(t)$, $\int_0^\pi \frac{\exp[-2f(t)] - 1}{\sin^2 t} \mathrm{d}t =$ $= 0$; the solution $u_1(t)$ of (q) determined by the conditions $u_1(0) = 0$, $u_1(0) = 1$ is then given by the relation $u_1(t) = \mathrm{e}^{f(t)} \sin t$. Therefore; *every simple centrosymmetric curve ⟨of class C^2⟩ can be expressed as*

$$\boldsymbol{u}(c_{11}u_1 + c_{12}u_2, c_{21}u_1 + c_{22}u_2),$$

where

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0, \quad u_1 = \mathrm{e}^{f(t)} \sin t, \quad u_2 = \mathrm{e}^{f(t)} \sin t \int_{\pi/2}^{t} \frac{\mathrm{e}^{-2f(\sigma)}}{\sin^2 \sigma} \mathrm{d}\sigma$$

and $f(t)$ satisfies the above conditions.

3. CENTROAFFINE INVARIANTS OF CURVES

Centroaffine arc length of a curve $\boldsymbol{u}[u_1(t), u_2(t)]$ from a point $U(t_0)$ to a point $U(t)$ in the sense of Blaschke [1, p. 15][1]) is defined as

$$(1) \quad l_{t_0}^t(u) = \int_{t_0}^{t} [\boldsymbol{u}'(\sigma); \boldsymbol{u}''(\sigma)]^{1/3} \mathrm{d}\sigma = \int_{t_0}^{t} [u_1'(\sigma)\, u_2''(\sigma) - u_1''(\sigma)\, u_2'(\sigma)]^{1/3} \mathrm{d}\sigma.$$

If a curve $\bar{\boldsymbol{u}}$ is obtained from a curve $\boldsymbol{u}$ by a centroaffine transformation with the determinant c then

$$l_{t_0}^t(\bar{\boldsymbol{u}}) = c^{1/3} l_{t_0}^t(\boldsymbol{u});$$

especially the length is the same for $c = 1$. A change of parameter $T = T(t)$, $T \in C^2$, $\mathrm{d}T/\mathrm{d}t \neq 0$ transforms a curve $\boldsymbol{u}$ into a new parametric expression $\boldsymbol{u}^*$ and it holds

$$l_{T(t_0)}^{T(t)}(\boldsymbol{u}^*) = l_{t_0}^t(\boldsymbol{u}).$$

1) The length is an invariant in the strict sense in the group of all afine transformations with determinant 1.

If a parametric expression $\boldsymbol{u}_q$ is considered then

$$(1_q) \qquad l_{t_0}^t(\boldsymbol{u}_q) = W^{1/3} \int_{t_0}^{t} \left[\frac{(\boldsymbol{u}'; \boldsymbol{u}'')}{(\boldsymbol{u}; \boldsymbol{u}')}\right]^{1/3} d\sigma = W^{1/3} \int_{t_0}^{t} \sqrt[3]{-q(\sigma)}\, d\sigma.$$

Let $\boldsymbol{u}_q(u_1, u_2)$ be a closed (or simple centrosymmetric, or simple closed,) curve with the parameter t introduced (according to Sec. 1) that the dispersion $\varphi(t)$ of (q) is in the form $\varphi^{[n]}(t) = t + \pi$ (or $\varphi(t) = t + \pi$, or $\varphi^{[2]}(t) = t + \pi$, resp.) where n is the first integer for which $\varphi^{[n]}(t) = t + \text{const}$. Then its length (without respect to orientation) is given by

$$(2) \qquad l(\boldsymbol{u}_q) = \left| W^{1/3} \int_0^{\pi} \sqrt[3]{-q(\sigma)}\, d\sigma \right| \text{ for even } n$$

$$l(\boldsymbol{u}_q) = 2 \left| W^{1/3} \int_0^{\pi} \sqrt[3]{-q(\sigma)}\, d\sigma \right| \text{ for odd } n.$$

O. Borůvka [3, p. 29] introduced his centroaffine arc length of a curve $\boldsymbol{u}(u_1, u_2)$ from a point $\boldsymbol{U}(t_0)$ to a point $\boldsymbol{U}(t)$ by the formula

$$(3) \qquad \mathscr{L}_{t_0}^t(\boldsymbol{u}) = \operatorname{sgn}(\boldsymbol{u}; \boldsymbol{u}') \int_{t_0}^{t} \sqrt{\left|\frac{(\boldsymbol{u}'; \boldsymbol{u}'')}{(\boldsymbol{u}; \boldsymbol{u}')}\right|}\, d\sigma.$$

If a curve $\boldsymbol{u}$ is obtained from a curve $\boldsymbol{u}$ by means of a centroaffine transformation with the determinant c, then

$$\mathscr{L}_{t_0}^t(\boldsymbol{u}) = \operatorname{sgn} c\, \mathscr{L}_{t_0}^t(\boldsymbol{u})$$

and if a change of parameter, $T = T(t)$, gives a new parametric expression $\boldsymbol{u}^*$ of a curve $\boldsymbol{u}$, then

$$\mathscr{L}_{T(t_0)}^{T(t)}(\boldsymbol{u}^*) = \mathscr{L}_{t_0}^t(\boldsymbol{u}).$$

Therefore $\mathscr{L}_{t_0}^t(\boldsymbol{u})$ is again a centroaffine invariant of a curve $\boldsymbol{u}$ which even changes only its sign under any centroaffine transformation.

For a parametric expression $\boldsymbol{u}_q$ of $\boldsymbol{u}$ there holds

$$(3q) \qquad \mathscr{L}_{t_0}^t(\boldsymbol{u}_q) = \operatorname{sgn} W \,.\, \int_{t_0}^{t} \sqrt{|q(\sigma)|}\, d\sigma.$$

If, moreover, the parameter t is introduced such as in formulas (2) then for a closed (or simple centrosymmetric, or. simple closed) curve $\boldsymbol{u}_q$, there holds (without respect to orientation)

(4) $$\mathscr{L}(\boldsymbol{u}_q) = \int_0^\pi \sqrt{|q(\sigma)|}\,\mathrm{d}\sigma \qquad \text{for even } n,$$

$$\mathscr{L}(\boldsymbol{u}_q) = 2\int_0^\pi \sqrt{|q(\sigma)|}\,\mathrm{d}\sigma \qquad \text{for odd } n.$$

If $u_1 \cos\Theta + u_2 \sin\Theta = p(\Theta)$ denotes a supporting line of a closed curve $\boldsymbol{u}$ containing the origin, then the length $\boldsymbol{L}(\boldsymbol{u})$ in the sense of Santaló [9] is given by the invariant (in the strict sense with respect to the group of centroaffine transformations with determinant 1) measure of straight lines not crossing the curve:

(5) $$\boldsymbol{L}(\boldsymbol{u}) = \int_0^{2\pi} \mathrm{d}\Theta \int_{p(\Theta)}^{\infty} \mathrm{d}p/p^3 = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\Theta}{p^2(\Theta)}.$$

The tangent of a curve $\boldsymbol{u}[u_1(t), u_2(t)]$ at a point $\boldsymbol{U}(t_0)$ is given by the equation

$$u_1'(t_0)\,[u_2 - u_2(t_0)] = u_2'(t_0)\,[u_1 - u_1(t_0)], \quad \text{or}$$

$$\frac{u_1u_2'(t_0) - u_2u_1'(t_0)}{\sqrt{u_1'^2(t_0) + u_2'^2(t_0)}} = \frac{u_1(t_0)\,u_2'(t_0) - u_1'(t_0)\,u_2(t_0)}{\sqrt{u_1'^2(t_0) + u_2'^2(t_0)}};$$

hence

$$-u_1'(t)/u_2'(t) = \operatorname{tg}\Theta(t),$$

$$p(t) = \frac{u_1(t)\,u_2'(t) - u_1'(t)\,u_2(t)}{\sqrt{u_1'^2(t) + u_2'^2(t)}}.$$

Then

$$\mathrm{d}\Theta(t) = -\frac{u_1''u_2' - u_1'u_2''}{u_1^2 + u_2^2}\,\mathrm{d}t$$

and

(5') $$\boldsymbol{L}(\boldsymbol{u}) = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\Theta}{p^2(\Theta)} = \frac{1}{2}\int_{t_0}^{t_1} \frac{|(\boldsymbol{u}'; \boldsymbol{u}'')|}{(\boldsymbol{u}; \boldsymbol{u}')^2}\,\mathrm{d}t,$$

where the values of $\Theta(t)$ just fill up the whole interval $\langle 0,2\pi\rangle$ if $t_0 \leq t \leq t_1$. The convexity of a curve $\boldsymbol{u}$ is equivalent to the relation $(\boldsymbol{u}; \boldsymbol{u}') . (\boldsymbol{u}'; \boldsymbol{u}'') \geq 0$.

If $\boldsymbol{u}_q$ is a parametric expression of a convex curve containing the origin and the parameter t is, according to Sec. 1, introduced so that the dispersion $\varphi(t)$ of (q) satisfies $\varphi[\varphi(t)] = t + \pi$, then

(5q) $$\boldsymbol{L}(\boldsymbol{u}_q) = -\frac{1}{2\,|W|}\int_0^\pi q(t)\,\mathrm{d}t,$$

because the convexity of the curve implies $q(t) \leq 0$.

If the curve is even centrosymmetric and the parameter is introduced so that the dispersion $\varphi(t)$ satisfies $\varphi(t) = t + \pi$, then $q(t)$ is periodic with period π and, therefore,

$$L(\boldsymbol{u}_q) = -\frac{1}{|W|}\int_0^{\pi} q(t)\,\mathrm{d}t. \tag{6}$$

The tree mentioned ways of definition of arc length of a curve as a centroaffine invariant yield an idea, how to introduce a more general definition containing the previous ones as special cases:

Centroaffine arc length of a curve $\boldsymbol{u}[u_1(t), u_2(t)]$ from a point $\boldsymbol{U}(t_0)$ to a point $\boldsymbol{U}(t)$ is

$$^{\alpha}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u}) = \operatorname{sgn}(\boldsymbol{u};\boldsymbol{u}')\int_{t_0}^{t}\frac{|(\boldsymbol{u}';\boldsymbol{u}'')|^{\alpha}}{|(\boldsymbol{u};\boldsymbol{u}')|^{3\alpha-1}}\,\mathrm{d}\sigma \tag{7}$$

or

$$^{\alpha}_{*}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u}) = \int_{t_0}^{t}\frac{\operatorname{sgn}(\boldsymbol{u}';\boldsymbol{u}'')\,.\,|(\boldsymbol{u}';\boldsymbol{u}'')|^{\alpha}}{|(\boldsymbol{u};\boldsymbol{u}')|^{3\alpha-1}}\,\mathrm{d}\sigma. \tag{7*}$$

If $\bar{\boldsymbol{u}}$ is obtained from $\boldsymbol{u}$ by a centroaffine transformation with the determinant c, and $\boldsymbol{u}^*$ is a new parametric expression of $\boldsymbol{u}$ after a change of parameter $\tau = \tau(t)$, then

$$^{\alpha}_{(*)}\mathfrak{L}^{t}_{t_0}(\bar{\boldsymbol{u}}) = \operatorname{sgn} c\,.\,|c|^{1-2\alpha}\,{}^{\alpha}_{(*)}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u}) \quad\text{and}\quad {}^{\alpha}_{(*)}\mathfrak{L}^{\tau(t)}_{\tau(t_0)}(\boldsymbol{u}^*) = {}^{\alpha}_{(*)}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u})$$

[where (*) means with or without the asterisk on both sides).

Therefore, *the arc length introduced by* (7) *or* (7*) *is an centroaffine invariant and*

$^{1/3}_{*}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u})$ *gives the arc length in the sense of Blaschke,*

$^{1/2}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u})$ *is the arc length in the sense of Borůvka,*

$\frac{1}{2}\,{}^{1}_{(*)}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u})$ *means the arc length in the sense of Santaló.*

If $\boldsymbol{u}_q$ *is a parametric expression of a curve, then*

$$^{\alpha}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u}_q) = \operatorname{sgn} W\,.\,|W|^{1-2\alpha}\,.\int_{t_0}^{t}|q(\sigma)|^{\alpha}\,\mathrm{d}\sigma \tag{7_q}$$

or

$$^{*}_{\alpha}\mathfrak{L}^{t}_{t_0}(\boldsymbol{u}_q) = -\operatorname{sgn} W\,|W|^{1-2\alpha}\,.\int_{t_0}^{t}\operatorname{sgn} q(\sigma)\,.\,|q(\sigma)|^{\alpha}\,\mathrm{d}\sigma. \tag{7^*_q}$$

4. EXTREMAL PROPERTIES OF CENTROSYMMETRIC CURVES

Let $\boldsymbol{u}_q$ be a parametric expression of a convex centrosymmetric curve and let the dispersion φ of (q) be of the form $\varphi(t) = t + \pi$ (it is possible according to Sec. 1). The area bounded by the curve $\boldsymbol{u}_q$ has the volume

$$\boldsymbol{P}(\boldsymbol{u}_q) = \left| \frac{1}{2} \int_0^{2\pi} (\boldsymbol{u}_q ; \boldsymbol{u}_q') \, \mathrm{d}t \right| = \pi \mid W \mid.$$

Let us seek a convex centrosymmetric curve with a fixed volume of its area which has the minimal length.

According to the general definition of arc length of the curve $\boldsymbol{u}_q$, there is

$$^{\alpha}\mathfrak{L}(\boldsymbol{u}_q) = 2 \mid W \mid^{1-2\alpha} . \int_0^{\pi} \mid q(\sigma) \mid^{\alpha} \mathrm{d}\sigma$$

and this length is the same as the length in the sense of Blaschke, or Borůvka, or twice length in the sense of Santaló for $\alpha = 1/3$, or $\alpha = 1/2$, or $\alpha = 1$. The convexity yields $q(t) \leq 0$.

We shall apply a result of paper [7]:

"Suppose a differential equation (q) with the dispersion $\varphi(t) = t + d$. Let $q \leq 0$, and let α be a real number, $0 < \alpha \leq 1$.

Then

$$\int_0^{d} \mid q(t) \mid^{\alpha} \mathrm{d}t \leq d(\pi/d)^{2\alpha},$$

the equality being reached only for $q(t) \equiv -\pi^2/d^2$."

Because of $\boldsymbol{P}(\boldsymbol{u}_q)$ *being constant* in our case, the value W is also a constant and, therefore, $^{\alpha}\mathfrak{L}(\boldsymbol{u}_q)$ $(0 < \alpha \leq 1)$ *reaches its maximum value* $2\pi \mid W \mid^{1-2\alpha}$ *exactly for* $q \equiv -1$, *i.e. exactly for ellipses with the centers at the origin.*

Thus, for $\alpha = 1/3$ we have obtained Blaschke's result [1, p. 61] for convex centrosymmetrie curves. For $\alpha = 1$ it holds $\boldsymbol{P}(\boldsymbol{u}_q) = \pi . \mid W \mid$ and $\boldsymbol{L}(\boldsymbol{u}_q) = 1/2 \, {}^{1}\mathfrak{L}(\boldsymbol{u}_q) \leq \pi / \mid W \mid$, the last equality being reached just for ellipses with the centers at the origin. This is then the result of Santaló [9, III. 19.3]:

"For a convex centrosymmetric curve there holds

$$\boldsymbol{P}(\boldsymbol{u}) . \boldsymbol{L}(\boldsymbol{u}) \leq \pi^2,$$

where the equality sets in only for ellipses with the centers at the origin."

Let us note that centroaffine length of a centrosymmetric curve in the sense of Borůvka is invariant in the strict sense with respect to any centroafine transformation, and then the minimal length is reached for ellipses without the assumption of a constant volume $\boldsymbol{P}(\boldsymbol{u})$.

5. FURTHER CENTROAFFINE INVARIANTS OF CURVES

Let $\boldsymbol{u}[u_1(t), u_2(t)]$ be a curve. According to W. Blaschke [1, p. 13] the new parameter s is introduced by the relation

$$s = \int_{t_0}^{t} (\boldsymbol{u}'; \boldsymbol{u}'')^{1/3}\, d\sigma$$

and the centroaffine curvature of the curve $\boldsymbol{u}^*[u_1^*(s), u_2^*(s)]$ at a point $\boldsymbol{U}^*(s)$ is defined as[1])

$$\boldsymbol{k} \overset{\text{def}}{=} \left(\frac{d^2\boldsymbol{u}^*}{ds^2} ; \frac{d^3\boldsymbol{u}^*}{ds^3} \right).$$

In the previous parameteric expression,

$$\boldsymbol{k} = (\boldsymbol{u}''; \boldsymbol{u}''')/(\boldsymbol{u}'; \boldsymbol{u}'')^{5/3} - \frac{1}{2}\frac{d^2}{dt^2}[(\boldsymbol{u}'; \boldsymbol{u}'')^{-2/3}].$$

This can be rewritten as

$$\boldsymbol{k} = \frac{1}{9(\boldsymbol{u}'; \boldsymbol{u}'')^{8/3}} [-5(\boldsymbol{u}'; \boldsymbol{u}''')^2 + 12(\boldsymbol{u}'; \boldsymbol{u}'')(\boldsymbol{u}''; \boldsymbol{u}''') + 3(\boldsymbol{u}'; \boldsymbol{u}'')(\boldsymbol{u}'; \boldsymbol{u}^{\mathrm{IV}})].$$

A calculation verifies that the centroaffine curvature at corresponding points does not change when the parameter is changed, and the centroaffine curvature $\bar{\boldsymbol{k}}$ of the curve $\boldsymbol{u}$ obtained from the curve $\boldsymbol{u}$ by a centroaffine transformation with the determinant c satisfies

$$\bar{\boldsymbol{k}} = c^{-2/3}\boldsymbol{k}$$

at the points of the same parameter.

If a parametric expression $\boldsymbol{u}_q$ of a curve is supposed, then

$$\boldsymbol{k} = -W^{-2/3} \cdot \frac{1}{9q^{8/3}} [9q^3 + 5q'^2 - 3qq''],$$

which can be rewritten as

$$\boldsymbol{k} = -W^{-2/3}\left[q^{1/3} + \frac{1}{2}(q^{-2/3})'' \right].$$

Remark. For the definition of curvature in Blaschke's sense the existence of u_1^{IV}, u_2^{IV} and $ds/dt \neq 0$ (hence $q(t) \neq 0$) must be supposed.

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1) see the note to the length in Blaschke's sense.

O. Borůvka defines his centroaffine curvature [3, p. 29] of a curve $\boldsymbol{u}$ at a point $\boldsymbol{U}(t)$ as

$$\mathscr{K} = \frac{\operatorname{sgn}(\boldsymbol{u}; \boldsymbol{u}')}{2} \sqrt{\left|\frac{(\boldsymbol{u}; \boldsymbol{u}')}{(\boldsymbol{u}'; \boldsymbol{u}'')}\right|} \left[3 \frac{(\boldsymbol{u}; \boldsymbol{u}'')}{(\boldsymbol{u}; \boldsymbol{u}')} - \frac{(\boldsymbol{u}'; \boldsymbol{u}''')}{(\boldsymbol{u}'; \boldsymbol{u}'')}\right].$$

It can be simply verified that the centroaffine curvature at the corresponding points does not change for arbitrary parametric expression.

For a parametric expression $\boldsymbol{u}^*_q(u^*_1, u^*_2)$ at a point $\boldsymbol{U}^*(t)$ we can obtain

$$\mathscr{K} = \frac{\operatorname{sgn} W(\boldsymbol{u}^*_1, \boldsymbol{u}^*_2)}{2} \cdot \frac{1}{\sqrt{|q(t)|}} \left[- \frac{q'(t) \, . \, W(u^*_1, u^*_2)}{q(t) \, . \, W(u^*_1, u^*_2)}\right] =$$

$$= \operatorname{sgn} W \, . \, \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{\sqrt{|q(t)|}}\right].$$

Remark. The existence of u'''_1, u'''_2 and $(\boldsymbol{u}'; \boldsymbol{u}'') \neq 0$ (or existence $q'(t)$ and $q \neq 0$) must be warranted here.

Generally, let $\mathfrak{J}$ be such a centroaffine invariant of a curve $\boldsymbol{u}_q(u_1, u_2$- which can be expressed as a following function of $n + 2$ variables at every point $\boldsymbol{U}(t)$:

$$\mathfrak{J} = f[W(u_1, u_2), q, q', \ldots, q^{(n)}].$$

Let us deal with conditions under which these functions can be centro affine invariants.

First, at the corresponding points the value of $\mathfrak{J}$ must not change when such a transformation of parameter, $\tau = \tau(t)$, $\tau \in C^2_{\mathrm{I}}$, $\mathrm{d}\tau/\mathrm{d}t \neq 0$ is introduced that the components u^*_1, u^*_2 of a curve in new parametric expression are again independent solutions of a differential equation of the type $\mathrm{d}^2y^*/\mathrm{d}\tau^2 = q^*(\tau)\, y^*$. This occurs only when $(\boldsymbol{u}^*; \mathrm{d}\boldsymbol{u}^*/\mathrm{d}\tau) = c^* =$ $=$ const. $\neq 0$. At the same time $0 \neq c = (\boldsymbol{u}; \mathrm{d}\boldsymbol{u}/\mathrm{d}t) = (\boldsymbol{u}^*; \mathrm{d}\boldsymbol{u}^*/\mathrm{d}\tau)\, .$ $.\, \mathrm{d}\tau/\mathrm{d}t = c^*\, \mathrm{d}\tau/\mathrm{d}t$; hence $\tau(t) = c_1 t + c_0$, $c_1 \neq 0$ are all transformations of parameter t under which the invariant $\mathfrak{J}$ must not change. At the corresponding points in these transformations there is $W = W(u_1, u_2) =$ $= (\boldsymbol{u}; \mathrm{d}\boldsymbol{u}/\mathrm{d}t) = (\boldsymbol{u}^*; \mathrm{d}\boldsymbol{u}^*/\mathrm{d}\tau)\, .\, c_1 = c_1 W^*(u^*_1, u^*_2) = c_1 W^*$, and further

$$q(t) = \frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} / u_1 = c_1^2 \frac{\mathrm{d}^2 u^*_1}{\mathrm{d}\tau^2} / u^*_1 = c_1^2 q^*(\tau), \qquad \frac{\mathrm{d}q(t)}{\mathrm{d}t} = c_1^3 \frac{\mathrm{d}q^*(\tau)}{\mathrm{d}\tau},$$

or generally

$$\frac{\mathrm{d}^i q(t)}{\mathrm{d}t^i} = c_1^{i+2} \frac{\mathrm{d}^i q^*(\tau)}{\mathrm{d}\tau^i}.$$

Then the first condition on f is

(8) $$f(z_{-1}, z_0, \ldots, z_n) = f(c z_{-1}, c^2 z_0, \ldots, c^{n+3} z_n)$$

for every real $c \neq 0$ and every real vector $(z_{-1}, \ldots, z_n)$.

The second and last condition on $\mathfrak{I}$: After a centroaffine transformation determined by a regular matrix C the value of $\mathfrak{I}$ at the points of the same parameter can be changed only by multiplying by a constant factor depending generally on the transformation but not on the parameter t. As $q(t)$ does not change under any centroaffine transformation and $W = \det C \,.\, W$, the condition is equivalent to satisfying the relation

$$f(kz_{-1}, z_0, \ldots, z_n) = h(k) \,.\, f(z_{-1}, z_0 \ldots, z_n) \tag{9}$$

for a real function h and any real numbers: $z_{-1}, z_0, \ldots, z_n, k \neq 0$.

Now, *let invariant* $\mathfrak{I}$ *be everywhere in further defined by a function* f *complying with the conditions* (8) *and* (9). It is obvious that *centroaffine curvatures in the sense of Blaschke and Borůvka satisfy the conditions for* $n = 2$ *and* $n = 1$. Invariants as $q(t)/W^2$, $q'(t)/W^3$, $q'^2(t)/q^3(t)$ and others can be also considered as curvatures.

6. CURVES WITH A PERIODIC CENTROAFFINE CURVATURE

In this section we shall need the following results of paper [6, p. 292] (see also [8]):

"Let $q \in C^{\circ}$ be a periodic function with period π.

There exist two independent solutions of the differential equation (q) that may be written in the form

$$y_1 = p_1(t), \qquad y_2 = p_2(t), \tag{*}$$

or in the form

$$y_1 = p_1(t), \qquad y_2 = p_2(t) + ctp_1(t), \tag{**}$$

$c \neq 0$ real, p_1, p_2 periodic or half-periodic[1]) functions with period π, $p_1 \in C^2$, $p_2 \in C^2$, exactly when the equation (q) has a non-trivial solution $y_1(t)$, periodic or half-periodic with period π. The solution $y_2(t)$ is then in the form (*) exactly when $y_1(t)$ is oscillatory and there holds

$$\int_0^{\pi} \left[\frac{1}{y_1^2(t)} - \sum_{i=1}^{n} \frac{1}{y_1'^2(a_i) \sin^2 (t - a_i)} \right] \mathrm{d}t = 0, \tag{10}$$

where $a_1, \ldots, a_n$ are all zeros of $y_1(t)$ on the interval $0 \leq t < \pi$."

With respect to the reference in Sec. 2, if $y_1(t) = e^{f(t)} \sin t$, then the condition (10) is equivalent to the relation

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[1]) A function $h(t)$ is half-periodic with period d if $h(t + d) = -h(t)$ for every t.

$$\int_0^{\pi} \frac{e^{-2f(t)} - 1}{\sin^2 t} \, dt = 0$$

as $n = 1$, $a_1 = 0$, $p_1(t)$ and $p_2(t)$ are half-periodic functions.

Now, let $q(t)$ be a periodic function with period d, $q(t) \in C^\circ$. Then $\mathfrak{J}(t)$ is again a periodic function with period d. According to the result of [5] quoted in Sec. 2, the periodicity of $q(t)$ is not sufficient for the differential equation to have the dispersion in the form $\varphi(t) = t + d$, $d = \text{const.}$ Especially, when f is chosen such that

$$f \in C^2, \quad f(0) = f'(0) = 0, \quad f(t + \pi) = f(t)$$

and

$$\int_0^{\pi} \frac{\exp[-2f(t)] - 1}{\sin^2 t} \, dt \neq 0,$$

then the differential equation (q) with the function q given by the relation

$$q(t) = f'' + f'^2 + 2f' \operatorname{cotg} t - 1 \tag{11}$$

has the solution $y_1(t) = e^{f(t)} \sin t$ with zeros in the same distances π. If $\varphi(t) = t + d$ is the dispersion of the differential equation (q), then necessarily $d = \pi$. But $\varphi(t) = t + \pi$ is not the dispersion of (q) as the condition $\int_0^{\pi} \frac{\exp[-2f(t)] - 1}{\sin^2 t} \, dt = 0$ is not satisfied. It means, that every solution not depending on y_1 has not yet zeros at the same distances, and especially, it is not bounded on the interval $(-\infty, \infty)$ according to the quoted result of [6]. Hence, the curve $\boldsymbol{u}_q(u_1, u_2)$, where u_1, u_2 are independent solutions of the differential equation with $q(t)$ given in (11) is not closed (because it is neither bounded), but its invariant $\mathfrak{J}(t)$ is a periodic function with period π.

As an example of a curve which is not closed (neither bounded) but which has a periodic invariant $\mathfrak{J}$ let $\boldsymbol{u}_q$ be mentioned, where

$$f(t) = 1/2 \ln \left(1 + \frac{\sin^2 t}{a}\right), \qquad a > 0;$$

then

$$\int_0^{\pi} \frac{\exp[-2f(t)] - 1}{\sin^2 t} \, dt = -\int_0^{\pi} \frac{dt}{a + \sin^2 t} \, dt \neq 0$$

and according to (11)

$$q(t) = \frac{a(3 - a) + 2(1 - 3a)\sin^2 t - 4\sin^4 t}{(a + \sin^2 t)^2}$$

(there is $q(t) < 0$ for $a > 3$).

We can summarize:

The periodicity of an invariant $\mathfrak{J}$ of a convex curve $\boldsymbol{u}$, ⟨then in special case, the periodicity of Blaschke's affine curvature or Borůvka's centroaffine curvature⟩ is not sufficient condition for the curve $\boldsymbol{u}$ to be closed.

The preceding considerations yield the assertion:

If one of components of a curve $\boldsymbol{u}_q(u_1, u_2)$ is periodic or half-periodic function with period π, then the curve $\boldsymbol{u}_q$ is closed or non-bounded (then it is neither closed) iff the component satisfies (10) *or not.*

Examples of non-closed curves with periodic curvatures are also curves which yield periodic differential equations (q) with characteristic numbers λ_1, λ_2 of property $|\lambda_i| \neq 1$ (see Floquet's theory e.g. [4, p. 102]).

Let us introduce some necessary conditions on curvature ($\boldsymbol{k}$ — in Blaschke's sense, $\mathcal{K}$ — in Borůvka's sense) for a curve $\boldsymbol{u}_q$ to be closed.

Let a parametric expression $\boldsymbol{u}_q(u_1, u_2)$ of a closed curve is such that the differential equation (q) has the dispersion $\varphi(t)$ satisfying $\varphi^{[n]}(t) = t + \pi$ (according to previous results, this can be always fulfilled). Further, the existence of necessary derivatines and $q(t) \neq 0$, if q is in a denominator, will be required. Then

1° $$\mathfrak{J}(t),\ \boldsymbol{k}(t) \quad \text{and} \quad \mathcal{K}(t)$$

are periodic functions with period π.

2° $$\int_0^t \mathcal{K}(\sigma)\,\mathrm{d}\sigma$$

is periodic with period π.

If $\boldsymbol{u}_q$ is even simple centrosymmetric and $n = 1$ (then the last condition can be always arranged) then also:

3° $$0 < W^{2/3} \int_0^\pi \boldsymbol{k}(t)\,\mathrm{d}t \leqq \pi$$

and the equality holds just for ellipses with the centers at the origin.

4° $$0 < \int_0^\pi \left[\int_0^t \operatorname{sgn} W \mathcal{K}(\sigma)\,\mathrm{d}\sigma + \frac{1}{\sqrt{|q(0)|}} \right]^{-2} \mathrm{d}t \leqq \pi$$

the equality being reached just for ellipses with the centers at the origin.

5°
$$\operatorname{sgn} W \int_0^t \mathscr{K}(\sigma)\, d\sigma$$

reaches at least four times the value $1 - 1/\sqrt{|q(0)|}$ *on the interval* $0 \leqq t < \pi$.

Proof: Under the given assumptions, the function $q(t)$ is periodic with period π, thus 1° must hold. And $\operatorname{sgn} W \int_0^t \mathscr{K}(\sigma)\, d\sigma = \frac{1}{\sqrt{|q(t)|}} - \frac{1}{\sqrt{|q(0)|}}$ implies 2°.

Further, $W^{2/3} \int_0^\pi k(t)\, dt = \int_0^\pi [q^{1/3} + 1/2(q^{-2/3})''] \, dt$. According to the periodicity of $q(t)$ with period π, the last integral is equal to $-\int_0^\pi q^{1/3}(t)\, dt$. As $q \neq 0$ and the curve u_q being closed, it holds $q(t) < 0$. With respect to the quoted result of [7] in Sec. 4, we have

$$0 < -\int_0^\pi q^{1/3}(t)\; dt = \int_0^\pi |q(t)|^{1/3}\, dt \leqq \pi,$$

the equality being reached just for ellipses with the centers at the origin. This proves 3°.

Now,

$$\int_0^\pi \left[\int_0^t \operatorname{sgn} W \mathscr{K}(\sigma)\, d\sigma + \frac{1}{\sqrt{|q(0)|}} \right]^{-2} dt = \int_0^\pi |q(t)|\, dt > 0$$

and according to the mentioned result of [7], it is $\int_0^\pi |q(t)|\, dt \leq \pi$, the equality holds just for ellipses with the centers at the origin. Thus 4° is proved.

At the end we have again $\operatorname{sgn} W \int_0^t \mathscr{K}(\sigma)\, d\sigma = \frac{1}{\sqrt{|q(t)|}} - \frac{1}{\sqrt{|q(0)|}}$.

O. Borůvka [3] has proved that every two functions q, q^* such that the corresponding differential equations (q), (q^*) have the same dispersion $\varphi(t)$ reach the same value at least four times on any interval $t_0 \leq t < \varphi(t_0)$. When $q^* \equiv -1$ [and then $\varphi(t) = t + \pi$] is taken then the function $1/\sqrt{|q(t)|}$ reaches at least four times the value 1 on the interval $0 \leq t < \pi$, q.e.d.

Remark. Let us note that there are other relations in [7] not demanding $q \neq 0$. Let us mention also that the estimation (four times) in 5° cannot be improved as an example is constructed in [5] which realizes the lowest limit.

7. CONSTRUCTIONS OF ALL CURVES OF PRESCRIBED PROPERTIES

Constructions of all closed (simple or not) curves which, as it has been shown, are characterized by the dispersion $\varphi(t)$ complying with $\varphi^{[n]}(t) = t + d$, and other curves connecting with those, can be derived from results of [6], where solutions of periodic differential equations (q) are studied. Especially, the construction of all differential equations (and then also curves) with dispersions of type $\varphi^{[n]}(t) = t + \pi$ are given here and sufficient and necessary conditions warranted dispersions of the type are derived.

Relations between dispersions of different differential equations and boundedness of their solutions (and then of corresponding curves, too) are studied in [8].

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