Valter Šeda

A comparison theorem in the theory of the second-order linear differential transformations

Archivum Mathematicum, Vol. 5 (1969), No. 1, 7--17

Persistent URL: http://dml.cz/dmlcz/104676

A COMPARISON THEOREM IN THE THEORY OF THE SECOND-ORDER LINEAR DIFFERENTIAL TRANSFORMATIONS

V. Šeda

Received June 20, 1968

In the paper a comparison theorem for the solutions of some nonlinear differential equations which have the fundamental meaning for the transformation theory of linear differential equations of the second order is proved. In some special cases this theorem will be sharpened. In deriving the results the transformation theory as it is mentioned in the book [2] will be used. To emphasize the inclusion of the Jacobi's differential equation into the theory of linear differential equations, we shall consider it in the form $y'' + q(t)\,y = 0$. The sign $\{x, t\}$ will have the usual meaning. The signs j, J will denote open intervals.

The comparison theorem will be proved step by step. As the first one will serve the

Lemma 1. *Let $t_0 \in j, x_0, x_0' \neq 0, x_0''$ be arbitrary numbers. Let $q_i(t) \in C_0(j)$, $i = 1,2$ and $q_1(t) \leqq q_2(t)$ for $t \in j$. Let $x_i(t)$, $i = 1,2$, be the solution of the differential equation*

$$\{x, t\} = q_i(t), \tag{1}$$

satisfying the initial conditions

$$x_i(t_0) = x_0, \qquad x_i'(t_0) = x_0', \qquad x_i''(t_0) = x_0''$$

and let j_i be its interval of definition. Then $j_2 \subset j_1$ and if $x_0' > 0$ ($x_0' < 0$) then

$$x_1'(t) \leqq x_2'(t) \qquad (x_1'(t) \geqq x_2'(t)), \qquad t \in j_2 \tag{2}$$

and

$$x_1(t) \geqq x_2(t) \qquad (x_1(t) \leqq x_2(t)) \text{ for } t \in j_2, \qquad t \leqq t_0,$$

$$x_1(t) \leqq x_2(t) \qquad (x_1(t) \geqq x_2(t)) \text{ for } t \in j_2, \qquad t \geqq t_0.$$

If for $t_1 \in j_2$, $t_1 < t_0$ and $t_2 \in j_2$, $t_2 > t_0$, respectively $x_1(t_1) = x_2(t_1)$ and $x_1(t_2) = x_2(t_2)$, respectively, then $x_1(t) = x_2(t)$, $q_1(t) = q_2(t)$ in $\langle t_1, t_0\rangle$ and $\langle t_0, t_2\rangle$, respectively.

Proof. Because with $x(t)$ is also $-x(t)$ a solution of (1) it is sufficient to prove the lemma only for $x_0' > 0$. Let, therefore, $x_0' > 0$ and let $u_i(t)$, $v_i(t)$ be a fundamental system of solutions of the equation

(3) $$y'' + q_i(t)\,y = 0, \qquad i = 1,2,$$

such that $x_i(t) = u_i(t)/v_i(t)$. We can assume that $u_1(t)$, $u_2(t)$ as well as $v_1(t)$, $v_2(t)$ satisfy the same initial conditions at t_0. The inequality (2) is equivalent to the inequality $v_2^2(t) \leqq v_1^2(t)$, $t \in j_2$. Consider the function $v_1(t)$. It can be written in the form

(4) $$v_1(t) = v_2(t) + \int_{t_0}^{t} K(t, \tau)\,[q_2(\tau) - q_1(\tau)]\,v_1(\tau)\,d\tau, \qquad t \in j,$$

where $K(t, \tau)$ as the function of t (τ being fixed) is the solution of (3) for $i = 2$ which satisfies the initial conditions $K(\tau, \tau) = 0$, $\dfrac{\partial K(\tau, \tau)}{\partial t} = 1$. Since $v_2(t) \neq 0$ in $j_2 = (a_2, b_2)$, $K(t, \tau) \leqq 0$ for $a_2 < t \leqq \tau \leqq t_0$ while $K(t, \tau) \geqq 0$ for $t_0 \leqq \tau \leqq t < b_2$ ($K(t, \tau) = 0$ only for $t = \tau$). If $v_2(t) > 0$ in j_2, then $v_1(t) > 0$ in the neighborhood of t_0. Then from (4) it follows that $v_1(t) \geqq v_2(t)$ first in that neighborhood and then in the whole interval j_2. Similarly we get the inequality $v_1(t) \leqq v_2(t)$, $t \in j_2$, if $v_2(t) < 0$ in j_2. In both cases $v_2^2(t) \leqq v_1^2(t)$, $t \in j_2$. Thus the inclusion $j_2 \subset j_1$ is also proved.

If $x_2(t_1) = x_1(t_1)$ for $t_1 < t_0, t_1 \in j_2$, then step by step we get $x_1'(t) = x_2'(t)$, $v_1^2(t) = v_2^2(t)$ and on basis of (4) $q_1(t) = q_2(t)$ in $\langle t_1, t_0\rangle$.

Lemma 2. *Let $t_0 \in j$, α_0, $\alpha_0' \neq 0$, α_0'' be arbitrary numbers. Let $q_i(t) \in C_0(j)$, $i = 1,2$, and $q_1(t) \leqq q_2(t)$ for $t \in j$. Let $\alpha_i(t)$, $i = 1,2$, be the solution of the differential equation*

(5) $$\{\alpha, t\} + \alpha'^2 = q_i(t)$$

satisfying the initial conditions

(6) $$\alpha_i(t_0) = \alpha_0, \qquad \alpha_i'(t_0) = \alpha_0', \qquad \alpha_i''(t_0) = \alpha_0''.$$

Then, if $\alpha_0' > 0$ $\quad(\alpha_0' < 0)$,

$$\alpha_1(t) \geqq \alpha_2(t) \qquad (\alpha_1(t) \leqq \alpha_2(t)) \qquad \text{for } t \in j, \qquad t \leqq t_0$$
$$\alpha_1(t) \leqq \alpha_2(t) \qquad (\alpha_1(t) \geqq \alpha_2(t)) \qquad \text{for } t \in j, \qquad t \geqq t_0.$$

If $\alpha_1(t_1) = \alpha_2(t_1)$ and $\alpha_1(t_2) = \alpha_2(t_2)$, respectively, for $t_1 \in j$, $t_1 < t_0$ and $t_2 \in j$, $t_2 > t_0$, respectively, then $\alpha_1(t) = \alpha_2(t)$, $q_1(t) = q_2(t)$ in $\langle t_1, t_0\rangle$ and $\langle t_0, t_2\rangle$, respectively.

Proof. Similarly as in the preceding case the lemma will be proved only for $\alpha_0' > 0$. Let $u_i(t)$, $v_i(t)$, $i = 1,2$, form a fundamental system of the solutions of the equation (3) with the property that $\operatorname{tg} \alpha_i(t) = u_i(t)/v_i(t)$ for such all $t \in j$ where $v_i(t) \neq 0$. Let $u_1(t)$, $u_2(t)$, as well as $v_1(t)$, $v_2(t)$, satisfy the same initial conditions at t_0. If $v_2(t) \neq 0$ for

$t \in (a_2, b_2)$ whereby $t_0 \in (a_2, b_2)$ and if $v_2(a_2) = v_2(b_2) = 0$ (provided $a_2, b_2 \in j$), then the lemma 2 in (a_2, b_2) follows from the lemma 1. Let us assume now that (a_3, b_3) is the maximal open interval in which the lemma 2 is true and let $b_3 < b$, where $j = (a, b)$. Then it must be $\alpha_1(b_3) = \alpha_2(b_3) = \alpha_3$. These cases can arise.

If $\alpha_1(t) = \alpha_2(t)$ for $t_0 \leqq t \leqq b_3$, then also $q_1(t) = q_2(t)$ in $\langle t_0, b_3\rangle$ and we can extend the validity of the lemma 2 to a larger interval than (a_3, b_3) is. In the case that $\alpha_1(t) = \alpha_2(t)$ for $t_0 \leqq t \leqq \bar{t}_0 < b_3$ we should take the point $\bar{t}_0$ as the initial point t_0. If suffices, therefore, to consider the case $\alpha_1(t) < \alpha_2(t)$ for $t_0 < t < b_3$. Here two possibilities can arise. If $\alpha_3 - \alpha_0 \geqq \pi$, then there exist the points $\bar{t}_2$, $\bar{\bar{t}}_2$, $t_0 \leqq \bar{\bar{t}}_2 \leqq \bar{t}_2 < b_3$ (whereby two equalities hold simultaneously) such that $\alpha_1(\bar{t}_2) = \alpha_2(\bar{\bar{t}}_2) = \alpha_3 - \pi$. Then the function $y_i(t) = \sin[\alpha_i(t) + \pi - \alpha_3]/\sqrt{\alpha_i'(t)}$ is a solution of the equation (3) for $i = 1,2$, with the property $y_1(\bar{t}_2) = y_1(b_3) = 0$, $y_2(\bar{\bar{t}}_2) = y_2(b_3) = 0$, $y_2(t) \neq 0$, $\bar{\bar{t}}_2 < t < b_3$. In the case $\bar{\bar{t}}_2 < \bar{t}_2$ this leads to the contradiction with the Sturm's comparison theorem ([1], p. 259) and in the case $\bar{\bar{t}}_2 = \bar{t}_2$ from that theorem we get the equality $q_1(t) = q_2(t)$ in $\langle t_0, b_3\rangle$ what is in contradiction with the assumption $\alpha_1(t) < \alpha_2(t)$ for $t_0 < t < b_3$.

If $\alpha_3 - \alpha_0 < \pi$, then for sufficiently small $\varepsilon > 0$ we consider the linear independent solutions $\bar{u}_i(t) = \sin\left[\alpha_i(t) - \alpha_3 + \frac{\pi}{2} - \varepsilon\right]/\sqrt{\alpha_i'(t)}$, $\bar{v}_i(t) = \cos\left[\alpha_i(t) - \alpha_3 + \frac{\pi}{2} - \varepsilon\right]/\sqrt{\alpha_i'(t)}$ of the equation (3) $(i = 1,2)$. One solution of the equation $\operatorname{tg} \alpha_i(t) = \bar{u}_i(t)/\bar{v}_i(t)$ is $\bar{\alpha}_i(t) = \alpha_i(t) - \alpha_3 + \frac{\pi}{2} - \varepsilon$, whereby $-\frac{\pi}{2} < \bar{\alpha}_i(t) < \frac{\pi}{2}$, $t_0 \leqq t \leqq b_3$. Therefore $\bar{v}_i(t) \neq 0$ in $\langle t_0, b_3\rangle$ and, by what was already proved, Lemma 2 is true in an open interval which contains $\langle t_0, b_3\rangle$. Hence $\bar{\alpha}_1(t) = \bar{\alpha}_2(t)$ in $\langle t_0, b_3\rangle$, what is in contradiction with $\alpha_1(t) < \alpha_2(t)$ for $t_0 < t < b_3$. Therefore $b_3 = b$ must be.

Similarly it can be shown that $a_3 = a$.

By using Lemma 2 the main theorem of the paper will be proved which reads as follows.

Theorem 1. *Let $t_0 \in j$, $X_0 \in J$, $X_0' \neq 0$, X_0'' be arbitrary numbers. Let $q_i(t) \in C_0(j)$, $Q_i(T) \in C_0(J)$, $i = 1,2$ and let $q_1(t) \leqq q_2(t)$ for $t \in j$, $Q_2(T) \leqq Q_1(T)$ for $T \in J$. Let $X_i(t)$, $i = 1,2$, be the solution of the equation*

$$\{X, t\} + Q_i(X)\, X'^2 = q_i(t) \tag{7}$$

satisfying the initial conditions

$$X_i(t_0) = X_0, \qquad X_i'(t_0) = X_0', \qquad X_i''(t_0) = X_0''$$

and let j_i *be its interval of definition. Then* $j_2 \subset j_1$. *Further, if* $X_0' > 0$ $(X_0' < 0)$ *then*

$$X_1(t) \geqq X_2(t) \qquad (X_1(t) \leqq X_2(t)) \quad \textit{for} \quad t \in j_2, \quad t \leqq t_0$$
$$X_1(t) \leqq X_2(t) \qquad (X_1(t) \geqq X_2(t)) \quad \textit{for} \quad t \in j_2, \quad t \geqq t_0. \tag{8}$$

If $X_1(t_1) = X_2(t_1)$ *and* $X_1(t_2) = X_2(t_2)$, *respectively, for* $t_1 \in j_2$, $t_1 < t_0$ *and* $t_2 \in j_2$, $t_2 > t_0$, *respectively, then* $X_1(t) = X_2(t)$, $q_1(t) = q_2(t)$ *in* $\langle t_1, t_0 \rangle$ *and in* $\langle t_0, t_2 \rangle$, *respectively, as well as* $Q_1(T) = Q_2(T)$ *in* $X_1(\langle t_1, t_0 \rangle)$ *and* $X_1(\langle t_0, t_2 \rangle)$, *respectively.*

Proof. By [2], p. 193, $X_i(t)$, $i = 1,2$, is the solution of the equation

$$A_i(X) = \alpha_i(t), \tag{9}$$

where $A_i(T)$ is the solution of the equation

$$\{A, T\} + A'^2 = Q_i(T), \tag{10}$$

which satisfies the initial conditions

$$A_i(X_0) = 0, \qquad A_i'(X_0) = 1, \qquad A_i''(X_0) = 0 \tag{11}$$

and $\alpha_i(t)$ is the solution of the equation (5) determined by the initial conditions

$$\alpha_i(t_0) = 0, \qquad \alpha_i'(t_0) = X_0', \qquad \alpha_i''(t_0) = X_0''. \tag{12}$$

Let $X_0' > 0$ $(X_0' < 0)$. By Lemma 2, $\alpha_2(t) \leqq \alpha_1(t) < 0$ $(0 < \alpha_1(t) \leqq \leqq \alpha_2(t))$ for $t \in j_2$, $t < t_0$ and there exists the solution $X_{22} = X_2(t)$ of the equation $A_2(X) = \alpha_2(t)$. From the continuity and the increase of $A_2(T)$ also the existence of the solution X_{21} of the equation $A_2(X) = \alpha_1(t)$ follows, whereby $X_{22} \leqq X_{21} (X_{21} \leqq X_{22})$. Again on basis of Lemma 2 $A_1(X_{21}) \leqq \alpha_1(t)$ $(A_1(X_{21}) \geqq \alpha_1(t))$ and since $A_1(T)$ is also continuous and increasing, there exists the solution $X_{11} = X_1(t)$ of the equation $A_1(X) = \alpha_1(t)$, whereby $X_{21} \leqq X_{11} (X_{11} \leqq X_{21})$. From this if follows that $X_1(t) \geqq X_2(t)$ $(X_1(t) \leqq X_2(t))$.

If $t \in j_2$, $t > t_0$, from Lemma 2 it follows that $0 < \alpha_1(t) \leqq \alpha_2(t)$ $(0 > \alpha_1(t) \geqq \alpha_2(t))$ and all preceding assertions in the parenthesis interchange their place with those outside the parenthesis. Hence $X_1(t) \leqq X_2(t)$ $(X_1(t) \geqq X_2(t))$ and at the same time we have proved $j_2 \subset j_1$. If there exists $t_1 \in j_2$, $t_1 < t_0$ such that $X_1(t_1) = X_2(t_1)$, then in the preceding considerations only the sign of equality holds at the point $t = t_1$ and therefore the equalities $\alpha_1(t_1) = \alpha_2(t_1)$ and $A_1[X_1(t_1)] = A_2[X_1(t_1)]$

must hold. From this, using Lemma 2, we get the last part of the assertion of the theorem. A similar result is true when $X_1(t_2) = X_2(t_2)$ for some $t_2 \in j_2$, $t_2 > t_0$.

In what follows we shall find a sufficient condition that certain inequality also hold between the first derivatives of the solutions $X_i(t)$ already mentioned in Theorem 1. It will be based on the following

Lemma 3. *Let the open intervals k_i, K_i, $i = 1, 2$, satisfy the relations $k_2 \subset k_1, K_1 \subset K_2$. Let $t_0 \in k_2$, $X_0 \in K_1$, $X_0' \neq 0$, X_0'' be arbitrary numbers. Let the functions $B_i(T) \in C_3(K_i)$, $b_i(t) \in C_3(k_i)$, $i = 1, 2$, have the following properties:*

1. $B_i(X_0) = b_i(t_0) = b_0$, $B_i'(X_0)\, X_0' = b_i'(t_0) = b_0' \neq 0$,
   $B_i''(X_0)\, X_0'^2 + B_i'(X_0)\, X_0'' = b_i''(t_0) = b_0''$.

2. *If* $b_0' > 0$ ($b_0' < 0$), *then* $0 < b_1'(t) \leqq b_2'(t)$ ($0 > b_1'(t) \geqq b_2'(t)$) *for* $t \in k_2$.

3. *If* $B_i'(X_0) > 0$ ($B_i'(X_0) < 0$), *then* $0 < B_2'(T) \leqq B_1'(T)$
   ($0 > B_2'(T) \geqq B_1'(T)$) *for* $T \in K_1$.

4. *Either a)* $B_2'(X_0) \,.\, B_2''(T) \geqq 0, \qquad T \in K_2, \quad T \leqq X_0$
   $B_2'(X_0) \,.\, B_2''(T) \leqq 0, \qquad T \in K_2, \quad T \geqq X_0$

   is true or

   b) $K_1 = K_2$, *and* $B_1'(X_0)\, B_1''(T) \geqq 0, \qquad T \in K_1, \quad T \leqq X_0$
   $B_1'(X_0)\, B_1''(T) \leqq 0, \qquad T \in K_1, \quad T \geqq X_0$

   is valid.

Then for the solution $X_i(t)$, $i = 1,2$ of the equation

$$B_i(X) = b_i(t) \tag{13}$$

it holds: If j_i is the interval of definition of $X_i(t)$, then $j_2 \subset j_1$. Moreover, if $X_0' > 0$ ($X_0' < 0$), then $0 < X_1'(t) \leqq X_2'(t)$ ($0 > X_1'(t) \geqq X_2'(t)$) for $t \in j_2$.

Proof. With respect to the fact that (13) is equivalent to the equation $-B_i(X) = -b_i(t)$, it suffices to consider the case $B_i'(X_0) > 0$.

Let $X_0' > 0$ ($X_0' < 0$). Then $b_0' > 0$ ($b_0' < 0$) and from the properties 2 and 3 of the functions $b_i(t)$, $B_i(T)$ the inequalities $b_2(t) \leqq b_1(t) < b_0$ ($b_2(t) \geqq b_1(t) > b_0$) for $t \in k_2$, $t < t_0$ and $b_0 < b_1(t) \leqq b_2(t)$ ($b_2(t) \leqq b_1(t) < < b_0$) for $t \in k_2$, $t > t_0$ follow. At the same time $B_1(T) \leqq B_2(T) < b_0$ for $T \in K_1$, $T < X_0$ and $B_1(T) \geqq B_2(T) > b_0$ for $T \in K_1$, $T > X_0$. In similar manner as in the proof of Theorem 1, from these inequalities the relation $j_2 \subset j_1$, and the inequalities (8) follow. Further from (13) the equality $X_i'(t) = b_i'(t)/B_i'[X_i(t)]$, $t \in j_i$, $i = 1,2$, follows, from where, using the inequalities 2, 3 and 4a, as well as the inequalities (8), we get

$$0 < X_1'(t) \leqq b_2'(t)/B_1'[X_1(t)] \leqq b_2'(t)/B_2'[X_1(t)] \leqq X_2'(t)$$
$$(X_2'(t) \leqq b_1'(t)/B_2'[X_2(t)] \leqq b_1'(t)/B_2'[X_1(t)] \leqq X_1'(t) < 0)$$

for $t \in j_2$.

If instead of inequalities 4a) the property 4b) of the function $B_1(T)$ is considered, then for $t \in j_2$

$$0 < X_1'(t) \leqq b_2'(t)/B_1'[X_1(t)] \leqq b_2'(t)/B_1'[X_2(t)] \leqq X_2'(t)$$
$$(X_2'(t) \leqq b_1'(t)/B_2'[X_2(t)] \leqq b_1'(t)/B_1'[X_2(t)] \leqq X_1'(t) < 0)$$

hold.

With respect to the meaning of the equation (9) where it suffices that $A_i(T)$ and $\alpha_i(t)$, $i = 1,2$, satisfy the initial condition
$\alpha_i(t_0) = A_i(X_0)$, $\alpha_i'(t_0) = A_i'(X_0)\, X_0'$, $\alpha_i''(t_0) = A_i''(X_0)\, X_0'^2 + A_i'(X_0)\, X_0''$ ([2], p. 194) it will be suitable to consider the solution $A_i(T)$ of the equation (9) and the solution $\alpha_i(t)$ of the equation (5), respectively, as the function $B_i(T)$ and $b_i(t)$, $i = 1,2$, respectively, in Lemma 3.

The solution $\alpha_i(t)$ of the equation (5) satisfying the initial conditions (6) can be also determined in the following manner.

Let for $i = 1,2$ $u_i(t)$, $v_i(t)$ be the fundamental system of the solutions of the equation (3) given by the conditions

(14) $$u_i(t_0) = 0 \qquad u_i'(t_0) = (\operatorname{sgn} \alpha_0') \sqrt{|\alpha_0'|}$$

(15) $$v_i(t_0) = 1/\sqrt{|\alpha_0'|} \quad v_i'(t_0) = -(\operatorname{sgn} \alpha_0')\, \alpha_0''/(2\sqrt{|\alpha_0'|^3}).$$

Then the function

(16) $$r_i(t) = \sqrt{u_i^2(t) + v_i^2(t)}, \qquad t \in j,$$

is the solution of the differential equation

(17) $$r'' + q_i(t)\, r = r^{-3} \qquad ([2], \text{p. } 32)$$

which satisfies the initial conditions

(18) $$r_i(t_0) = 1/\sqrt{|\alpha_0'|}, \qquad r_i'(t_0) = -(\operatorname{sgn} \alpha_0')\, \alpha_0''/(2\sqrt{|\alpha_0'|^3}).$$

Therefore the function $\alpha_0 + (\operatorname{sgn} \alpha_0') \int_{t_0}^{t} \mathrm{d}\tau/r_2^i(\tau)$ is the solution of the differential equation (5) ([2], p. 35) which satisfies the conditions (6) and hence on basis of the Uniqueness theorem ([2], p. 193) we get the relation

(19) $$\alpha_i(t) = \alpha_0 + (\operatorname{sgn} \alpha_0') \int_{t_0}^{t} \mathrm{d}\tau/r_2^i(\tau), \qquad t \in j.$$

Using this we shall prove

Lemma 4. *Let all assumptions of Lemma 2 be satisfied and let* $\alpha_i(t)$, $i = 1,2$, *have the same meaning as in that lemma. Let* $(a_2, b_2) \subset j$ *be the* *maximal interval containing t_0 such that the solution $v_2(t)$ of the differential equation* (3) *for $i = 2$ determined by the conditions* (15) *is positive in* (a_2, b_2). *Then, if $\alpha_0' > 0$, ($\alpha_0' < 0$) then the inequalities*

$$(20) \qquad 0 < \alpha_1'(t) \leqq \alpha_2'(t)\ (0 > \alpha_1'(t) \geqq \alpha_2'(t)), \qquad t \in (a_2, b_2),$$

hold.

Proof. With regard to the equality (19) the inequality (20) is equivalent to the inequality $r_1^2(t) \geqq r_2^2(t)$, $t \in (a_2, b_2)$, where $r_i(t)$, $i = 1,2$, is given by the relation (16) and $u_i(t)$, $v_i(t)$ mean the solutions of the differential equation (3) satisfying (14) and (15), respectively. In the proof of Lemma 1 the inequality $v_1^2(t) \geqq v_2^2(t)$, $t \in (a_2, b_2)$, was proved. If the notations from that proof are used, we can write

$$(21) \qquad u_1(t) = u_2(t) + \int_{t_0}^{t} K(t, \tau)\, [q_2(\tau) - q_1(\tau)]\, u_1(\tau)\, d\tau, \quad t \in j.$$

If $\alpha_0' > 0$, then $u_1(t) < 0$ and $u_1(t) > 0$, respectively, for $t_0 - \varepsilon < t < t_0$ and for $t_0 < t < t_0 + \varepsilon$, respectively, where $\varepsilon > 0$ is a small number. Then from (21) it follows that $u_1(t) \leqq u_2(t) \leqq 0$ for $t \in (a_2, t_0)$ and $u_1(t) \geqq$ $\geqq u_2(t) \geqq 0$ for $t \in \langle t_0, b_2)$. Hence $u_1^2(t) \geqq u_2^2(t)$ for $t \in (a_2, b_2)$. The same inequality will be got if $\alpha_0' < 0$.

If $q_2(t) \leqq 0$ in j and $\alpha_0'' = 0$, then the solution $v_2(t)$ of the equation (3) for $i = 2$ determined by the conditions (15) has no zero in j, as it follows from the identity $(v_2(t)\, v_2'(t))' = -q_2(t)\, v_2^2(t) + v_2'^2(t)$, $t \in j$. Hence then $(a_2, b_2) = j$. At the same time the solution $r_2(t)$ of the equation (17) for $i = 2$ satisfying the initial conditions (18) is convex and thus, on basis of (19), for $\alpha_2(t)$ the inequalities

$$\alpha_2'(t_0)\, \alpha_2''(t) \geqq 0, \qquad t \in j, \qquad t \leqq t_0$$
$$\alpha_2'(t_0)\, \alpha_2''(t) \leqq 0, \qquad t \in j, \qquad t \geqq t_0$$

are true. A similar result also holds for the solution $A_1(T)$ of the differential equation (10) for $i = 1$ if $A_1''(X_0) = 0$ and $Q_1(T) \leqq 0$, $T \in J$.

Hence, if $Q_2(T) \leqq Q_1(T) \leqq 0$, $T \in J$, $q_1(t) \leqq q_2(t) \leqq 0$, $t \in j$, as well as $X_0'' = 0$, then the solution $A_i(T)$ of the equation (10) which is determined by the initial conditions (11) and the solution $\alpha_i(t)$ of the equation (5) given by the initial conditions (12), respectively, satisfies, on basis of the last consideration, and Lemma 4, all assumptions of Lemma 3. From that the following sharpening of Theorem 1 in a special case follows.

Theorem 2. *Let $t_0 \in j$, $X_0 \in J$, $X_0' \neq 0$ be arbitrary numbers. Let $q_i(t) \in C_0(j)$, $Q_i(T) \in C_0(J)$, $i = 1,2$, and $q_1(t) \leqq q_2(t) \leqq 0$ for $t \in j$, $Q_2(T) \leqq Q_1(T) \leqq 0$ for $T \in J$. Let $X_i(t)$, $i = 1,2$, be the solution of the differential equation* (7) *satisfying the initial conditions*

$$X_i(t_0) = X_0, \qquad X_i'(t_0) = X_0', \qquad X_i''(t_0) = 0$$

and j_i be its interval of definition. Then for $X_0' > 0$ ($X_0' < 0$), the inequalities $0 < X_1'(t) \leqq X_2'(t)$ ($0 > X_1'(t) \geqq X_2'(t)$) for $t \in j_2$ hold.

A further sharpening of Theorem 1 is based on the following

Lemma 5. *Let $t_0 \in j = (a, b)$, α_0 be arbitrary numbers. Let $q_i(t) \in C_0(j)$, $i = 1,2$, have these properties:*

1. $0 < c^2 = q_1(t) \leqq q_2(t)$ *for all* $t \in j$.
2. *The closed interval* $\langle t_0 - \pi/(2\sqrt{q_2(t_0)}),\ t_0 + \pi/(2\sqrt{q_2(t_0)}) \rangle = \langle a_2, b_2 \rangle \subset j$.
3. $q_2(t) = q_2(t_0)$ *for all* $t \in \langle a_2, b_2 \rangle$, $q_2(t)$ *is nonincreasing in* $(a, a_2\rangle$ *and nondecreasing in* $\langle b_2, b)$.

Let $\alpha_i(t)$, $i = 1,2$, be the solution of the differential equation (5) satisfying the initial conditions

$$(22) \qquad \alpha_i(t_0) = \alpha_0, \qquad \alpha_i'(t_0) = c, \qquad \alpha_i''(t_0) = 0.$$

Then, for $c > 0$ ($c < 0$) the inequalities

$$0 < \alpha_1'(t) \leqq \alpha_2'(t) \qquad (0 > \alpha_1'(t) \geqq \alpha_2'(t)) \quad \textit{for} \quad t \in j$$

hold.

Proof. It is sufficient to prove the lemma only for $c > 0$. Hence, let $c > 0$. From Lemma 4 the validity of Lemma 5 follows in (a_2, b_2). The solution of the differential equation (5) for $i = 1$ satisfying the initial conditions (22) is $\alpha_1(t) = \alpha_0 + c(t - t_0)$. Therefore

$$(23) \qquad c \leqq \alpha_2'(t), \qquad t \in (a_2, b_2).$$

Since $q_2(t)$ is nonincreasing in $(a, b_2\rangle$ and nondecreasing in $\langle a_2, b)$, on basis of a theorem in [2], p. 115, the fundamental central dispersion $\varphi(t)$ of the differential equation (3) for $i = 2$ satisfies the inequalities $\varphi'(t) \geqq 1$ for $t \in (a, a_2\rangle$ and $\varphi'(t) \leqq 1$ for $t \in \langle a_2, b)$. Using the last inequalities in the equality $\alpha_2'[\varphi(t)] = \alpha_2'(t)/\varphi'(t)$, which follows from the Abelian functional equation ([2], p. 118), we get that the inequality (23) will be also satisfied in further intervals to the left and to the right from (a_2, b_2) in which the nontrivial solution $v_2(t)$ of the differential equation (3) for $i = 2$ satisfying the condition $v_2'(t_0) = 0$ is different from 0. By the continuity of $v_2(t)$ (23) also holds at the zeros of that function.

Similarly will be proved

Lemma 6. *Let $t_0 \in j = (a, b)$, α_0 be arbitrary numbers. Let $q_i(t) \in C_0(j)$, $i = 1,2$, have these properties:*

1. $q_1(t) \leqq c^2 = q_2(t) > 0$ *for all* $t \in j$.

2. *The closed interval* $\langle t_0 - \pi/(2\sqrt{q_1(t_0)}),$
   $t_0 + \pi/(2\sqrt{q_1(t_0)}) \rangle = \langle a_1, b_1 \rangle \subset j.$

3. $q_1(t) = q_1 t_0) > 0$ *for all* $t \in \langle a_1, b_1 \rangle$, $q_1(t)$ *is nondecreasing in* $(a, a_1\rangle$ *and nonincreasing in* $\langle b_1, b)$.

Let $\alpha_i(t)$, $i = 1{,}2$, *be the solution of the differential equation* (5) *satisfying the initial conditions* (22). *Then, for* $c > 0$ $(c < 0)$ *the inequalities*

$$0 < \alpha_1'(t) \leqq \alpha_2'(t) \qquad (0 > \alpha_1'(t) \geqq \alpha_2'(t)) \quad \text{for } t \in j$$

hold.

Proof. The assertion of the lemma will be proved only in $\langle a, b_1 \rangle$. Further the proof continues similarly as that of the preceding lemma. Let $c > 0$. The inequality (20) in $\langle a_1, b_1 \rangle$ is equivalent to the inequality $r_1^2(t) \geqq r_2^2(t)$ in that interval, where $r_i(t)$, $i = 1{,}2$, is given by the relation (16) and $u_i(t)$, $v_i(t)$ are the solutions of the equation (3) satisfying the initial conditions (14) and (15), respectively, for $\alpha_0' = c$, $\alpha_0'' = 0$. A simple calculation shows that $r_1^2(t) = 1/c + (c^2 - c_1^2)/(c_1^2 c) \sin^2 c_1(t - t_0) \geqq 1/c = r_2^2(t)$, $t \in j$, where $c_1^2 = q_1(t_0)$.

Combining Lemma 5 and Lemma 6, on basis of Lemma 3, we get some sharpenigs of Theorem 1. From them only the following will be mentioned as

Theorem 3. *Let* $t_0 \in j = (a, b)$, $X_0 \in J = (A, B)$ *be arbitrary numbers. Let* $q_i(t) \in C_0(j)$, $Q_i(T) \in C_0(J)$, $i = 1{,}2$, *have the following properties:*

1. *The functions* $q_1(t)$, $q_2(t)$ *have the properties* 1.—3. *from Lemma 5.*

2. $Q_2(T) \leqq C^2 = Q_1(T) > 0$ *for all* $T \in J$.

3. *The closed interval* $\langle X_0 - \pi/(2\sqrt{Q_2(X_0)}),$
   $X_0 + \pi/(2\sqrt{Q_2(X_0)}) \rangle = \langle A_2, B_2 \rangle \subset J.$

4. $Q_2(T) = Q_2(X_0) > 0$ *for all* $T \in \langle A_2, B_2 \rangle$, $Q_2(T)$ *is nondecreasing in* $(A, A_2\rangle$ *and nonincreasing in* $\langle B_2, B)$.

Let $X_i(t)$, $i = 1{,}2$, *be the solution of the differential equation* (7) *satisfying the initial conditions*

$$X_i(t_0) = X_0, \qquad X_i'(t_0) = c/C, \qquad X_i''(t_0) = 0$$

and let j_i *be its interval of definition. Then for* $c/C > 0$ $(c/C < 0)$ *the inequalities* $0 < X_1'(t) \leqq X_2'(t)$ $(0 > X_1'(t) \geqq X_2'(t))$ *for* $t \in j_2$ *hold.*

Further the following theorem is true.

Theorem 4. *Let all assumptions of Theorem 1 be satisfied and let the solution* $X_i(t)$ *and the interval* j_i, $i = 1{,}2$, *have the same meaning as in that* *theorem. Let, further,* $Q_1(T_1) \geqq Q_2(T_2)$ *for all* $T_2 \leqq T_1 \leqq X_0$, *as well as for all* $X_0 \leqq T_1 \leqq T_2$. *Let* $Q_2(T) \leqq 0$ *for* $T \in J$. *Then for* $X_0' > 0$ $(X_0' < 0)$ *the following inequalities are true:*

$$(24) \qquad 0 < X_1'(t) \leqq X_2'(t) \; (0 > X_1'(t) \geqq X_2'(t)) \qquad \text{for} \quad t \in j_2$$

$$(25) \qquad \begin{array}{ll} X_1''(t)/X_1'(t) \geqq X_2''(t)/X_2'(t) & t \in j_2, \quad t \leqq t_0 \\ X_1''(t)/X_1'(t) \leqq X_2''(t)/X_2'(t) & t \in j_2, \quad t \geqq t_0 \end{array}$$

and

$$(26) \qquad \begin{array}{ll} X_1''(t) \leqq X_2''(t), & t \in j_2, \quad t \geqq t_0 \\ (X_1''(t) \leqq X_2''(t)), & t \in j_2, \quad t \leqq t_0. \end{array}$$

Proof. If we denote $X_i''(t)/(2X_i'(t)) = z_i(t)$, $t \in j_i$, $i = 1,2$, then $z_i(t)$ satisfies the differential equation $z' = q_i(t) + z^2 + f_i(t)$, where $f_i(t) = - Q_i[X_i(t)]\, X_0'^2 \exp\left(4 \int_{t_0}^{t} z_i(\tau)\, d\tau\right)$.

Assume, first, that $q_1(t) < q_2(t)$, $t \in j$. Then $z_1(t_0) = z_2(t_0)$ and $z_1'(t_0) < z_2'(t_0)$, hence there exists a neighbourhood of the point t_0 in which $z_1(t) > z_2(t)$ for $t < t_0$ and $z_1(t) < z_2(t)$ for $t > t_0$. Let (t_a, t_b) be the maximal open interval in which these inequalities are valid and let $t_b < b_2$, where $j_2 = (a_2, b_2)$. Then it must be $z_1(t_b) = z_2(t_b)$. At the same time on basis of Theorem 1 and of properties of the function $Q_i(T)$, $f_1(t) < f_2(t)$ for $t_a < t < t_0$, as well as for $t_0 < t < t_b$. Further there exists a constant $M > 0$ such that $|z_i(t)| \leqq M$ for $t \in \langle t_0, t_b \rangle$. Hence for $t \in (t_0 - \varepsilon, t_b + \varepsilon)$, $z \in (-M - 1,\ M + 1)$, $\varepsilon > 0$ being sufficiently small, $q_1(t) + z^2 + f_1(t) < q_2(t) + z^2 + f_2(t)$ is true and at the same time both functions satisfy a Lipschitz condition in the variable z there. By the Corollary to Comparison theorem ([1], p. 23) from the equality $z_1(t_b) = z_2(t_b)$ the equality $z_1(t) = z_2(t)$ in $\langle t_0, t_b \rangle$ follows. By a simple transformation we get that the mentioned corollary (with the opposite inequality) is also valid to the left from t_0 and hence, if $t_a > a$, then $z_1(t) = z_2(t)$ for $t \in \langle t_a, t_0 \rangle$. In both cases we have come to a contradiction. Therefore the inequalities (25) are valid in j_2. From them after integration we get the inequalities (24) and (26), step by step.

If $q_1(t) \leqq q_2(t)$, let us consider the solution $X_{1n}(t)$, $n = 1, 2, 3, \ldots$, of the differential equation

$$\{X, t\} + Q_1(X)\, X'^2 = q_1(t) - 1/n$$

satisfying the same initial conditions as $X_1(t)$. Then the inequalities (24), (25) and (26) are valid where instead of $X_1(t)$ will stand $X_{1n}(t)$. From these inequalities, as well as from Theorem 1, it follows, that the values of the functions $X_{1n}(t)$, $X_{1n}'(t)$, $X_{1n}''(t)$, as well as those of the functions

$X_1(t)$, $X_1'(t)$, $X_1''(t)$ for t from the compact subinterval of the interval j_2 lie in a compact set of the space of the variables (t, X, X', X'') which does not contain $X' \neq 0$. Further the Uniqueness theorem holds for the solutions of the equation (7) ([2], p. 193). From that, using the Theorem on continuous dependence of solution on the parameter, [3], p. 58, we get the uniform convergence $X_{1n}(t)$ to $X_1(t)$, $X_{1n}'(t)$ to $X_1'(t)$ and $X_{1n}''(t)$ to $X_1''(t)$ on each compact subinterval of the interval j_2. Thus the inequalities (24), (25) and (26) are proved also in this case.

REFERENCES

[1] G. Birkhoff, G.—C. Rota, *Ordinary Differential Equations*, Ginn and Company, Boston 1962.

[2] O. Borůvka, *Lineare Differentialtransformationen 2. Ordnung*, VEB Deutscher Verlag der Wissenschaften, Berlin 1967.

[3] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, Inc. New York 1955.

Department of mathematics
A. Komenský University
Bratislava, Šmeralova 2, ČSSR