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SIMULTANEOUS NONDETERMINISTIC GAMES (III)

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REMARKS

The preceding two parts (§§ 0—3; §§ 4, 5) of this discourse appeared in this journal in T5 (1969), 29—60, and T6 (1970), 115—144. The third part contains the introduction of some special aims and of certain properties of aims (§ 6a, b), and the most important general min-max results for aims (§ 6c/1—3). These min-max results are very strong [as they guarantee that plain strategies (excepting the case of the passive player in § 6c/3, where somewhat "stronger" kinds of strategies must be used) are sufficient for in some sense optimal playing at aims belonging to certain important sizable classes of aims], and involve significant particular cases investigated formerly in [1] (and other Berge's works), [4], [5], [18], as we shall show in § 6e, where the connections among the corresponding results of the quoted references and those of this article will be presented, too. Remarks to § 6a-c will be contained in § 6d, in the fourth part. The results of § 6c will serve for deriving theorems on pay-off functions (mainly on the so-called Bergean ones, cf. [18], and § 10 of [4]) in § 9, on topological games (cf. [7], [8]) in § 8, and for proving theorems belonging to the so-called descriptive point of view (cf. § 9 in [4], and [11]) in § 7.

Preliminary (and, usually, considerably less general) versions of many results and considerations of the whole discourse are contained in my papers [4], [5], [18], which were published as research memoranda of the research programme headed by doc. dr. V. Polák. Within the framework of this programme, I have besides investigated some problems concerning the actual playing of certain SN-games: As it was already shown by Zermelo in [20], ⌜two—player initial finite games with perfect information and with (antagonistic) WDR (= win—draw—loss; any infinite play is drawn) pay—offs⌝ (instead of ⌜...⌝ we shall say only "*PI-games*") have significant strategic properties; for having these properties, the so-called board games (special actual PI-games, as, e.g., chess, checkers etc.), having been known for several thousand years, are to be considered as the best among all *actually played* games. Nevertheless, the initial situation in any PI-game (except those having only one play starting from the initial position) is asymmetric for the players; thus, PI-games are not "fair", which is a certain flaw of them. But not only PI-games but also ⌞initial finite complete games with WDR pay-offs⌟ (instead of ⌞...⌟ we shall say only "*I-games*"; PI-games can be considered as particular I-games, cf. § 4d) have those significant strategic properties ([19], § 2.2); thus, it is natural to search for *symmetric* (i.e. having an automorphism which changes the players and, of course, preserves the initial position) I-games, expecially symmetric I-games with the character in some sense near to that of well-known board games (as chess etc.). Constructing the former games — which might be considered as the best among all *possible* games — has shown to be a somewhat complicated problem (while it is not very difficult to construct "well playable" symmetric I-games if no special preliminary conditions to their character are given); this problem, together with certain connected questions (in particular, the applications of the so-called antagonistic coordinative arbitration) were investigated circumstantially in [19], where I have also presented a well-playable concrete symmetric I-game (called Symmetrized Chess and being near to the usual chess), as an example of the constructions given there. (The main results of [19] will be published separately in a journal article.)

Thus, one may conclude that investigating SN-games (and their special subclasses, mainly antagonistic complete games) not only is advantageous from the mathematical point view (cf. § 0), but also leads to the introduction of quite new actually playable games being in principle better than the games known till now.

§ 6. THE FUNDAMENTAL MIN-MAX RESULTS

0. Convention (for § 6). In § 6, let (P, P_0) be a *type* (§ 1.1); the basic aims (§ 6.6), the r-aims (§ 6.13.2), the properties in § 6.1 etc. will be introduced to the given type. Of course, also the sets $Z := P - P_0$, $\boldsymbol{Z}$, $\boldsymbol{P}$, memory relations (cf. §§ 1.1-2) etc. are considered as defined to (P, P_0).

Further, **in § 6a, b** $\boldsymbol{A}$ denotes an aim (§ 1.1), $\mathfrak{A}$ means an aim-collection (§ 3.4), $\Gamma \in \text{Corr}(P, P)$ is some (fixed) P_0-ended graph (§§ 2.16, 2.18.2), $\boldsymbol{X} := \boldsymbol{X}_\Gamma$ (§ 2.26.2).

Remark. In contradistinction to the preceding two parts, we shall always use the denotations u_1, u_i, v_ι, $\mathfrak{U}_j$, $\mathfrak{B}_2$ (etc.) instead of $\boldsymbol{u_1}$, $\boldsymbol{u_i}$, $\boldsymbol{v_\iota}$, $\boldsymbol{\mathfrak{U}_j}$, $\boldsymbol{\mathfrak{B}_2}$ (cf. §§ 3.2, 3.3), respectively, as the composition of the thick indices in the latter expressions would be too laborious for the printing house. (That replacement does not lead to ambiguity.)

a) The properties (I, Γ), $(\mathrm{I}^\circ, \Gamma)$, (I_0, Γ), $(\mathrm{I}_0^\circ, \Gamma)$

The important (cf. § 3.10—11) property (I, Γ) was introduced in § 3.9.2. Now we define several other usable properties at aims.

1.1. Definition. We say that $\boldsymbol{A}$ has the *property* (K, Γ), where $K \in \{\mathrm{I}_0, \mathrm{I}^\circ, \mathrm{I}_0^\circ\}$ (three-symbol set), iff for each $\boldsymbol{x} \in \boldsymbol{X}$ and each k with $0 \leq k < 1 + l(\boldsymbol{x})$ there holds the corresponding statement $[K, \Gamma]$ (depending on $\boldsymbol{x}$, k), where

$$[\mathrm{I}_0, \Gamma] := \boldsymbol{x} \in \boldsymbol{A} \Rightarrow \boldsymbol{x}^{[k]} \in \boldsymbol{A}$$
$$[\mathrm{I}^\circ, \Gamma] := \boldsymbol{x} \in \boldsymbol{A} \Leftarrow \boldsymbol{x}^{[k]} \in \boldsymbol{A}$$
$$[\mathrm{I}_0^\circ, \Gamma] := \boldsymbol{x} \in \boldsymbol{A} \Leftrightarrow \boldsymbol{x}^{[k]} \in \boldsymbol{A}$$

1.2. Definition, remark. For $\boldsymbol{x}$, $\boldsymbol{y} \in \boldsymbol{X}$, $0 \leq k < 1 + \min(l(\boldsymbol{x}), l(\boldsymbol{y}))$ we define (cf. 1.1, § 3.9.2):

$$[\mathrm{I}, \Gamma] := [\boldsymbol{x} \in \boldsymbol{A} \wedge (x_0, \ldots, x_k) = (y_0, \ldots, y_k) \wedge (\boldsymbol{x}^{[k]} \notin \boldsymbol{A} \vee \boldsymbol{y}^{[k]} \in \boldsymbol{A})] \Rightarrow \boldsymbol{y} \in \boldsymbol{A}$$

Then $\boldsymbol{A}$ has the *property* (I, Γ) iff $[\mathrm{I}, \Gamma]$ holds for all those $\boldsymbol{x}$, $\boldsymbol{y}$, k.

1.3. Definition, remark. We put $\bar{K} := \begin{cases} \mathrm{I}^\circ \\ \mathrm{I}_0 \\ K \end{cases}$ if $K \begin{cases} = \mathrm{I}_0 \\ = \mathrm{I}^\circ \\ \in \{\mathrm{I}, \mathrm{I}_0^\circ\} \end{cases}$; thus $\bar{\bar{K}} = K$. (K, Γ) (where $K \in \{\mathrm{I}^\circ, \mathrm{I}_0, \mathrm{I}_0^\circ, \mathrm{I}\}$) is said to be an aim basic Γ-property (or only: a *basic property*), and $(\bar{K}, \Gamma)$ is called its *dual* one. Instead of $(K, \mathbf{L})$ we shall write (K), too.

1.4. Definition. We say that $\mathfrak{A}$ has the property (K, Γ) iff each $\boldsymbol{A} \in \mathfrak{A}$ has this basic property. (Cf. § 3.9.3.) We say that an aim function $\boldsymbol{f}$ [an aim correspondence $\boldsymbol{p}^\varepsilon$] (§ 5.8) has the property (K, Γ) iff the collection im $\boldsymbol{f}$ $(:= \{\boldsymbol{f}(a) \mid a \in \text{dom}\, \boldsymbol{f}\})$ $[\{\boldsymbol{p}^\varepsilon A \mid A \subset P\}]$ has this property.

2.0. Definition. A collection $\mathfrak{U}$ is said to be $\cup$*-directed* [$\cap$*-directed*] iff to each $A_1, A_2 \in \mathfrak{U}$ there exists $A \in \mathfrak{U}$ such that $A_1 \cup A_2 \subset A$ $[A \subset A_1 \cap A_2]$.

2.1. Theorem. *Let* (K, Γ) *be a basic property. There holds:*

(*o*) $\boldsymbol{A}$ *has* $(K, \Gamma) \Leftrightarrow \boldsymbol{A} \cap \boldsymbol{X}$ *has* (K, Γ)

(*i*) $\boldsymbol{A}$ *has* $(\mathrm{I}_0^\circ, \Gamma) \Leftrightarrow \boldsymbol{A}$ *has the properties* $(\mathrm{I}^\circ, \Gamma)$, (I_0, Γ)

(*ii*) $\boldsymbol{A}$ *has* $(\mathrm{I}_0^\circ, \Gamma) \Rightarrow \boldsymbol{A}$ *has* (I, Γ)

(*iii*) $\boldsymbol{A}$ *has* $(K, \Gamma) \Leftrightarrow \boldsymbol{P} - \boldsymbol{A}$ *has* $(\overline{K}, \Gamma) \Leftrightarrow \boldsymbol{X} - \boldsymbol{A}$ *has* $(\overline{K}, \Gamma)$

(*iv*) $\boldsymbol{A}$ *has* $(K) \Rightarrow \boldsymbol{A}$ *has* (K, Γ)

(*v*) $\emptyset$ *and* $\boldsymbol{P}$ *have the property* (I_0°)
$\boldsymbol{X}$ *has the properties* (I_0), (I), $(\mathrm{I}_0^\circ, \Gamma)$
$\boldsymbol{P} - \boldsymbol{X}$ *has the properties* (I°), (I), $(\mathrm{I}_0^\circ, \Gamma)$

(*vi*/1) $\boldsymbol{A}$ *has* $(\mathrm{I}_0, \Gamma) \Leftrightarrow \boldsymbol{A} \cap \boldsymbol{X}$ *has* (I_0)

(*vi*/2) $\boldsymbol{A}$ *has* $(\mathrm{I}^\circ, \Gamma) \Leftrightarrow \boldsymbol{A} \cup (\boldsymbol{P} - \boldsymbol{X})$ *has* (I°)

(*vii*) $\mathfrak{A}$ *has* $(K, \Gamma) \wedge K \neq \mathrm{I} \Rightarrow \bigcup_{\boldsymbol{A} \in \mathfrak{A}} \boldsymbol{A}$ *and* $\bigcap_{\boldsymbol{A} \in \mathfrak{A}} \boldsymbol{A}$ *have* (K, Γ)

(*viii*/1) $\mathfrak{A}$ *has* $(\mathrm{I}, \Gamma) \wedge \mathfrak{A}$ *is* $\cup$*-directed* $\Rightarrow \bigcup_{\boldsymbol{A} \in \mathfrak{A}} \boldsymbol{A}$ *has* (I, Γ)

(*viii*/2) $\mathfrak{A}$ *has* $(\mathrm{I}, \Gamma) \wedge \mathfrak{A}$ *is* $\cap$*-directed* $\Rightarrow \bigcap_{\boldsymbol{A} \in \mathfrak{A}} \boldsymbol{A}$ *has* (I, Γ)

(*Here we put* $\bigcap_{\boldsymbol{A} \in \emptyset} \boldsymbol{A} := \boldsymbol{P}$.)

Proof.

(*o*): Let $\boldsymbol{x} \in \boldsymbol{X}$, $0 \leq k < 1 + l(\boldsymbol{x})$. Then, of course, $\boldsymbol{x} \in \boldsymbol{A}$ iff $\boldsymbol{x} \in \boldsymbol{A} \cap \boldsymbol{X}$, and the same may be said about $\boldsymbol{x}^{[k]}$, for evidently $\boldsymbol{x}^{[k]} \in \boldsymbol{X}$. By means of this trivial remark the assertion (*o*) can be obtained immediately from 1.1, 1.2.

(*i*) is trivial, since $[\mathrm{I}_0^\circ, \Gamma] \Leftrightarrow [\mathrm{I}^\circ, \Gamma] \wedge [\mathrm{I}_0, \Gamma]$ (for all $\boldsymbol{A} \subset \boldsymbol{P}$, $\boldsymbol{x} \in \boldsymbol{X}$, $0 \leqslant k < 1 + l(\boldsymbol{x})$).

(*ii*): Let $\boldsymbol{A}$ have the property $(\mathrm{I}_0^\circ, \Gamma)$. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, $0 \leqslant k < 1 + \min(l(\boldsymbol{x}), l(\boldsymbol{y}))$, then $\boldsymbol{x} \in \boldsymbol{A} \wedge (\boldsymbol{x}^{[k]} \notin \boldsymbol{A} \vee \boldsymbol{y}^{[k]} \in \boldsymbol{A}) \Leftrightarrow (\boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{x}^{[k]} \notin \boldsymbol{A}) \vee (\boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{y}^{[k]} \in \boldsymbol{A}) \Rightarrow \boldsymbol{y}^{[k]} \in \boldsymbol{A} \Rightarrow \boldsymbol{y} \in \boldsymbol{A}$ ($\boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{x}^{[k]} \notin \boldsymbol{A}$ cannot occur, etc.); consequently, $\boldsymbol{A}$ has the property (I, Γ).

(*iii*): Let $\boldsymbol{x} \in \boldsymbol{X}$, $0 \leqslant k < 1 + l(\boldsymbol{x})$. If $\boldsymbol{A}$ has the property $(\mathrm{I}^\circ, \Gamma)$, then $\boldsymbol{x} \in \boldsymbol{P} - \boldsymbol{A}$ implies $\boldsymbol{x}^{[k]} \in \boldsymbol{P} - \boldsymbol{A}$; thus $\boldsymbol{P} - \boldsymbol{A}$ has (I_0, Γ). If $\boldsymbol{A}$ has (I_0, Γ), then $\boldsymbol{x}^{[k]} \in \boldsymbol{P} - \boldsymbol{A}$ implies $\boldsymbol{x} \in \boldsymbol{P} - \boldsymbol{A}$; thus $\boldsymbol{P} - \boldsymbol{A}$ has $(\mathrm{I}^\circ, \Gamma)$. Consequently, if $\boldsymbol{A}$ has $(\mathrm{I}_0^\circ, \Gamma)$, then $\boldsymbol{P} - \boldsymbol{A}$ has $(\mathrm{I}_0^\circ, \Gamma)$ (cf. (i)).

Now let $\boldsymbol{A}$ have the property (I, Γ). Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, $0 \leqslant k < 1 + \min(l(\boldsymbol{x}), l(\boldsymbol{y}))$. Let $\boldsymbol{x} \in \boldsymbol{P} - \boldsymbol{A}$, $(x_0, \ldots, x_k) = (y_0, \ldots, y_k)$, and let either $\boldsymbol{x}^{[k]} \notin \boldsymbol{P} - \boldsymbol{A}$ or $\boldsymbol{y}^{[k]} \in \boldsymbol{P} - \boldsymbol{A}$. If $\boldsymbol{y} \notin \boldsymbol{P} - \boldsymbol{A}$, i.e. $\boldsymbol{y} \in \boldsymbol{A}$, then this and the above suppositions imply:

$$\boldsymbol{y} \in \boldsymbol{A} \wedge (x_0, \ldots, x_k) = (y_0, \ldots, y_k) \wedge (\boldsymbol{y}^{[k]} \notin \boldsymbol{A} \vee \boldsymbol{x}^{[k]} \in \boldsymbol{A});$$

consequently (cf. 1.2 with $\boldsymbol{x} := \boldsymbol{y}$, $\boldsymbol{y} := \boldsymbol{x}$) $\boldsymbol{x} \in \boldsymbol{A}$, but this is a contradiction for we have supposed $\boldsymbol{x} \in \boldsymbol{P} - \boldsymbol{A}$. Therefore $\boldsymbol{y} \in \boldsymbol{P} - \boldsymbol{A}$. Hence $\boldsymbol{P} - \boldsymbol{A}$ has the property (I, Γ).

Thus, if $\boldsymbol{A}$ has a basic property (K, Γ), then $\boldsymbol{P} - \boldsymbol{A}$ has $(\overline{K}, \Gamma)$. Hence, if $\boldsymbol{P} - \boldsymbol{A}$ has the property $(\overline{K}, \Gamma)$, then $\boldsymbol{A} = \boldsymbol{P} - (\boldsymbol{P} - \boldsymbol{A})$ has $(\overline{\overline{K}}, \Gamma) = (K, \Gamma)$. Finally, $\boldsymbol{X} - \boldsymbol{A} = (\boldsymbol{P} - \boldsymbol{A}) \cap \boldsymbol{X}$, hence (see (o)) $\boldsymbol{P} - \boldsymbol{A}$ has $(\overline{K}, \Gamma)$ iff $\boldsymbol{X} - \boldsymbol{A}$ has $(\overline{K}, \Gamma)$.

(*iv*): This follows immediately from 1.1, 1.2 ($\boldsymbol{X} = \boldsymbol{X}_\Gamma \subset \boldsymbol{P} = \boldsymbol{X}_{\mathrm{L}}$).

(*v*): Evidently, $\emptyset$ has the property (I_0°), $\boldsymbol{X}$ has the properties (I_0) and $(\mathrm{I}_0^\circ, \Gamma)$. If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{P}$, $0 \leqslant k < 1 + \min(l(\boldsymbol{x}), l(\boldsymbol{y}))$, $\boldsymbol{x} \in \boldsymbol{X}$, then $\boldsymbol{x}^{[k]} \notin \boldsymbol{X}$ cannot occur, and if moreover $\boldsymbol{y}^{[k]} \in \boldsymbol{X}$, $(x_0, \ldots, x_k) = (y_0, \ldots, y_k)$, then, clearly, $\boldsymbol{y} \in \boldsymbol{X}$, too. Thus $\boldsymbol{X}$ has the property (I). The other assertions in (v) follow, e.g., from the above proved ones by means of (iii).

(*vi*/1): If $\boldsymbol{A} \cap \boldsymbol{X}$ has the property (I_0), then $\boldsymbol{A} \cap \boldsymbol{X}$ has (I_0, Γ) (see (iv) with $\boldsymbol{A} := \boldsymbol{A} \cap \boldsymbol{X}$), hence (cf. (o)) $\boldsymbol{A}$ has (I_0, Γ). On the other hand, if $\boldsymbol{A}$ has (I_0, Γ), then $\boldsymbol{A} \cap \boldsymbol{X}$ has (I_0, Γ), hence if some $\boldsymbol{x} \in \boldsymbol{P}$ belongs to $\boldsymbol{A} \cap \boldsymbol{X}$ and if $0 \leqslant k < 1 + l(\boldsymbol{x})$, then ($\boldsymbol{x} \in \boldsymbol{X}$ and hence) $\boldsymbol{x}^{[k]} \in \boldsymbol{A} \cap \boldsymbol{X}$; thus $\boldsymbol{A} \cap \boldsymbol{X}$ has the property (I_0).

(*vi*/2): This follows from (vi/1) and (iii).

(*vii*): This assertion follows immediately from the definition 1.1.

(*viii*/1): Let $\mathfrak{A}$ have (I, Γ) and be $\cup$-directed. Let $\boldsymbol{B}$ be the union of $\mathfrak{A}$. Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, $0 \leqslant k < 1 + \min(l(\boldsymbol{x}), l(\boldsymbol{y}))$. Let $\boldsymbol{x} \in \boldsymbol{B}$, $(x_0, \ldots, x_k) = (y_0, \ldots, y_k)$ and either $\boldsymbol{x}^{[k]} \notin \boldsymbol{B}$ or $\boldsymbol{y}^{[k]} \in \boldsymbol{B}$. There exists $\boldsymbol{A}_1 \in \mathfrak{A}$ such that $\boldsymbol{x} \in \boldsymbol{A}_1$; if $\boldsymbol{x}^{[k]} \notin \boldsymbol{B}$, then $\boldsymbol{x}^{[k]} \notin \boldsymbol{A}_1$, hence $\boldsymbol{y} \in \boldsymbol{A}_1 \subset \boldsymbol{B}$ (since $\boldsymbol{A}_1$ has (I, Γ)), if $\boldsymbol{y}^{[k]} \in \boldsymbol{B}$, then $\boldsymbol{y}^{[k]} \in \boldsymbol{A}_2$ for some $\boldsymbol{A}_2 \in \mathfrak{A}$, but $\mathfrak{A}$ is $\cup$-directed, hence there exists $\boldsymbol{A} \in \mathfrak{A}$ such that $\boldsymbol{A}_1 \cup \boldsymbol{A}_2 \subset \boldsymbol{A}$, hence $\boldsymbol{x} \in \boldsymbol{A}$, $\boldsymbol{y}^{[k]} \in \boldsymbol{A}$, $\boldsymbol{y} \in \boldsymbol{A} \subset \boldsymbol{B}$. Thus always $\boldsymbol{y} \in \boldsymbol{B}$. Therefore, $\boldsymbol{B}$ has the property (I, Γ).

(*viii*/2): If $\mathfrak{A}$ has (I, Γ) and is $\cap$-directed, then $\mathfrak{B} := \{\boldsymbol{P} - \boldsymbol{A} \mid \boldsymbol{A} \in \mathfrak{A}\}$ has (I, Γ) (see (iii)) and, clearly, is $\cup$-directed. Hence the union $\boldsymbol{C}$ of $\mathfrak{B}$ has the property (I, Γ) (see (viii/1)), thus $\boldsymbol{P} - \boldsymbol{C}$ has the property (I, Γ) (again (iii)), but $\boldsymbol{P} - \boldsymbol{C}$ is the intersection of $\mathfrak{A}$.

Q. E. D.

2.2. Counter-Examples. Let $P := \{1, 2\}$, $P_0 := \emptyset$, let Γ be such that $\Gamma x = \{x\}$ for each $x \in P$. Hence $\boldsymbol{X} = \{(1, 1, 1, \ldots), (2, 2, 2, \ldots)\}$. Clearly, $\boldsymbol{X}$ has not (I°), consequently (2.2 (iii)) $\boldsymbol{P} - \boldsymbol{X}$ has not (I_0); cf. 2.2 (v). Nevertheless, $\boldsymbol{X}$ and $\boldsymbol{P} - \boldsymbol{X}$ have (I) (2.2 (v)); cf. 2.2 (ii).

Let $\boldsymbol{A} := \{(1, 1, 1, \ldots), (1, 2, 2, 2, \ldots)\}$. Then $\boldsymbol{A} \cap \boldsymbol{X} = \{(1, 1, 1, \ldots)\}$, hence $\boldsymbol{A}$ has the property (I_0°, Γ) and thus also the other three basic Γ-properties (2.2 (o), (i), (ii)). On the other hand, $\boldsymbol{A}$ has no basic **L**-property (K) [in fact: if $\boldsymbol{x} := (1, 2, 2, 2, \ldots)$, $\boldsymbol{y} := (1, 2, 1, 1, 1, \ldots)$, then $\boldsymbol{x} \in \boldsymbol{A}$, $(x_0, x_1) = (y_0, y_1)$, $\boldsymbol{x}^{[1]} \notin \boldsymbol{A}$, but $\boldsymbol{y} \notin \boldsymbol{A}$, hence $\boldsymbol{A}$ has not (I); clearly, $\boldsymbol{A}$ has not any other basic **L**-property]; cf. 2.2 (iv).

Let $\boldsymbol{A}_1 := \{(1, 2, 1, 2, \ldots), (2, 1, 2, 1, \ldots)\}$, $\boldsymbol{A}_2 := \{(1, 1, 1, \ldots)\}$, $\boldsymbol{B}_m := \boldsymbol{P} - \boldsymbol{A}_m$ ($m = 1, 2$). It can be simply verified that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have the property (I) (and (I_0), too), hence (2.2 (iii)) $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ have (I), $\boldsymbol{A}_1 \cup \boldsymbol{A}_2$ has not (I) [in fact, if $\boldsymbol{x} := (1,1,1,\ldots)$, $\boldsymbol{y} := (1, 1, 2, 1, 2, \ldots)$, then $\boldsymbol{x} \in \boldsymbol{A}_1 \cup \boldsymbol{A}_2$, $(x_0, x_1) = (y_0, y_1)$, $\boldsymbol{y}^{[1]} \in \boldsymbol{A}_1 \cup \boldsymbol{A}_2$, but $\boldsymbol{y} \notin \boldsymbol{A}_1 \cup \boldsymbol{A}_2$], hence $\boldsymbol{B}_1 \cap \boldsymbol{B}_2$ ($= \boldsymbol{P} - (\boldsymbol{A}_1 \cup \boldsymbol{A}_2)$; 2.2 (iii)) has not (I); cf. 2.2 (vii) — (viii).

b) The basic aims, the r-aims, and their properties

3.1. Convention. If some $\boldsymbol{x} = (x_k)_{0 \leqslant k < 1 + l(\boldsymbol{x})} \in \boldsymbol{P}$ is considered, then we shall use also the denotation $\boldsymbol{x} = (x_k)$, and, moreover, if $l(\boldsymbol{x}) < k_0 < \omega_0$, then we introduce formally $x_{k_0} := x_{l(\boldsymbol{x})}$ and $\boldsymbol{x}^{[k_0]} := (x_{l(\boldsymbol{x})})$. Thus, if $\boldsymbol{x} = (x_k) \in \boldsymbol{P}$, then we have defined x_n for *each* integer $n \geqslant 0$.

3.2. Convention. In the following $\bigwedge$ ($\bigvee$) means the *universal* (*existential*) *logical quantifier applied to non-negative integers*. E.g., $\bigwedge_{m>k} P(m, k)$ means: "for all integers m being greater than (fixed non-negative integer) k $P(m, k)$ is valid", while e.g. $\bigwedge_{k<m} P(m, k)$ means: "for all non-negative integers k being lesser than (fixed non-negative integer) m $P(m, k)$ is valid". Similarly, the short denotations as $\bigcup_k$, $\bigcap_{k>n}$ etc. will concern non-negative integers.

$\neg$ will be used as the symbol of *negation*.

4. Definition, remarks. For $A, B \subset P$ we put

$$\mathbf{L}_A^B := (B \times A, P, P),$$

i.e., $\mathbf{L}_A^B$ ($\in \operatorname{Corr}(P, P)$) is a graph (§ 2.18.1) such that

$$\mathbf{L}_A^B x = \begin{cases} B \\ \emptyset \end{cases} \quad \text{if} \quad x \in \begin{cases} A \\ P - A \end{cases}.$$

Further we introduce

$$\mathbf{L}_A := \mathbf{L}_A^P, \qquad \mathbf{L}^B := \mathbf{L}_P^B.$$

Evidently, there holds:

(0) $$\mathbf{L} = \mathbf{L}_Z^P,$$

(1) $$\mathbf{L}_P^P = \mathbf{L}_P = \mathbf{L}^P,$$

(2) $$\mathbf{L}_\emptyset^\emptyset = \mathbf{L}_\emptyset = \mathbf{L}^\emptyset,$$

(3) $$\mathbf{L}_A^B = \mathbf{L}_A \cap \mathbf{L}^B$$

(4) $$\mathbf{L}_{P-A}^{P-B} = (\mathbf{L}_P - \mathbf{L}_A) \cap (\mathbf{L}^P - \mathbf{L}^B),$$

(4′) $$\mathbf{L}_{P-A} = \mathbf{L}_P - \mathbf{L}_A, \; \mathbf{L}^{P-B} = \mathbf{L}^P - \mathbf{L}^B,$$

(the operations $\cap$, $-$ on the right sides of (3), (4), (4′) are used in the sense introduced in § 2.5.2).

5. Definition, remark. Under a ***parameter*** we shall mean an ordered pair $\gamma = (\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}}) \in \operatorname{Corr}(P, P) \times \operatorname{Corr}(P, P)$; $\bar{\gamma} := (\mathbf{L}_P - \Gamma^{\mathrm{I}}, \mathbf{L}_P - \Gamma^{\mathrm{II}})$ is said to be the *complementary parameter* (to γ). Evidently then

(5) $$\bar{\bar{\gamma}} = \gamma.$$

6. Definition. (The ***basic aims.***) Let $\gamma = (\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}})$ be a parameter. For $\varepsilon \in \{\triangle, \square, 0, 1, 2, \ldots\}$ we shall introduce the aims $\boldsymbol{q}^\varepsilon(\gamma) = \boldsymbol{q}^\varepsilon(\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}})$, $\boldsymbol{q}_\varepsilon(\gamma) = \boldsymbol{q}_\varepsilon(\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}})$:

$$\boldsymbol{q}^k(\gamma) := \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{P}, \quad x_{k+1}\Gamma^{\mathrm{I}}x_k \wedge \bigwedge_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m\},$$

$$\boldsymbol{q}_k(\gamma) := \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{P}, \quad x_{k+1}\Gamma^{\mathrm{I}}x_k \vee \bigvee_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m\},$$

$$\boldsymbol{q}^\triangle(\gamma) := \bigcup_k \boldsymbol{q}^k(\gamma),$$

$$\boldsymbol{q}_\triangle(\gamma) := \bigcap_k \boldsymbol{q}_k(\gamma),$$

$$\boldsymbol{q}^\square(\gamma) := \bigcap_n \bigcup_{k>n} \boldsymbol{q}^k(\gamma),$$

$$\boldsymbol{q}_\square(\gamma) := \bigcup_n \bigcap_{k>n} \boldsymbol{q}_k(\gamma),$$

$$\mathfrak{M} := \{\boldsymbol{q}^\varepsilon(\gamma) \mid \gamma \in \operatorname{Corr}(P, P) \times \operatorname{Corr}(P, P), {}^\varepsilon \in \{{}^\triangle, {}_\triangle, {}^\square, {}_\square\}\},$$

(where, and also in the following, the ε-symbolism is used in the same manner as in § 5.8). Elements of $\mathfrak{M}$ will be called the ***basic aims.***

7. Theorem. *Let γ be a parameter, $\varepsilon \in \{{}^\triangle, {}^\square, {}^0, {}^1, \ldots, {}_\triangle, {}_\square, {}_0, {}_1, \ldots\}$.*
Then

(i) $$\boldsymbol{q}^\varepsilon(\gamma) = \boldsymbol{P} - \boldsymbol{q}_\varepsilon(\bar{\gamma}).$$

Further (for any aim $\boldsymbol{A}$)

(ii) $$\boldsymbol{A} \in \mathfrak{M} \Leftrightarrow \boldsymbol{P} - \boldsymbol{A} \in \mathfrak{M}.$$

Proof. For any $x, y \in P$ and an arbitrary $\Gamma^\circ \in \mathrm{Corr}(P, P)$ there holds: $\neg y\Gamma^\circ x$ iff $y(\mathrm{L}_P - \Gamma^\circ)x$. This trivial remark shows that (i) is true with $^\varepsilon := {}^k$ and $^\varepsilon := {}_k$ ($k = 0, 1, 2, \ldots$). Now the verity of (i) with the other $^\varepsilon$ follows immediately from definition 5. (i) and the corresponding definitions 5, 6 imply that (ii) holds. Q. E. D.

8. Theorem. *For every parameter* $\gamma = (\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}})$ *there holds:*

(i/1) $$\boldsymbol{q}_\triangle(\gamma) = \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{P}, (\bigwedge_k x_{k+1}\Gamma^{\mathrm{I}}x_k) \vee \bigvee_k (x_{k+1}\Gamma^{\mathrm{II}}x_k \wedge \bigwedge_{m<k} x_{m+1}\Gamma^{\mathrm{I}}x_m)\},$$

(i/2) $$\boldsymbol{q}^\triangle(\gamma) = \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{P}, (\bigvee_k x_{k+1}\Gamma^{\mathrm{I}}x_k) \wedge \bigwedge_k (x_{k+1}\Gamma^{\mathrm{II}}x_k \vee \bigvee_{m<k} x_{m+1}\Gamma^{\mathrm{I}}x_m)\},$$

((ii/1) $$\boldsymbol{q}^\square(\gamma) = \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{P}, (\bigwedge_k x_{k+1}\Gamma^{\mathrm{II}}x_k) \wedge \bigwedge_n \bigvee_{k>n} x_{k+1}\Gamma^{\mathrm{I}}x_k\},$$

(ii/2) $$\boldsymbol{q}_\square(\gamma) = \{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{P}, (\bigvee_k x_{k+1}\Gamma^{\mathrm{II}}x_k) \vee \bigvee_n \bigwedge_{k>n} x_{k+1}\Gamma^{\mathrm{I}}x_k\}.$$

Proof. It will be sufficient to prove only e.g. (i/1), (ii/1) (cf. the preceding proof and 7 (i)). Let $\boldsymbol{x} = (x_k) \in \boldsymbol{P}$ be fixed. We introduce (at $\boldsymbol{x}$) these four statements:

$$\mathrm{V}_1 := \bigwedge_k (x_{k+1}\Gamma^{\mathrm{I}}x_k \vee \bigvee_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m),$$

$$\mathrm{V}_2 := (\bigwedge_k x_{k+1}\Gamma^{\mathrm{I}}x_k) \vee \bigvee_k (x_{k+1}\Gamma^{\mathrm{II}}x_k \wedge \bigwedge_{m<k} x_{m+1}\Gamma^{\mathrm{I}}x_m),$$

$$\mathrm{V}_3 := \bigwedge_n \bigvee_{k>n} (x_{k+1}\Gamma^{\mathrm{I}}x_k \wedge \bigwedge_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m),$$

$$\mathrm{V}_4 := (\bigwedge_k x_{k+1}\Gamma^{\mathrm{II}}x_k) \wedge \bigwedge_n \bigvee_{k>n} x_{k+1}\Gamma^{\mathrm{I}}x_k.$$

It is evident that if (for each $\boldsymbol{x}$) there holds $\mathrm{V}_1 \Leftrightarrow \mathrm{V}_2$ [$\mathrm{V}_3 \Leftrightarrow \mathrm{V}_4$], then (i/1) [(ii/1)] is valid.

1. If $x_{r+1}\Gamma^{\mathrm{II}}x_r \wedge \bigwedge_{m<r} x_{m+1}\Gamma^{\mathrm{I}}x_m$ for some r, then $(\bigwedge_{k>r} \bigvee_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m) \wedge \bigwedge_{k<r} x_{k+1}\Gamma^{\mathrm{I}}x_k$, hence $\bigwedge_k (x_{k+1}\Gamma^{\mathrm{I}}x_k \vee \bigvee_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m)$. The latter statement also holds if $\bigwedge_k x_{k+1}\Gamma^{\mathrm{I}}x_k$. Thus V_2 implies V_1. On the other hand, if V_1 holds, then either $\neg \bigvee_k x_{k+1}\Gamma^{\mathrm{II}}x_k$, then $\bigwedge_k \neg \bigvee_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m$, hence $\bigwedge_k x_{k+1}\Gamma^{\mathrm{I}}x_k$ which implies V_2, or $\bigvee_k x_{k+1}\Gamma^{\mathrm{II}}x_k$, then there exists the integer $r := \min\{k \mid x_{k+1}\Gamma^{\mathrm{II}}x_k\}$, hence $\bigwedge_{k<r} \neg x_{k+1}\Gamma^{\mathrm{II}}x_k$, consequently $\bigwedge_{k<r} \neg \bigvee_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m$, therefore (cf. V_1) $\bigwedge_{k<r} x_{k+1}\Gamma^{\mathrm{I}}x_k$, thus $x_{r+1}\Gamma^{\mathrm{II}}x_r \wedge \bigvee_{m<r} x_{m+1}\Gamma^{\mathrm{I}}x_m$, hence again V_2 holds. Thus we have proved $\mathrm{V}_1 \Leftrightarrow \mathrm{V}_2$.

2. If V_3 is valid, then $\bigwedge_n \bigvee_{k>n} x_{k+1}\Gamma^{\mathrm{I}}x_k$ and $\bigwedge_n \bigvee_{k>n} \bigwedge_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m$, hence $\bigwedge_n x_{n+1}\Gamma^{\mathrm{II}}x_n$. Thus V_4 holds. On the other hand, if V_4 holds, then $\bigwedge_n \bigvee_{k>n} (x_{k+1}\Gamma^{\mathrm{I}}x_k \wedge \bigwedge_m x_{m+1}\Gamma^{\mathrm{II}}x_m)$, consequently $\bigwedge_n \bigvee_{k>n} (x_{k+1}\Gamma^{\mathrm{I}}x_k \wedge \bigwedge_{m<k} x_{m+1}\Gamma^{\mathrm{II}}x_m)$, hence V_3 is valid. Thus $\mathrm{V}_3 \Leftrightarrow \mathrm{V}_4$ has been proved.

Q. E. D.

9. Remarks. Let q^{ε} (where $^{\varepsilon}$ is some of those occurring in the head of 6) be that mapping of Corr $P \times$ Corr P (= the set of all parameters) into exp $\mathbf{P}$ which maps any parameter γ to $q^{\varepsilon}(\gamma)$. Thus $q^{\varepsilon} \in (\exp \mathbf{P})^{\mathrm{Corr}\, P \times \mathrm{Corr}\, P}$; q^{ε} is a certain aim-function (§ 5.8). Let $E :=$ Corr $P \times$ Corr P, $F := \mathbf{P}$ in the situation considered in § 2.4; then we may introduce the induced operations $\bigcup$, $\bigcap$, $-$ onto $(\exp F)^E = (\exp \mathbf{P})^{\mathrm{Corr}\, P \times \mathrm{Corr}\, P}$ by § 2.5.2. It is easy to see that then

$$(6) \qquad q^{\triangle} = \bigcup_k q^k, \qquad q_{\triangle} = \bigcap_k q_k,$$

$$(7) \qquad q^{\square} = \limsup_k q^k, \quad q_{\square} = \liminf_k q_k,$$

where $\limsup_k$ [$\liminf_k$] means, of course, $\bigcap_n \bigcup_{k \geqslant n}$ [$\bigcup_n \bigcap_{k \geqslant n}$].

10. Theorem. *Let γ be a parameter. Then*

$q^{\square}(\gamma)$ has the property (I_0),

$q_{\square}(\gamma)$ has the property (I^0).

Proof. These assertions follow immediately from theorem 8, (ii/1-2). Q.E.D.

11. Theorem. *Each basic aim has the property* (I).

Proof. Let $\gamma = (\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}})$ be a parameter. Let $\mathbf{x}, \mathbf{y} \in \mathbf{P}$, $0 \leqslant m < 1 + \min(l(\mathbf{x}), l(\mathbf{y}))$, $(x_0, \ldots, x_m) = (y_0, \ldots, y_m)$, let $\mathbf{x} \in \mathbf{A} := q^{\varepsilon}(\gamma)$, where $^{\varepsilon}$ will be chosen in the following.

1. Let $^{\varepsilon} := {}^{\triangle}$. Then (cf. 6) there exists k_1 such that $x_{k_1+1}\, \Gamma^{\mathrm{I}}\, x_{k_1} \wedge \bigwedge_{k<k_1} x_{k+1} \Gamma^{\mathrm{II}} x_k$. If $k_1 < m$, then $y_{k_1+1} \Gamma^{\mathrm{I}} y_{k_1} \wedge \bigwedge_{k<k_1} y_{k+1} \Gamma^{\mathrm{II}} y_k$ (since $\bigwedge_{k \leqslant m} y_k = x_k$ and $k_1 + 1 \leqslant m$), hence $\mathbf{y} \in \mathbf{A}$. If $k_1 \geqslant m$, then $x_{k_1+1} \Gamma^{\mathrm{I}} x_{k_1} \wedge \bigwedge_{m \leq k < k_1} x_{k+1} \Gamma^{\mathrm{II}} x_k$, hence $\mathbf{x}^{[m]} \in \mathbf{A}$; if, moreover, $\mathbf{y}^{[m]} \in \mathbf{A}$, then there exists $k_2 \geqslant m$ such that $y_{k_2+1} \Gamma^{\mathrm{I}} y_{k_2} \wedge \bigwedge_{m \leqslant k < k_2} y_{k+1} \Gamma^{\mathrm{II}} y_k$, but $\bigwedge_{k<m} y_{k+1} \Gamma^{\mathrm{II}} y_k$ ($\bigwedge_{k \leqslant m} y_k = x_k$ etc.), hence $y_{k_2+1} \Gamma^{\mathrm{I}} y_{k_2} \wedge \bigwedge_{k<k_2} y_{k+1} \Gamma^{\mathrm{II}} y_k$, thus $\mathbf{y} \in \mathbf{A}$. Therefore, $\mathbf{A}$ has the property (I).

2. Let $^{\varepsilon} := {}^{\square}$. Then $\mathbf{x}^{[m]} \in \mathbf{A}$ (cf. 10) and, in particular, $\bigwedge_{k<m} x_{k+1} \Gamma^{\mathrm{II}} x_k$, $\bigwedge_{k<m} y_{k+1} \Gamma^{\mathrm{II}} y_k$ (8(ii/1)). If, moreover, $\mathbf{y}^{[m]} \in \mathbf{A}$, then (see again 8(ii/1)) $(\bigwedge_{k \geqslant m} y_{k+1} \Gamma^{\mathrm{II}} y_k) \wedge \bigwedge_{n \geqslant m} \bigvee_{k \geqslant n} y_{k+1} \Gamma^{\mathrm{I}} y_k$, but then (cf. above) $(\bigwedge_k y_{k+1} \Gamma^{\mathrm{II}} y_k) \wedge \bigwedge_n \bigvee_{k \geqslant n} y_{k+1} \Gamma^{\mathrm{I}} y_k$, hence $\mathbf{y} \in \mathbf{A}$. Therefore, $\mathbf{A}$ has the property (I).

3. If ε is either $\triangle$ or $\square$, then $q^{\varepsilon}(\bar{\gamma})$ has the property (I) (see 1, 2), hence $q_{\varepsilon}(\gamma) = \mathbf{P} - q^{\varepsilon}(\bar{\gamma})$ has the property (I) (7(i); 2.1(iii) with $\Gamma := \mathbf{L}$).

Q.E.D.

12.0. Remark. It is possible to derive several other properties of the basic aims by means of theorems 2.1, 10, 11. On the other hand, simple examples show that it may happen that e.g. $q^{\triangle}(\gamma)$ has neither $(\mathrm{I}^{\circ}, \Gamma)$ nor (I_0, Γ), $q^{\square}(\gamma)$ has not $(\mathrm{I}^{\circ}, \Gamma)$, etc. Nevertheless, there holds

12.1. Lemma. *Let $\gamma = (\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}})$ be a parameter, $\varepsilon \in \{\triangle, \square\}$. Then*

(i) $\Gamma \subset \Gamma^{\mathrm{II}} \Rightarrow q^{\varepsilon}(\gamma)$ *has the property* $(\mathrm{I}^{\circ}, \Gamma)$

(ii) $\mathbf{L}_P - \Gamma \supset \Gamma^{\mathrm{II}} \Rightarrow q_{\varepsilon}(\gamma)$ *has the property* (I_0, Γ)

Proof. (i) follows immediately from 6 ($\varepsilon := \triangle$), 8(ii/1) ($\varepsilon := \square$). If $\mathbf{L}_P - \Gamma \supset \Gamma^{\mathrm{II}}$, then $\Gamma \subset \mathbf{L}_P - \Gamma^{\mathrm{II}}$, hence $\boldsymbol{q}^{\varepsilon}(\mathbf{L}_P - \Gamma^{\mathrm{I}}, \mathbf{L}_P - \Gamma^{\mathrm{II}}) = \boldsymbol{q}^{\varepsilon}(\bar{\gamma})$ has the property (I^0, Γ), and $\boldsymbol{q}_{\varepsilon}(\gamma) = \boldsymbol{P} - \boldsymbol{q}^{\varepsilon}(\bar{\gamma})$ has the property (I_0, Γ) (12.1(i), 7(i), 2.1(iii)). Q.E.D.

12.2. Remarks. Clearly, $\mathbf{L}_P - \Gamma \supset \Gamma^{\mathrm{II}}$ is equivalent to the condition $\Gamma^{\mathrm{II}} \cap \Gamma = \mathbf{L}_{\emptyset}$. Thus 10 and 12.1 imply especially:

12.3. Corollary. *For any parameter* $\gamma = (\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}})$ *there holds:*

(i) $\Gamma \subset \Gamma^{\mathrm{II}} \Rightarrow \boldsymbol{q}^{\square}(\gamma)$ has the property $(\mathrm{I}_0^{\circ}, \Gamma)$,

(ii) $\Gamma^{\mathrm{II}} \cap \Gamma = \mathbf{L}_{\emptyset} \Rightarrow \boldsymbol{q}_{\square}(\gamma)$ has the property $(\mathrm{I}_0^{\circ}, \Gamma)$.

13.1. Definition, remarks. Under an ***r-parameter*** we shall mean a pair $\varkappa = (\mathfrak{K}, \Gamma^{\circ})$ where $\mathfrak{K}$ is a countable (i.e. card $\mathfrak{K} \leqslant \aleph_0$) collection in P and $\Gamma^{\circ} \in \mathrm{Corr}\,(P, P)$. To such an r-parameter $\varkappa$ we define the complementary one

$$\bar{\varkappa} := (\{P - K \mid K \in \mathfrak{K}\}, \mathbf{L}_P - \Gamma^{\circ}),$$

(i.e. $\bar{\varkappa} = (\exp P - \bar{\mathfrak{K}}, \mathbf{L}_P - \Gamma^{\circ})$, where $^-$ is the operation introduced in § 4.2 ($Q := P$)); evidently

(8) $$\bar{\bar{\varkappa}} = \varkappa.$$

We say that an r-parameter $\varkappa = (\mathfrak{K}, \Gamma^{\circ})$ is a $\cup$*-parameter* [an $\cap$*-parameter*] iff $\mathfrak{K}$ is $\cup$-directed [$\cap$-directed]. Evidently

(9) $$\varkappa \text{ is a } \cup\text{-parameter} \Leftrightarrow \bar{\varkappa} \text{ is an } \cap\text{-parameter}$$

13.2. Definition, remarks.

(i) Let $\varkappa = (\mathfrak{K}, \Gamma_0)$ be a $\cup$-parameter. Then we put

$$\boldsymbol{r}_{\square}(\varkappa) := \boldsymbol{r}_{\square}(\mathfrak{K}, \Gamma^{\circ}) := \bigcup_{K \in \mathfrak{K}} \boldsymbol{q}_{\triangle}(\mathbf{L}_K, \Gamma^{\circ}).$$

(ii) Let $\varkappa = (\mathfrak{K}, \Gamma^{\circ})$ be an $\cap$-parameter. Then we put

$$\boldsymbol{r}^{\square}(\varkappa) := \boldsymbol{r}^{\square}(\mathfrak{K}, \Gamma^{\circ}) := \bigcap_{K \in \mathfrak{K}} \boldsymbol{q}^{\triangle}(\mathbf{L}_K, \Gamma^{\circ}).$$

(where the intersection is defined to be equal to $\boldsymbol{P}$ if $\mathfrak{K} = \emptyset$). By means of (4′) and 7(i) we conclude

(10/1) $$\boldsymbol{r}^{\square}(\varkappa) = \boldsymbol{P} - \boldsymbol{r}_{\square}(\bar{\varkappa}) \text{ for each } \cap\text{-parameter } \varkappa,$$

(10/2) $$\boldsymbol{r}^{\square}(\varkappa) = \boldsymbol{P} - \boldsymbol{r}^{\square}(\bar{\varkappa}) \text{ for each } \cup\text{-parameter } \varkappa.$$

Elements of $\{\boldsymbol{r}_{\square}(\varkappa) \mid \varkappa \text{ is a } \cup\text{-parameter}\}$ [$\{\boldsymbol{r}^{\square}(\varkappa) \mid \varkappa \text{ is an } \cap\text{-parameter}\}$] will be called $\cup$*-aims* [$\cap$*-aims*]. $\cup$-aims and $\cap$-aims will be also called ***r-aims.*** From (10) there follows

(11) $$\boldsymbol{A} \text{ is a } \cup\text{-aim} \Leftrightarrow \boldsymbol{P} - \boldsymbol{A} \text{ is an } \cap\text{-aim}$$

(11′) $$\boldsymbol{A} \text{ is an r-aim} \Leftrightarrow \boldsymbol{P} - \boldsymbol{A} \text{ is an r-aim}$$

13.3. Lemma. *Let* $\varkappa_{\iota} = (\mathfrak{K}_{\iota}, \Gamma_{\iota})$ ($\iota = 1, 2$) *be complementary* r*-parameters* (i.e. $\varkappa_2 = \overline{\varkappa_1}$), *let* $\varkappa_1' := (\mathfrak{K}_1 \cup \{\emptyset\}, \Gamma_1)$, $\varkappa_2' := (\mathfrak{K}_2 \cup \{P\}, \Gamma_2)$. *Then* $\varkappa_1', \varkappa_2'$ *are complementary parameters, and the following four statements are mutually equivalent:* $\varkappa_1$ *is a* $\cup$*-parameter;* $\varkappa_1'$ *is a* $\cup$*-parameter;* $\varkappa_2$ *is an* $\cap$*-parameter;* $\varkappa_2'$ *is an* $\cap$*-parameter. Further, if*

some (and, consequently, each) of these four statements is valid, then $\mathbf{r}_{\square}(\varkappa_1) = \mathbf{r}_{\square}(\varkappa_1')$, $\mathbf{r}^{\square}(\varkappa_2) = \mathbf{r}^{\square}(\varkappa_2')$.
(This lemma follows immediately from the above definitions and remarks).

14.1. Lemma. *Let* $C \subset A$, $D \subset B$. *Then* $\mathbf{L}_C^D \subset \mathbf{L}_A^B$.

Proof. This follows immediately from § 6.4 (as $D \times C \subset B \times A$). Q.E.D.

14.2. Lemma. *Let* $\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}}, \Gamma^{\mathrm{III}}, \Gamma^{\mathrm{IV}} \in \mathrm{Corr}\,(P, P)$, *let* ε *be some of those occurring in the head of* 6. *If* $\Gamma^{\mathrm{III}} \subset \Gamma^{\mathrm{I}}$, $\Gamma^{\mathrm{IV}} \subset \Gamma^{\mathrm{II}}$, *then* $\mathbf{q}^{\varepsilon}(\Gamma^{\mathrm{III}}, \Gamma^{\mathrm{IV}}) \subset \mathbf{q}^{\varepsilon}(\Gamma^{\mathrm{I}}, \Gamma^{\mathrm{II}})$.

Proof. In fact, $y\Gamma^{\mathrm{III}}x \Rightarrow y\Gamma^{\mathrm{I}}x$, $y\Gamma^{\mathrm{IV}}x \Rightarrow y\Gamma^{\mathrm{II}}x$ for any $x, y \in P$, hence the assertion follows immediately from the definition 6. Q.E.D.

14.3. Remark. *If* $\varkappa = (\mathfrak{K}, \Gamma^{\circ})$ *is an* $\cap$*-parameter and* $\varkappa' := ([\mathfrak{K}]_P, \Gamma^{\circ})$, *then* $\mathbf{r}_{\square}(\varkappa) = \mathbf{r}_{\square}(\varkappa')$ (namely, "$\supset$" follows from $\mathfrak{K} \subset [\mathfrak{K}]_P$, while "$\subset$" can be derived by means of 14.1-2, cf. 13.2 (ii) and § 2.12).

15. Lemma. *Each r-aim has the property* (I).

Proof. Let $\mathbf{A}$ be a $\cup$-aim, i.e. $\mathbf{A} = \mathbf{r}_{\square}(\varkappa)$ for some $\cup$-parameter $\varkappa = (\mathfrak{K}, \Gamma^{\circ})$. $\mathfrak{K}$ is $\cup$-directed, hence $\{\mathbf{q}_{\triangle}(\mathbf{L}_K, \Gamma^{\circ}) \mid K \in \mathfrak{K}\}$ is $\cup$-directed (14.1, 14.2), each $\mathbf{q}_{\triangle}(\mathbf{L}_K, \Gamma^{\circ})$ has the property (I) (theorem 11), hence $\mathbf{r}_{\square}(\varkappa)$ has the property (I) (13.2(i), 2.1(viii/1) with $\Gamma := \mathbf{L}$). This, (11) and 2.1(iii) imply that also each $\cap$-aim has the property (I). Q.E.D.

c) The main min-max results

Meta-remarks. We shall apply the idea of active and passive aims (§ 5c) in the following three cases (§§ 6c/1, 2, 3). The corresponding considerations would comprehend a certain sizable common part; we do not wish to present it three times, therefore we shall use an advantageous (though somewhat unusual) way of exposition (such a way was used in [4], too): we shall present not the actual three texts (corresponding to those cases), but a certain *text schema* (containing, in fact that common part) and further three *lists of text substitutions*. The exact wordings of actual texts are to be obtained from the text schema by replacing all occurrences of the variables α and αth by their particular values (cf. the formative rules) and by the replacement of all text variables by the corresponding particular texts from the αth list.

Thus, we may say that the *proper text* of this paper consists, on the one hand, of the explicitely presented passages, and, on the other hand, of those (not explicitely presented!) passages which are to be formed in the above mentioned way, while the *forming means* (i.e. the text schema and the lists of text substitutions) and also the *meta-text* (i.e. these meta-remarks on and the rules of that formation) do not belong to the proper text and are designated in another manner than its parts. (The prefix "meta" is used only in the above mentioned connections in this paper.)

THE FORMATIVE RULES. α will denote the *serial variable* ($\alpha = 1, 2, 3$),

α th := $\begin{cases}\text{first} \\ \text{second for } \alpha = \\ \text{third}\end{cases}$ $\begin{cases}1 \\ 2. \\ 3\end{cases}$ The proper text of § 6c/α is to be obtained by the replacement of each α and αth by its actual value and by the replacement of each *text variable* (there are nine text variables) $\langle \Xi \rangle$ (where Ξ is some auxiliary designation, "*identifier*") by the particular text presented in the αth list beyond the corresponding expression "$\langle \Xi \rangle :=$ ".

THE TEXT SCHEMA

c/α) The αth main min-max results

16/α. Suppositions, definition, remarks. In § 6c/α let $\{j, i\} = \{1, 2\}$ (j and i are fixed), let $\mathscr{U} = (u_1, u_2)$ be a regular weakly complete pair of elements of Corr (P, exp P), let (P, P_0) be the type of $\mathscr{U}$. We put $v_\iota := [u_\iota]_P(\iota = 1, 2)$ (the "player ι's" game correspondence), thus $v_{3-\iota} = v_\iota'$ (cf. § 4.16). ⟨***the choice of*** $\mathbf{A}_1, \mathbf{A}_2$⟩

Let $\sim_\iota := \overset{\circ}{\sim}$ (§ 1.2) (i.e., roughly speaking, the *active player* uses only his plain strategies; cf. § 5.11). ⟨***the choice of*** $\sim_j$⟩

17.0/α. Definition, remark. ⟨***the introduction of*** w_1, w_2⟩

17/1/α. Lemma. *There holds:*

(*i*) $w_j = \overline{w_\iota}$,

(*ii*) w_1 *and* w_2 *are Mcorrespondences.*

Proof. ⟨***the proof of*** 17.1/α⟩ Q.E.D.

17.2/α. Definition. We define $w_\iota^* \in \mathrm{Corr}\,(P, \exp P)$ $(\iota = 1, 2)$ by

$$w_j^* := \mathbf{1} \cap w_j, \qquad w_\iota^* := \mathbf{1} \cup w_\iota,$$

(where $\mathbf{1}$ is defined by § 2.18.1).

17.3/α. Lemma. *There holds:*

(*i*) $$w_j^* = \overline{w_\iota^*},$$

(*ii*) w_1^* *and* w_2^* *are* M*-correspondences.*

Proof. In fact, $\overline{w_\iota^*} = \overline{\mathbf{1} \cup w_\iota} = \overline{\mathbf{1}} \cap \overline{w_\iota} = \mathbf{1} \cap w_j = w_j^*$ (17.2/α, 17.1/α (i), § 4(24), 47)). (ii) follows from 17.1/α (ii) (cf., e.g., § 2.12, § 2 (30)). Q.E.D.

18.0/α. Definition. We put

$$\mathfrak{A}_j := \{X \mid X \subset P \;\; X \subset w_j X\},$$

$$\mathfrak{A}_\iota := \{X \mid X \subset P,\, X \supset w_\iota X\}.$$

18.1/α. Remark. Clearly, for $\iota = 1, 2$

$$\mathfrak{A}_\iota = \{X \mid X \subset P,\, X = w_\iota^* X\},$$

i.e., $\mathfrak{A}_\iota$ is the set of all fixpoints of the M-correspondence w_ι^*.

18.2/α. Remark. w_ι and w_ι^* $(\iota = 1, 2)$ are M-correspondences, hence they have fixpoints (§ 5.15.0); moreover $\bigcap\limits_{\substack{X \subset P \\ X \supset w_\iota X}} X$ and $\bigcap\limits_{\substack{X \subset P \\ X \supset w_\iota^* X}} X$ are the smallest fixpoints of w_ι and w_ι^*, respectively (§ 5.15.0*)); but $X \supset w_\iota X$ iff $X \supset w_\iota^* X$, therefore w_ι and w_ι^* have the same smallest fixpoint. Analogously, w_j and w_j^* have the same greatest fixpoint $\bigcup\limits_{\substack{X \subset P \\ X \subset w_j X}} X = \bigcup\limits_{\substack{X \subset P \\ X \subset w_j^* X}} X$. (Of course, the equations between those fixpoints follow from § 5 (13′), (14), too, but the latter statements were derived by means of the transfinite iterations ("successive approximations"), while we shall need to have derived those equations without using the notion of transfinite iteration, cf. remark 19/α.)

19/α. Remark. 16/α and 18/α have introduced a particular case of the situation considered in § 5c. To this particular case there will be related the conditions (C) (P*), (A*) defined in § 5.11. $\mathfrak{A}_j$ was chosen in such a way that it is simple to verify (P*) (see 21/α). $\mathfrak{A}_\iota$ is uniquely determined by $\mathfrak{A}_j$ if (C) holds (then $\mathfrak{A}_\iota = \exp P - \overline{\mathfrak{A}_j}$,

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*) where the words "smallest" and "greatest" are to be replaced mutually.

see § 5.11), but we have also $\mathfrak{A}_i$ explicitely defined (while (C) will be proved in 20/α) as the set of fixpoints of a certain M-correspodence, which is important especially at the following application of the method of "successive approximations". The condition (A*), which here consists in finding $A \in \mathfrak{A}_i$ and $\sigma_i \in \mathring{S}(u_i)$ such that $s(x, \sigma_i) \subset \subset \mathbf{A}_i$ for each $x \in A$, will be proved twice, namely by means of the mutually independent uses of both the methods mentioned in § 5.12: the "(I)-property method" (22/α) and the "successive approximations method" (23/α); the auxiliary results and considerations obtained at applying these two methods are important, too.

20/α. Lemma. *The condition* (C) *is satisfied.*

Proof. In fact, for each $X \subset P$ $P - w_j^* X = P - \overline{w_i^*} X = w_i^*(P - X)$ (17.3/α(i), § 4(30)), therefore $X \in \mathfrak{A}_j \Leftrightarrow X = w_j^* X \Leftrightarrow P - X = w_i^*(P - X) \Leftrightarrow P - X \in \mathfrak{A}_i$ (18.1/α). Q.E.D.

21/α. Lemma. *The condition* (P*) *is satisfied.*

Proof. ⟨***the proof of the satisfaction of*** (P*)⟩ Q.E.D.

22/α. THE "(I)-PROPERTY METHOD".

22.1/α. Lemma. $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *have the property* (I).

Proof. This follows immediately from 16/α and ⟨***the property*** (I)⟩. Q.E.D.

22.2/α. Corollaries. $\mathbf{A}_i$ has the property (I) (22.1/α), hence it has the property (I, Γ) (theorem 2.1 (iv)) where Γ may be the graph of u_i. Therefore, a plainly $\{\mathbf{A}_i\}$-absolute u_i-strategy exists (§ 3.11), i.e., there is $\sigma_i \in \mathring{S}(u_i)$ such that $s(x, \sigma_i) \subset \mathbf{A}_i$ for each $x \in \mathring{u}_i\mathbf{A}_i$ (§ 3.5). Consequently, $\mathring{u}_i\mathbf{A}_i = \{x \mid x \in P,\ s(x, \sigma_i) \subset \mathbf{A}_i\}$ (cf. § 3.2); if $\sigma^* \in \mathring{S}(u_i)$ and $\sigma \mid Z \cap \mathring{u}_i\mathbf{A}_i = \sigma^* \mid Z \cap \mathring{u}_i\mathbf{A}_i$, then $s(x, \sigma^*) \subset \mathbf{A}_i$ for each $x \in \mathring{u}_i\mathbf{A}_i$ (§ 3.10), i.e., then σ^* is a plainly $\{\mathbf{A}_i\}$-absolute u_i-strategy, too.

22.3/α. Lemma. $\mathring{u}_i\mathbf{A}_i \in \mathfrak{A}_i$.

Proof. Let $A := \mathring{u}_i\mathbf{A}_i$; it is sufficient to prove $w_i A \subset A$ (18.0/α). Let $\sigma_i \in \mathring{S}(u_i)$ be such that $s(x, \sigma_i) \subset \mathbf{A}_i$ for each $x \in A$ (see 22.2/α). ⟨***the remainder of the proof of*** 22.3/α⟩ Q.E.D.

22.4/α. Corollary. (of 22.3-2/α; cf. 19/α). (A*) *is satisfied.*

23/α. THE "SUCCESSIVE APPROXIMATIONS METHOD".

23.1/α. Definition, remarks. In § 6.23/α we shall denote (for shortness)

$$w := w_i.$$

w is an M-correspondence (17.1/α (ii)), hence the sets $w^\infty \emptyset$, $w^\xi \emptyset$ (for any ordinal number ξ) are defined (see § 5.15.1); we denote

$$A^\infty := w^\infty \emptyset, \quad A^\xi := w^\xi \emptyset.$$

Thus, $A^\xi = \bigcup_{0 \leqq \eta < \xi} wA^\eta$ for every ξ (§ 5.15.1). Let $\xi_0 := \min\{\xi \mid \xi$ is an ordinal number $A^\xi = A^{\xi+1}\}$; indeed, ξ_0 exists (§ 5(10)), and

$$\emptyset = A^0 \subsetneqq A^1 \subsetneqq \ldots \subsetneqq A^{\xi_0} = A^{\xi_0+1} = \ldots = A^\infty,$$

(§ 5(1), (7)). Further, for each $x \in A^\infty$ we put

$$\eta(x) := \min\{\eta \mid \eta \text{ is an ordinal number, } x \in A^\eta\}.$$

There holds especially:

a) A^∞ is the smallest fixpoint of $w(=w_\iota)$ and also of w_ι^* (§ 5 (14), (13′)); consequently,

$$A^\infty \in \mathfrak{A}_\iota.$$

b) If $x \in A^\infty$, then $0 < \eta(x) \leqslant \xi_0$ (cf. above) and $\eta(x)$ is not limit (§ 5 (5); cf. above); consequently, $\eta(x) - 1$ exists, and (cf. § 5 (3))

$$x \in A^{\eta(x)} - A^{\eta(x)-1} = wA^{\eta(x)-1} - A^{\eta(x)-1}.$$

23.2/α. Remarks, definition. ⟨***the definition of*** σ_0⟩

23.3/α. Lemma. $\mathfrak{s}(x, \sigma_0) \subset \mathbf{A}_\iota$ *for each* $x \in A^\infty$ (*where* σ_0 *is the strategy introduced in* 23.2/α).

Proof. ⟨***the proof of* 23.3/α**⟩ Q.E.D.

23.4/α. Corollary (of 23.2/α, 23.1/α a); cf. 19/α). (A*) is satisfied.

24/α. Remark. Thus we have proved the satisfaction of (C), (P*), (A*) at the above introduced situation (cf. 19/α). Now the main result of those of § 6c/α, namely theorem 25/α, follows immediately from § 5.11 (excepting the statement (iv), which follows from (iv′) by means of 18.0-2/α).

25/α. THE αth MAIN MIN-MAX THEOREM. *Let*

$$E_\iota := \sim_\iota \mathbf{u}_\iota \mathbf{A}_\iota \quad \textit{for} \quad \iota = 1, 2$$

(16/α; *in particular,* $E_\iota = \mathring{\mathbf{u}}_\iota \mathbf{A}_\iota$). *Then there holds:*

(*i*) $$E_1 \cup E_2 = P,\ E_1 \cap E_2 = \emptyset.$$

(*ii*) $$E_\iota = \mathbf{u}_\iota \mathbf{A}_\iota.$$

(*iii*) *There exists a* $(\sim_\iota, \{\mathbf{A}_\iota\})$*-absolute* u_ι*-strategy.*

(*iv′*) $$\left.\begin{matrix} E_\iota \\ E_\jmath \end{matrix}\right\} \textit{ is the } \left\{\begin{matrix} \textit{smallest} \\ \textit{greatest} \end{matrix}\right. (\textit{under} \subset) \textit{ set of } \left\{\begin{matrix} \mathfrak{A}_\iota \\ \mathfrak{A}_\jmath \end{matrix}\right. .$$

(*iv*) $$\left.\begin{matrix} E_\iota \\ E_\jmath \end{matrix}\right\} \textit{ is the } \left\{\begin{matrix} \textit{smallest} \\ \textit{greatest} \end{matrix}\right. \textit{fixpoint of } \left\{\begin{matrix} w_\iota \\ w_\jmath \end{matrix}\right. \textit{ and also of } \left\{\begin{matrix} w_\iota^* \\ w_\jmath^* \end{matrix}\right. .$$

THE LISTS OF TEXT SUBSTITUTIONS

THE FIRST LIST OF TEXT SUBSTITUTIONS

(α := 1, αth := first)

⟨***the choice of*** $\mathbf{A}_1, \mathbf{A}_2$⟩ := Let $\gamma_\iota \doteq (\Gamma^{\mathrm{I}}_\iota, \Gamma^{\mathrm{II}}_\iota)$ $(\iota = 1, 2)$ be mutually complementary parameters (i.e. $\gamma_2 = \overline{\gamma_1}$), let

$$\mathbf{A}_\jmath := \mathbf{q}_\triangle(\gamma_\jmath),\ \mathbf{A}_\iota := \mathbf{q}^\triangle(\gamma_\iota);$$

consequently (cf. 7(i)), $\mathbf{A}_2 = \mathbf{P} - \mathbf{A}_1$.

⟨***the choice of*** $\sim_\jmath$⟩ := Let $\sim_\jmath := \mathring{\sim}$, i.e., the *passive player* uses only his plain strategies, too.

$\langle$***the introduction of*** $w_1, w_2\rangle :=$ For $\iota = 1, 2$ we put

$$P_0^\iota := \{x \mid x \in P_0 \cap \Gamma_\iota^{\mathrm{I}} x\};$$

evidently, $P_0^2 = P_0 - P_0^1$.

Let w_j, w_i be those elements of Corr (P, exp P) for which

$$xw_jD \Leftrightarrow x \in P_0^j \cup v_j(\Gamma_j^{\mathrm{I}}x \cap (D \cup \Gamma_j^{\mathrm{II}}x)),$$
$$xw_iC \Leftrightarrow x \in P_0^i \cup v_i(\Gamma_i^{\mathrm{I}}x \cup (C \cap \Gamma_i^{\mathrm{II}}x)),$$

for any $x \in P$, $D, C \subset P$.

$\langle$***the proof of*** **17.1**/$\alpha\rangle :=$ v_j and v_i are M-correspondences, which easily implies that w_j and w_i are M-correspondences, too (cf. § 2(13)). Thus (ii) holds. Let $x \in P$, $D \subset P$, $C := P - D$. Then (cf. 16/1, § 4(30), (31)) $x\overline{w_i}D \Leftrightarrow \neg\, xw_iC \Leftrightarrow$ $\Leftrightarrow (x \in P_0 \wedge \neg\, xw_iC) \vee (x \in Z \wedge \neg\, xw_iC) \Leftrightarrow (x \in P_0 \wedge \neg\, x \in P_0^i) \vee (x \in v_iP \wedge$ $\wedge \neg\, xv_i(\Gamma_i^{\mathrm{I}}x \cup (C \cap \Gamma_i^{\mathrm{II}}x))) \Leftrightarrow x \in P_0^j \vee xv_i'(P - (\Gamma_i^{\mathrm{I}}x \cup (C \cap \Gamma_i^{\mathrm{II}}x))) \Leftrightarrow x \in P_0^j \cup$ $\cup\, v_j(\Gamma_j^{\mathrm{I}}x \cap (D \cup \Gamma_j^{\mathrm{II}}x)) \Leftrightarrow xw_jD$. Hence $\overline{w_i} = w_j$, i.e., (i) holds.

$\langle$***the proof of the satisfaction of*** **(P*)**$\rangle :=$ Let $B \in \mathfrak{A}_j$, i.e., $B \subset P$ and $B \subset w_jB$. For each $z \in Z \cap B$ there holds $z \in [u_j]_P(\Gamma_j^{\mathrm{I}}z \cap (B \cup \Gamma_j^{\mathrm{II}}z))$ (16/1, 17.0/1); hence, there exists $\sigma \in \mathring{S}(u_j)$ such that $\sigma z \subset \Gamma_j^{\mathrm{I}}z \cap (B \cup \Gamma_j^{\mathrm{II}}z)$ for each $z \in Z \cap B$. For this σ and any $x \in B$, $\mathbf{x} = (x_k) \in \mathrm{s}(x, \sigma)$ (cf. 3.1) there holds: If $x_k \in B$ for some number k, then $x_{k+1}\Gamma_j^{\mathrm{I}}x_k$ [namely, if $x_k \in B$, then either $x_k \in P_0 \cap B$, then $x_{k+1} = x_k \in P_0 \cap$ $\cap\, B \subset P_0 \cap w_jB = P_0^j$, hence $x_{k+1}\Gamma_j^{\mathrm{I}}x_k$ (17.0/1), or $x_k \in Z \cap B$, then $x_{k+1} \in \sigma x_k \subset$ $\subset \Gamma_j^{\mathrm{I}}x_k$, thus again $x_{k+1}\Gamma_j^{\mathrm{I}}x_k$]. Therefore, if $\bigwedge_k x_k \in B$, then $\bigwedge_k x_{k+1}\Gamma_j^{\mathrm{I}}x_k$, hence $\mathbf{x} \in$ $\in \mathbf{q}_\triangle(\gamma_j)$ (see 8(i/1)). On the other hand, if $\neg \bigwedge_k x_k \in B$, then there exists k_0 such that $(\bigwedge_{k < k_0} x_k \in B) \wedge x_{k_0+1} \notin B$ (namely, $x_0 = x \in B$); thus $\bigwedge_{k < k_0} x_{k+1}\Gamma_j^{\mathrm{I}}x_k$ and, further, $x_{k_0} \in$ $\in B \cap Z$ [$x_{k_0} \in B$; $x_{k_0} \in B \cap P_0$ implies $x_{k_0+1} = x_{k_0} \in B$, which is a contradiction], therefore, $x_{k_0+1} \in \sigma_j x_{k_0} - B \subset (\Gamma_j^{\mathrm{I}}x_{k_0} \cap (B \cup \Gamma_j^{\mathrm{II}}x_{k_0})) - B \subset \Gamma_j^{\mathrm{II}}x_{k_0}$, hence $x_{k_0+1}\Gamma_j^{\mathrm{II}}x_{k_0}$; thus, again $\mathbf{x} \in \mathbf{q}_\triangle(\gamma_j)$ (see 8(i/1)). In such a way we have proved that always $\mathbf{x} \in \mathbf{q}_\triangle(\gamma_j) = \mathbf{A}_j$. Therefore, $\mathrm{s}(x, \sigma) \subset \mathbf{A}_j$ for each $x \in B$. Consequently, (P*) is satisfied.

$\langle$***the property*** **(I)**$\rangle :=$ theorem 11

$\langle$***the remainder of the proof of*** **22.3**/$\alpha\rangle :=$ Let $y \in w_iA$, i.e., $y \in P_0^i \cup v_i(\Gamma_i^{\mathrm{I}} \cup$ $\cup\, (A \cap \Gamma_i^{\mathrm{II}}y))$. Let $y \notin A$. If $y \in P_0$, then $y \in P_0 \cap w_iA = P_0^i$, $y\Gamma_i^{\mathrm{I}}y$, hence $(y) \in \mathbf{q}^\triangle(\gamma_i)$, $\mathrm{s}(y, \sigma_i) = \{(y)\} \subset \mathbf{q}^\triangle(\gamma_i) = \mathbf{A}_i$ (see 6), $y \in \mathring{\mathbf{u}}_i\mathbf{A}_i = A$, which is a contradiction (as $y \notin A$). On the other hand, if $y \in Z$, then, evidently, there exists $\sigma^* \in \mathring{S}(u_i)$ such that $\sigma^* \mid Z \cap A = \sigma_i \mid Z \cap A$, $\sigma^*y \subset \Gamma_i^{\mathrm{I}}y \cup (A \cap \Gamma_i^{\mathrm{II}}y)$; hence (see 22.2/1) $\mathrm{s}(x, \sigma^*) \subset \mathbf{A}_i$ for each $x \in A$, and that contradiction $y \in A$ can be obtained again: Let $\mathbf{x} = (x_k) \in \mathrm{s}(y, \sigma^*)$. Then $l(\mathbf{x}) > 0$ (as $x_0 = y \in Z$), $x_1 \in \sigma^*x_0 = \sigma^*y \subset \Gamma_i^{\mathrm{I}}y \cup (A \cap \Gamma_i^{\mathrm{II}}y)$; hence, either $x_1 \in \Gamma_i^{\mathrm{I}}y$, then $x_1\Gamma_i^{\mathrm{I}}x_0$, $\mathbf{x} \in \mathbf{q}^\triangle(\gamma_i) = \mathbf{A}_i$ (see 6), or $x_1 \in A \cap \Gamma_i^{\mathrm{II}}y$, but then $\mathbf{x}^{[1]} \in \mathrm{s}(x_1, \sigma^*) \subset \mathbf{A}_i$ (as $x_1 \in A$), but $x_1\Gamma_i^{\mathrm{II}}x_0$, hence (cf. 6) $\mathbf{x} \in \mathbf{A}_i$, too. Hence $\mathrm{s}(y, \sigma^*) \subset \mathbf{A}_i$, $y \in \mathring{\mathbf{u}}_i\mathbf{A}_i = A$ (the contradiction).

Therefore, $y \in A$. Consequently, $w_iA \subset A$.

$\langle$***the definition of*** $\sigma_0\rangle :=$ Let $\sigma_0 \in \mathring{S}(u_i)$ be such that $\sigma_0 z \subset \Gamma_i^{\mathrm{I}}z \cup (A^{\eta(z)-1} \cap \Gamma_i^{\mathrm{II}}z)$ for each $z \in Z \cap A^\infty$; such σ_0 exists [if $z \in Z \cap A^\infty$, then $z \in Z \cap wA^{\eta(z)-1}$ (23.1/1 b)), hence $z \in [u_i]_P(\Gamma_i^{\mathrm{I}}z \cup (A^{\eta(z)-1} \cap \Gamma_i^{\mathrm{II}}z))$ (17.0/1, 16/1)].

$\langle$***the proof of* 23.3/α**$\rangle$:= Let $x \in A^\infty$, $\mathbf{x} = (x_k) \in s(x, \sigma_0)$. Let $\mathbf{x} \notin \mathbf{A}_\iota$. Let V_m be this assertion: $x_0, \ldots, x_m \in \mathring{A}^\infty$, $\eta(x_0) > \cdots > \eta(x_m)$, $\bigwedge_{k<m} (x_{k+1}\Gamma^{II}_\iota x_k \wedge \neg\, x_{k+1}\Gamma^{I}_\iota x_k)$. V_0 is true. If V_m holds for some m, then $\neg\, x_{m+1}\Gamma^{I}_\iota x_m$ (otherwise $\mathbf{x} \in \mathbf{q}^m(\gamma_\iota) \subset \mathbf{A}_\iota$, a contradiction), hence $x_m \in Z$ (as otherwise $x_m \in P_0 \cap A^\infty = P_0 \cap w_\iota A^\infty = P^\iota_0$, $x_{m+1} = x_m \in \Gamma^{I}_\iota x_m$, cf. 23.1/1, 17.0/1, 16/1), $x_{m+1} \in \sigma_0 x_m - \Gamma^{I}_\iota x_m \subset A^{\eta(x_m)-1} \cap \Gamma^{II}_\iota x_m$, $\eta(x_{m+1}) \leqslant \eta(x_m) - 1 < \eta(x_m)$; therefore, V_{m+1} is valid. Thus, V_k holds for each k, hence $(\eta(x_k))_{0 \leqslant k < \omega_0}$ is an infinite decreasing sequence of ordinal numbers, which is impossible. This contradiction shows that $\mathbf{x} \in \mathbf{A}_\iota$. Consequently, $s(x, \sigma_0) \subset \mathbf{A}_\iota$ for each $x \in A^\infty$.

THE SECOND LIST OF TEXT SUBSTITUTIONS

(α := 2, αth := second)

$\langle$***the choice of*** $\mathbf{A}_1, \mathbf{A}_2$$\rangle$:= Let $\gamma_\iota = (\Gamma^{I}_\iota, \Gamma^{II}_\iota)$ $(\iota = 1, 2)$ be mutually complementary parameters (i.e. $\gamma_2 = \overline{\gamma_1}$), let

$$\mathbf{A}_j := \mathbf{q}^{\square}(\gamma_j), \quad \mathbf{A}_\iota := \mathbf{q}_{\square}(\gamma_\iota);$$

consequently (cf. 7(i)), $\mathbf{A}_2 = \mathbf{P} - \mathbf{A}_1$.

$\langle$***the choice of*** $\sim_j$$\rangle$:= Let $\sim_j := \mathring{\sim}$, i.e., the *passive player* uses only his plain strategies, too.

$\langle$***the introduction of*** $w_1, w_2$$\rangle$:= Let w_j, w_ι be those elements of Corr $(P, \exp P)$ for which

$$w_j D = \mathring{\mathbf{u}}_j \mathbf{q}^{\triangle}(\Gamma^{I}_j \cap \Gamma^{II}_j \cap \mathbf{L}^D, \Gamma^{II}_j),$$
$$w_\iota C = \mathring{\mathbf{u}}_\iota \mathbf{q}_{\triangle}(\Gamma^{I}_\iota \cup \Gamma^{II}_\iota \cup \mathbf{L}^C, \Gamma^{II}_\iota),$$

for any $D, C \subset P$; here we might write also $\mathbf{u}_\iota \mathbf{q} \ldots$ instead of $\mathring{\mathbf{u}}_\iota \mathbf{q} \ldots$ $(\iota = 1, 2)$, too (cf. 25/1(ii)).

$\langle$***the proof of* 17.1/α**$\rangle$:= (ii) follows immediately from § 2(13), § 3(3), § 6.14.1-2. Let $D \subset P$, $C := P - D$. Using § 4(30) (with $Q := P$), § 6.25/1(i), § 6.7(i) and the fact that $(\Gamma^{I}_j \cap \Gamma^{II}_j \cap \mathbf{L}^D, \Gamma^{II}_j)$, $(\Gamma^{I}_\iota \cup \Gamma^{II}_\iota \cup \mathbf{L}^C, \Gamma^{II}_\iota)$ are mutually complementary parameters (cf. 5, § 6(4′), (1) etc.), we obtain: $\overline{w_\iota D} = P - w_\iota C = P - \mathring{\mathbf{u}}_\iota \mathbf{q}_{\triangle}(\Gamma^{I}_\iota \cup \Gamma^{II}_\iota \cup \mathbf{L}^C, \Gamma^{II}_\iota) = \mathring{\mathbf{u}}_j \mathbf{q}^{\triangle}(\Gamma^{I}_j \cap \Gamma^{II}_j \cap \mathbf{L}^D, \Gamma^{II}_j) = w_j D$. Hence $\overline{w_\iota} = w_j$, i.e., (i) holds.

$\langle$***the proof of the satisfaction of* (P*)**$\rangle$:= Let $B \in \mathfrak{A}_j$, i.e., $B \subset P$, $B \subset w_j B$. Let $\mathbf{B} := \mathbf{q}^{\triangle}(\Gamma^{I}_j \cap \Gamma^{II}_j \cap \mathbf{L}^B, \Gamma^{II}_j)$. Then there exists $\sigma \in \mathring{S}(u_j)$ such that $s(x, \sigma) \subset \mathbf{B}$ for each $x \in \mathring{\mathbf{u}}_j \mathbf{B} = w_j B$ (see 25/1(iii); cf. part IV). Let $x \in B$, $\mathbf{x} = (x_k) \in s(x, \sigma)$. Let some k_0 be such that $x_{k_0} \in B$; then $x_{k_0} \in w_j B$, therefore $\mathbf{x}^{[k_0]} \in s(x_{k_0}, \sigma) \subset \mathbf{B}$, hence there exists $r \geqslant 0$ such that $x_{k_0+r+1}(\Gamma^{I}_j \cap \Gamma^{II}_j \cap \mathbf{L}^B)\, x_{k_0+r} \wedge \bigwedge_{s<r} x_{k_0+s+1}\Gamma^{II}_j x_{k_0+s}$ (see 6), which implies (cf. 4) $x_{k_0+r+1}\Gamma^{I}_j x_{k_0+r} \wedge (\bigwedge_{s \leqslant r} x_{k_0+s+1}\Gamma^{II}_j x_{k_0+s}) \wedge x_{k_0+r+1} \in B$. From this and from $x_0 = x \in B$ it follows (by induction) that there exists a sequence $0 = k_1 < k_2 < \ldots$ such that for $m = 1, 2, \ldots$ there holds $x_{k_{m+1}}\Gamma^{I}_j x_{k_{m+1}-1} \wedge (\bigwedge_{k_m \leqslant k < k_{m+1}} x_{k+1}\Gamma^{II}_j x_k) \wedge x_{k_{m+1}} \in B$. Consequently, $(\bigwedge_k x_{k+1}\Gamma^{II}_j x_k) \wedge \bigwedge_n \bigvee_{k \geqslant n} x_{k+1}\Gamma^{I}_j x_k$, hence (see 8(ii/1)) $\mathbf{x} \in \mathbf{q}^{\square}(\gamma_j) = \mathbf{A}_j$. Therefore, $s(x, \sigma) \subset \mathbf{A}_j$ for each $x \in B$.

$\langle$***the property* (I)**$\rangle$:= theorem 11

$\langle$***the remainder of the proof of* 22.3/α**$\rangle$:= Let $\mathbf{C} := \mathbf{q}_\triangle(\Gamma_i^{\mathrm{I}} \cup \Gamma_i^{\mathrm{II}} \cup \mathbf{L}^A, \Gamma_i^{\mathrm{II}})$, $C := \mathring{\boldsymbol{u}}_i\mathbf{C} (= w_i A)$, let σ be some plainly $\{\mathbf{C}\}$-absolute u_i-strategy (see 25/1 (iii) or 22.2/1 (with $\boldsymbol{A}_i := \mathbf{C}$)). Let σ^* be that plain u_i-strategy for which $\sigma^* \mid Z \cap A = \sigma_i \mid Z \cap A$, $\sigma^* \mid Z - A = \sigma \mid Z - A$. Then $\mathrm{s}(x, \sigma^*) \subset \boldsymbol{A}_i$ for each $x \in A$ (see 22.2/2).

Let $x \in C$, $\mathbf{x} = (x_k) \in \mathrm{s}(x, \sigma^*)$. Then there occurs just one of the following cases (α), (β):

(α) $\bigwedge_k x_k \notin A$. Then $\mathbf{x} \in \mathrm{s}(x, \sigma) \subset \mathbf{C}$ (namely: $\sigma^* \mid Z - A = \sigma \mid Z - A$, σ is plainly $\{\mathbf{C}\}$-absolute, $x \in C = \mathring{\boldsymbol{u}}_i\mathbf{C}$) and: either $\bigvee_k x_{k+1}\Gamma_i^{\mathrm{II}}x_k$, then $\mathbf{x} \in \mathbf{q}_\square(\Gamma_i^{\mathrm{I}}, \Gamma_i^{\mathrm{II}}) = \boldsymbol{A}_i$ (see 8(ii/2)), or $\bigwedge_k \neg\, x_{k+1}\Gamma_i^{\mathrm{II}}x_k$, then (cf. 8(i/1)) $\bigwedge_k (\neg\, x_{k+1}\Gamma_i^{\mathrm{II}}x_k \wedge x_{k+1}(\Gamma_i^{\mathrm{I}} \cup \Gamma_i^{\mathrm{II}} \cup \mathbf{L}^A)\, x_k \wedge x_{k+1} \notin A)$, hence (cf. 4) $\bigwedge_k x_{k+1}\Gamma_i^{\mathrm{I}}x_k$, thus (see again 8(ii/2)) $\mathbf{x} \in \boldsymbol{A}_i$.

(β) There exists k_0 such that $x_{k_0} \in A$; then (see above) $\mathbf{x}^{[k_0]} \in \mathrm{s}(x_{k_0}, \sigma^*) \subset \boldsymbol{A}_i$, hence $\mathbf{x} \in \boldsymbol{A}_i$ ($\boldsymbol{A}_i = \mathbf{q}_\square(\gamma_i)$ has the property (I°), as we have mentioned in 10).

Thus we have proved that $\mathrm{s}(x, \sigma^*) \subset \boldsymbol{A}_i$ for each $x \in C$, consequently $w_i A = C \subset \mathring{\boldsymbol{u}}_i\boldsymbol{A}_i = A$, i.e., $w_i A \subset A$.

$\langle$***the definition of*** $\sigma_0\rangle$:= For any non-limit ordinal number η such that $0 < \eta \leqslant \xi_0$ let $\sigma^\eta \in \mathring{S}(u_i)$ be such that $\mathrm{s}(x, \sigma^\eta) \subset \mathbf{q}_\triangle(\Gamma_i^{\mathrm{I}} \cup \Gamma_i^{\mathrm{II}} \cup \mathbf{L}^{A^{\eta-1}}, \Gamma_i^{\mathrm{II}})$ for all $x \in A^\eta$ [such σ^η exists: $A^\eta = wA^{\eta-1} = \mathring{\boldsymbol{u}}_i\mathbf{q}_\triangle(\Gamma_i^{\mathrm{I}} \cup \Gamma_i^{\mathrm{II}} \cup \mathbf{L}^{A^{\eta-1}}, \Gamma_i^{\mathrm{II}})$, see 23.1/2b), 17.0/2; further, cf. 25/1(iii)]. Let $\sigma_0 \in \mathring{S}(u_i)$ be such that for each $z \in Z \cap A^\infty$ there holds (cf. 23.1/2b)) $\sigma_0 z = \sigma^{\eta(z)}z$.

$\langle$***the proof of* 23.3/α**$\rangle$:= Let $x \in A^\infty$, $\mathbf{x} = (x_k) \in \mathrm{s}(x, \sigma_0)$. If $\bigvee_k x_{k+1}\Gamma_i^{\mathrm{II}}x_k$, then $\mathbf{x} \in \mathbf{q}_\square(\gamma_i) = \boldsymbol{A}_i$ (see 8(ii/2)). In the following let $\bigwedge_k \neg\, x_{k+1}\Gamma_i^{\mathrm{II}}x_k$ (the other possibility); then there holds $\bigwedge_k (x_k \in A^\infty \Rightarrow x_{k+1} \in A^{\eta(x_k)})$.

[*Proof:* Let there exist m such that $x_m \in A^\infty$ but $x_{m+1} \notin A^{\eta(x_m)}$; there holds $x_m \notin P_0$ (otherwise $x_{m+1} = x_m \in A^{\eta(x_m)}$, a contradiction) and (see 17.0/2, 23.1/2b)) $A^{\eta(x_m)} = w_i A^{\eta(x_m)-1} = \mathring{\boldsymbol{u}}_i\mathbf{C}$ where $\mathbf{C} := \mathbf{q}_\triangle(\Gamma_i^{\mathrm{I}} \cup \Gamma_i^{\mathrm{II}} \cup \mathbf{L}^{A^{\eta(x_m)-1}}, \Gamma_i^{\mathrm{II}})$, hence $\mathrm{s}(x_{m+1}, \sigma^{\eta(x_m)}) \not\subset \mathbf{C}$ (otherwise $x_{m+1} \in \mathring{\boldsymbol{u}}_i\mathbf{C} = A^{\eta(x_m)}$, a contradiction), thus there exists $\mathbf{y} = (y_k) \in \mathrm{s}(x_{m+1}, \sigma^{\eta(x_m)}) - \mathbf{C}$. Now we put $\mathbf{x}' := (x_m, x_{m+1}, y_1, y_2, \ldots)$; then $\mathbf{x}' \in \mathrm{s}(x_m, \sigma^{\eta(x_m)}) \subset \mathbf{C}$ (see 23.2/2), but $\bigwedge_k \neg\, x_{k+1}\Gamma_i^{\mathrm{II}}x_k$ (the supposition), hence (see 8(i/1)) $\mathbf{y} = (\mathbf{x}')^{[1]} \in \mathbf{C}$, which is a contradiction (as $\mathbf{y} \notin \mathbf{C}$).]

From this and from $x_0 = x \in A^\infty$ it follows that $\bigwedge_k x_k \in A^\infty$ and that $(\eta(x_k))_{0 \leq k < \omega_0}$ is a nonincreasing sequence of ordinal numbers. Consequently, there exists n such that ($x_n \in A^\infty$ and) $\eta(x_n) = \eta(x_{n+1}) = \ldots$. Then, clearly (cf. 23.2/2), $\mathbf{x}^{[n]} \in \mathrm{s}(x_n, \sigma^{\eta(x_n)}) \subset \mathbf{q}_\triangle(\Gamma_i^{\mathrm{I}} \cup \Gamma_i^{\mathrm{II}} \cup \mathbf{L}^{A^{\eta(x_n)-1}}, \Gamma_i^{\mathrm{II}})$, but $\bigwedge_k \neg\, x_{k+1}\Gamma_i^{\mathrm{II}}x_k$ (the supposition) and $\bigwedge_{k \geqslant n} x_k \notin A^{\eta(x_n)-1}$ (as $\eta(x_k) = \eta(x_n)$ for $k \geqslant n$), hence (cf. 4, 8(i/1)) $\bigwedge_{k \geqslant n} x_{k+1}\Gamma_i^{\mathrm{I}}x_k$, thus (see 8(ii/2)) $\mathbf{x} \in \mathbf{q}_\square(\gamma_i) = \boldsymbol{A}_i$.

In such a way we have proved that $\mathrm{s}(x, \sigma_0) \subset \boldsymbol{A}_i$ for each $x \in A^\infty$.

THE THIRD LIST OF TEXT SUBSTITUTIONS

($\alpha := 3$, αth := third)

$\langle$***the choice of*** $\boldsymbol{A}_1, \boldsymbol{A}_2\rangle$:= Let $\varkappa_\iota = (\mathfrak{K}_\iota, \Gamma_\iota)$ ($\iota = 1, 2$) be mutually complementary r-parameters (i.e. $\varkappa_2 = \overline{\varkappa_1}$), let $\varkappa_j$ be an $\cap$-parameter and $\varkappa_i$ be a $\cup$-parameter (cf. § 6(9)). We shall suppose (without loss of generality, see 13.3) that $\mathfrak{K}_1 \neq \emptyset \neq \mathfrak{K}_2$. Let

$$\boldsymbol{A}_j := \boldsymbol{r}^{\square}(\varkappa_j), \quad \boldsymbol{A}_i := \boldsymbol{r}_{\square}(\varkappa_i);$$

consequently (cf. § 6(10)), $\boldsymbol{A}_2 = \boldsymbol{P} - \boldsymbol{A}_1$. (Of course, the symbol $\varkappa$ itself (i.e., without index) will be used in the sense introduced in § 1.2.)

$\langle$***the choice of*** $\sim_j\rangle$:=

Let Γ be the graph of u_j, let $\boldsymbol{X} := \boldsymbol{X}_\Gamma$ (§ 2.26.2).

Let χ be a mapping of $\boldsymbol{Z}$ into $\{\emptyset, P\}$ such that: if $\boldsymbol{z} = (z_0, \ldots, z_n) \in \boldsymbol{Z}$, $\chi\boldsymbol{z} = \emptyset$, then there exists $K \in \mathfrak{K}_j$ such that $\bigvee_{k<n} [K \cap \{z_0, \ldots, z_k\} = \emptyset \wedge \neg\, z_{k+1}\Gamma_j z_k]$. (Clearly, if $\boldsymbol{x} = (x_k) \in \boldsymbol{P}$, $\chi(x_0, \ldots, x_k) = \emptyset$ for some k ($< l(\boldsymbol{x})$), then $\boldsymbol{x} \notin \boldsymbol{A}_j$.)

Let π be a mapping of $\boldsymbol{Z}$ into $[\mathfrak{K}_j]_P$ (§ 2.12) such that for each $\boldsymbol{x} = (x_k) \in \boldsymbol{X}$ with $l(\boldsymbol{x}) = \omega_0$ and for each $\boldsymbol{z} = (z_0, \ldots, z_n) \in \boldsymbol{Z}$ having the property $\bigwedge_{m<n} z_{m+1}\Gamma z_m$ there holds:

(i) $$\bigwedge_k [\{x_k, x_{k+1}\} \cap \pi(x_0, \ldots, x_k) = \emptyset \Rightarrow \pi(x_0, \ldots, x_k) = \pi(x_0, \ldots, x_{k+1})],$$

(ii) $$z_n \in \pi(z_0, \ldots, z_n) \Rightarrow \text{for each } K \in \mathfrak{K}_j \bigvee_{m \leqq n} z_m \in K,$$

(iii) $$[\bigwedge_k \bigvee_{m>k} x_m \in \pi(x_0, \ldots, x_k)] \Rightarrow \text{for each } K \in \mathfrak{K}_j \bigvee_m x_m \in K.$$

Let $\sim_j$ be that (binary) relation on $\boldsymbol{Z}$ for which

$$\boldsymbol{z}^1 \sim_j \boldsymbol{z}^2 \Leftrightarrow \varkappa(\boldsymbol{z}^1) = \varkappa(\boldsymbol{z}^2) \wedge \pi\boldsymbol{z}^1 \cap \chi\boldsymbol{z}^1 = \pi\boldsymbol{z}^2 \cap \chi\boldsymbol{z}^2$$

for any $\boldsymbol{z}^1, \boldsymbol{z}^2 \in \boldsymbol{Z}$ (cf. § 1.2). Evidently, $\sim_j$ is a memory relation.

Remarks.

0) The above given definition of $\sim_j$ is somewhat complicated, but it involves various natural particular choices of $\sim_j$, cf. remarks 2 and 3. In contradistinction to the cases $\alpha = 1$ and $\alpha = 2$, here it is not possible to choose $\sim_j := \mathring{\sim}$ generally (although there are particular cases at which the latter choice is possible, see remark 2), as it is shown by this

Example ([4], § 9.3.2). Let $P := \{0, 1, 2, \ldots\}$, $P_0 = \emptyset$, $0u_j := \{\{1\}, \{2\}, \ldots\}$, $xu_j :=$ $:= \{0\}$ for $x \in P - \{0\}$, $0u_i := \{1, 2, \ldots\}$, $xu_i := \{0\}$ for $x \in P - \{0\}$. Let $\boldsymbol{A}_\iota$, $\mathfrak{A}_\iota$ etc. (cf. 16-18/3) be introduced to $\mathfrak{K}_j := \{\{k, k+1, k+2, \ldots\} \mid k = 0, 1, 2, \ldots\}$, $\Gamma_j := \mathbf{L}_P$ (which determines $\varkappa_i$, cf. 13.1). Clearly, then $\mathfrak{A}_j = \{B \mid B \subset P$, card $B \in \{0, \aleph_0\}\}$, $\boldsymbol{u}_j\boldsymbol{A}_j = \emptyset \neq P = \mathring{\boldsymbol{u}}_j\boldsymbol{A}_j$ (cf. 25/3(ii)); especially, $P \in \mathfrak{A}_j$, but $s(x, \sigma) \notin \boldsymbol{A}_j$ for each $x \in P$ and each $\sigma \in \mathring{S}(u_j)$ (cf. (P*)).

1) By definition, at $\boldsymbol{z} \in \boldsymbol{Z}$ the player j retains the last position $\varkappa(\boldsymbol{z})$ of $\boldsymbol{z}$ and the set $\pi\boldsymbol{z} \cap \chi\boldsymbol{z}$. If $\pi\boldsymbol{z} \cap \chi\boldsymbol{z} = \emptyset$, then either $\chi\boldsymbol{z} = \emptyset$ or $\pi\boldsymbol{z} = \emptyset$ (as $\chi\boldsymbol{z} \in \{\emptyset, P\}$), and $\boldsymbol{x} \notin \boldsymbol{A}_j$ for any variant $\boldsymbol{x}$ having $\boldsymbol{z}$ as an initial segment (cf. the definition of χ; if $\pi\boldsymbol{z} = \emptyset$, then $\emptyset \in \mathfrak{K}_j$ and $\boldsymbol{A}_j = \emptyset$). If $\pi\boldsymbol{z} \cap \chi\boldsymbol{z} \neq \emptyset$, then $\chi\boldsymbol{z} = P$, $\pi\boldsymbol{z} \cap \chi\boldsymbol{z} = \pi\boldsymbol{z}$, and, roughly speaking, $\pi\boldsymbol{z}$ can be considered as the set which could be attained under abiding Γ_j in the following course of play (cf. the proof of 21/3).

2) $\sim_j$ is uniquely determined by π and χ. If $\pi\mathbf{z} \cap \chi\mathbf{z}$ depends only on $\varkappa(\mathbf{z})$, then $\sim_j = \overset{\circ}{\sim}$. There are three important particular cases at which such a choice of π and of χ is possible:

a) $\emptyset \in \mathfrak{K}_j$ (then $\boldsymbol{A}_j = \emptyset$; cf. 1)), i.e., $\mathfrak{K}_j$ is a singular collection. Then it is possible to take the constant mapping of $\boldsymbol{Z}$ onto $\{\emptyset\}$ as π (and to choose χ arbitrarily in compliance with the corresponding definition). The case a) is a particular case of the following one:

b) $\bigcap_{K\in\mathfrak{K}_j} K \in \mathfrak{K}_j$. Then it is possible to choose the constant mapping of $\boldsymbol{Z}$ onto $\{\bigcap_{K\in\mathfrak{K}_j} K\}$ as π, and the constant mapping of $\boldsymbol{Z}$ onto $\{P\}$ as χ.

c) $\boldsymbol{X}$ does not contain infinite variants. Then it is possible to choose the constant mapping of $\boldsymbol{Z}$ onto $\{P\}$ as χ, and to put $\pi\mathbf{z} := \pi_0\varkappa(\mathbf{z})$ where π_0 is a mapping of Z into $[\mathfrak{K}_j]_P$ such that $z \notin \pi_0 z$ if $z \in Z - \bigcap_{K\in\mathfrak{K}_j} K$.

3) Always it is possible to take the constant mapping of $\boldsymbol{Z}$ onto $\{P\}$ as χ. Let us mention two general examples of practically suitable mappings π.

In the following two examples, let $(K_n)_{0\leqslant n<1+l}$ $(0 < l \leqslant \omega_0)$ be a sequence of sets such that $\{K_n \mid 0 \leqslant n < 1 + l = \mathfrak{K}_j\}$. (Such a sequence exists, as $0 < \operatorname{card} \mathfrak{K}_j \leqslant \aleph_0$.)

3.1) Example. Let π_1 be the mapping of $\boldsymbol{Z}$ such that for any $\mathbf{z} = (z_0, \ldots, z_m) \in \boldsymbol{Z}$ there holds

$$\pi_1\mathbf{z} = \begin{cases} K_{\min\{n|0\leqslant n<1+l,\ K_n\cap\{z_0,\ldots,z_m\}=\emptyset\}} \\ P \end{cases} \text{if} \begin{cases} \bigvee_{n<1+l} K_n \cap \{z_0, \ldots, z_m\} = \emptyset \\ \bigwedge_{n<1+l} K_n \cap \{z_0, \ldots, z_m\} \neq \emptyset \end{cases}.$$

This defines just one mapping π_1 of $\boldsymbol{Z}$ into $\mathfrak{K}_j \cup \{P\} \subset [\mathfrak{K}_j]_P$. It is clear that those conditions (i) — (iii) are satisfied with $\pi := \pi_1$.

3.2) Example. Let the symbol $(z_0, \ldots, z_{-1})$ mean the empty sequence. We put $\mu(z_0, \ldots, z_{-1}) := 0$, $l((z_0, \ldots, z_{-1})) := -1$. Let m be arbitrary; if for any $\mathbf{z} \in \boldsymbol{Z} \cup \{(z_0, \ldots, z_{-1})\}$ with $l(\mathbf{z}) < m$ an integer $\mu\mathbf{z}$ is defined, then we put for each $\mathbf{z} = (z_0, \ldots, z_m) \in \boldsymbol{Z}$

$$\mu\mathbf{z} := \min\{n \mid \mu(z_0, \ldots, z_{m-1}) \leqslant n < 1 + l,\ z_m \notin K_n\},$$

where $\min \emptyset := 1 + l$. Clearly, this inductive definition introduces an integer $\mu\mathbf{z}$ (such that $0 \leqslant \mu\mathbf{z} < 1 + l$) to any $\mathbf{z} \in \boldsymbol{Z}$. We introduce $K_{1+l} := \boldsymbol{P}$. Let π_2 be the mapping of $\boldsymbol{Z}$ such that $\pi_2\mathbf{z} = K_{\mu\mathbf{z}}$ for each $\mathbf{z} \in \boldsymbol{Z}$. Thus π_2 is a mapping of $\boldsymbol{Z}$ into $\mathfrak{K}_j \cup \{P\} \subset [\mathfrak{K}_j]_P$. It is not difficult to verify that those conditions (i) — (iii) are satisfied with $\pi := \pi_2$.

⟨***the introduction of*** w_1, w_2⟩ := Let w_j, w_t be those elements of Corr $(\boldsymbol{P}, \exp \boldsymbol{P})$ for which

$$w_jD = \bigcap_{K\in\mathfrak{K}_j} \mathring{\boldsymbol{u}}_j\boldsymbol{q}^{\triangle}(\mathbf{L}_{K\cap D}, \Gamma_j),$$

$$w_tC = \bigcup_{K\in\mathfrak{K}_t} \mathring{\boldsymbol{u}}_t\boldsymbol{q}_{\triangle}(\mathbf{L}_{K\cup C}, \Gamma_t),$$

for any $D, C \subset P$; here we might write also $\boldsymbol{u}_\iota\boldsymbol{q}\ldots$ instead of $\mathring{\boldsymbol{u}}_\iota\boldsymbol{q}\ldots$ $(\iota = 1, 2)$, too (cf. 25/1(ii)).

⟨***the proof of*** **17.1** $/\alpha$⟩ := (ii) follows immediately from § 2 (13), § 3 (3), § 6.15.1 — 2. Let $D \subset P$, $C := P - D$. Using § 4 (30) (with $Q := P$), § 6.25/1 (i), § 6.7 (i), § 6.13.1

and the fact that $(\mathbf{L}_{K\cup C}, \Gamma_i)$, $(\mathbf{L}_{(P-K)\cap D}, \Gamma_j)$ are mutually complementary parameters for any $K \subset P$ (cf. 5, §6(4′) etc.), we obtain: $\overline{w_i}D = P - w_iC = P - \bigcup_{K\in\mathfrak{K}_i} \mathring{u}_i\mathbf{q}_\triangle(\mathbf{L}_{K\cup C}, \Gamma_i) = \bigcap_{K\in\mathfrak{K}_i} \mathring{u}_j\mathbf{q}^\triangle(\mathbf{L}_{(P-K)\cap D}, \Gamma_j) = \bigcap_{K\in\mathfrak{K}_j} \mathring{u}_j\mathbf{q}^\triangle(\mathbf{L}_{K\cap D}, \Gamma_j) = w_jD$. Hence $\overline{w_i} = w_j$, i.e., (i) holds.

⟨***the proof of the satisfaction of*** (**P***)⟩ := Let $B \in \mathfrak{A}_j$ in this proof. For any $K \subset P$ we denote $\mathbf{B}_K := \mathbf{q}^\triangle(\mathbf{L}_{K\cap B}, \Gamma_j)$, $B_K := \mathring{u}_j\mathbf{B}_K$. Thus $B \subset w_jB = \bigcap_{K\in\mathfrak{K}_j} B_K$. For any $K \subset P$ let $\sigma_K \in \mathring{S}(u_j)$ be such that $\mathrm{s}(x, \sigma_K) \subset \mathbf{B}_K$ for each $x \in B_K$ (such σ_K exists, see 25/1 (iii); cf. part IV). Again, let Γ be the graph of u_j.

There holds:

(L1) $$B_{K_0} = w_jB \text{ for any } K_0 \in [\mathfrak{K}_j]_P.$$

(L2) $$P_0 \cap w_jB \subset \bigcap_{K\in\mathfrak{K}_j} K$$

(L3) If $K \in [\mathfrak{K}_j]_P$, $z \in (Z \cap w_jB) - K$ and $y \in \sigma_K z$, then $y \in w_jB$ and $y\Gamma_j z$.

Proofs (*of* (*L*1) — (*L*3)):

(*L*1): Let $K_0 \in [\mathfrak{K}_j]_P$. There exists $K_0' \in \mathfrak{K}_j$ such that $K_0' \subset K_0$. There holds $B_{K_0} \supset B_{K_0'} \supset \bigcap_{K\in\mathfrak{K}_j} B_K = w_jB$ (§ 6.14, § 3 (3)). Let $K \in \mathfrak{K}_j$. Let $\boldsymbol{\sigma}$ be the mapping of $\mathbf{Z}$ such that for each $\mathbf{z} = (z_0, \ldots, z_m) \in \mathbf{Z}$ there holds:

$$\boldsymbol{\sigma}\mathbf{z} = \begin{cases}\sigma_{K_0}z_m \\ \sigma_K z_m\end{cases} \text{ if } \{z_0, \ldots, z_m\} \cap K_0 \cap B \begin{cases} = \emptyset \\ \neq \emptyset\end{cases}.$$

Clearly, $\boldsymbol{\sigma} \in S(u_j)$. Let $x \in B_{K_0}$, let $\mathbf{x} = (x_k) \in \mathrm{s}(x, \sigma)$. Supposing $\bigwedge_k x_k \notin K_0 \cap B$, we can obtain a contradiction (then $\mathbf{x} \in \mathrm{s}(x, \sigma_{K_0}) \subset \mathbf{B}_{K_0}$ (as $x \in B_{K_0}$ etc.), hence, among others, $\bigvee_k x_k \in K_0 \cap B$). Thus, there exists $k_0 := \min\{k \mid x_k \in K_0 \cap B\}$; there holds $k_0 \leqslant l(\mathbf{x})$ and $\mathbf{x}^{[k_0]} \in \mathrm{s}(x_{k_0}, \sigma_K) \subset \mathbf{B}_K$ (namely, $x_{k_0} \in K_0 \cap B \subset B \subset w_jB \subset B_K$). Thus, there holds:

(α_1) $$\bigwedge_{k<k_0} x_k \notin K_0 \cap B,$$

(α_2) $$\bigvee_{k\geqslant k_0} (x_k \in K \cap B \vee \bigwedge_{\substack{m<k\\ m\geq k_0}} x_{m+1}\Gamma_j x_m).$$

Further, $\bigwedge_{k<k_0} x_{k+1} \in \sigma_{K_0}x_k$, hence there exists $\mathbf{y} = (y_k) \in \mathrm{s}(x, \sigma_{K_0})$ such that

(β) $$\bigwedge_{k\leqslant k_0} x_k = y_k.$$

There holds $\mathbf{y} \in \mathrm{s}(x, \sigma_{K_0}) \subset \mathbf{B}_{K_0}$ (as $x \in B \subset B_{K_0}$ etc.), hence

(γ) $$\bigvee_k (y_k \in K_0 \cap B \wedge \bigwedge_{m<k} y_{m+1}\Gamma_j y_m).$$

But (α_1), (β), (γ) give $\bigvee_{k\geqslant k_0} (y_k \in K_0 \cap B \wedge \bigwedge_{m<k} y_{m+1}\Gamma_j y_m)$, hence $\bigwedge_{m<k_0} y_{m+1}\Gamma_j y_m$, i.e., $\bigwedge_{m<k_0} x_{m+1}\Gamma_j x_m$ (see (β)), which together with (α_2) gives $\bigvee_k (x_k \in K \cap B \wedge \bigwedge_{m<k} x_{m+1}\Gamma_j x_m)$.

Hence $\mathbf{x} \in \mathbf{B}_K$. Consequently, $s(x, \boldsymbol{\sigma}) \subset \mathbf{B}_K$, hence $x \in \boldsymbol{u}_j\mathbf{B}_K = \mathring{\boldsymbol{u}}_j\mathbf{B}_K = B_K$ (see 25/1 (ii), 16/1). Thus, for any $K \in \mathfrak{K}_j$ and each $x \in B_{K_0}$ there holds $x \in B_K$. Hence $B_{K_0} \subset \subset \bigcap_{K \in \mathfrak{K}_j} B_K = w_jB$. This completes the proof of (L1).

(*L2*): If $K \in \mathfrak{K}_j$ and $x \in w_jB$, then $s(x, \sigma_K) \subset \mathbf{B}_K$ (as $w_jB \subset B_K$); if, moreover, $x \in P_0$, then $s(x, \sigma_K) = \{(x)\} \subset \mathbf{B}_K = \mathbf{q}^{\triangle}(\mathbf{L}_{K \cap B}, \Gamma_j)$, hence, among others, $x \in K$. Thus $P_0 \cap w_jB \subset K$ for each $K \in \mathfrak{K}_j$.

(*L3*): Let $K \in [\mathfrak{K}_j]_P$, $z \in (Z \cap w_jB) - K$ (or only $z \in w_jB - K$, see (L2)), let $y \in \sigma_K z$. There exists $\mathbf{x} = (x_k) \in s(z, \sigma_K)$ such that $x_1 = y$, further $s(z, \sigma_K) \subset \mathbf{B}_K$ (as $z \in w_jB \subset B_K$, cf. the proof of (L1)), thus $\bigvee_k (x_k \in K \cap B \wedge \bigwedge_{m<k} x_{m+1}\Gamma_j x_m)$, but $x_0 = = z \notin K$, hence $\bigvee_{k \geqslant 1} (x_k \in K \cap B \wedge \bigwedge_{m<k} x_{m+1}\Gamma_j x_m)$. Thus $\mathbf{x}^{[1]} \in \mathbf{B}_K$, and, moreover, $x_1\Gamma_j x_0$, i.e., $y\Gamma_j z$. Further, $y \in B_K$ [if $y \notin B_K$, then there exists $\mathbf{y} = (y_k) \in s(y, \sigma_K) - \mathbf{B}_K$, but now we can choose $\mathbf{x} = (x_k) := (z, y, y_1, y_2, \ldots)$, then $\mathbf{x} \in s(z, \sigma_K)$, $x_1 = y$, hence $\mathbf{x}^{[1]} \in \mathbf{B}_K$ (see above), but $\mathbf{x}^{[1]} = \mathbf{y} \notin \mathbf{B}_K$, which is a contradiction], but $B_K = = w_jB$ (see (L1)), hence $y \in w_jB$, which completes the proof of (L3).

Q. E. D. ((*L*1) — (*L*3))

Let $\boldsymbol{\sigma}$ be the mapping of $\mathbf{Z}$ such that $\boldsymbol{\sigma}\mathbf{z} = \sigma_{\pi\mathbf{z} \cap \chi\mathbf{z}}\chi(\mathbf{z})$ for each $z \in \mathbf{Z}$. Evidently, $\boldsymbol{\sigma} \in \sim_j S(u_j)$.

Let $x \in w_jB$, let $\mathbf{x} = (x_k) \in s(x, \boldsymbol{\sigma})$. There holds

$$\mathbf{x} \in s(x, \boldsymbol{\sigma}) \subset \mathbf{X}_\Gamma. \tag{1}$$

For obtaining a contradiction we shall suppose $\mathbf{x} \notin \mathbf{A}_j$.

We introduce (for any k)

$$\mathrm{V}_k := x_k \in w_jB \wedge k \leqslant l(\mathbf{x}) \wedge \bigwedge_{m<k} x_{m+1}\Gamma_j x_m.$$

Let V_k hold for some k. Then there exists $K_0 \in \mathfrak{K}_j$ such that $K_0 \cap \{x_0, \ldots, x_k\} = \emptyset$ (otherwise the supposition $\bigwedge_{m<k} x_{m+1}\Gamma_j x_m$ implies $\mathbf{x} \in \mathbf{A}_j$, which is a contradiction), in particular, $x_k \notin \bigcap_{K \in \mathfrak{K}_j} K$. Hence, $x_k \in Z$ (otherwise $x_k \in P_0 \cap w_jB \subset \bigcap_{K \in \mathfrak{K}_j} K$ (see (L2), which has been eliminated), thus $k + 1 \leqslant l(\mathbf{x})$. Further, $\chi(x_0, \ldots, x_k) = P$ (as $\bigwedge_{m<k} x_{m+1}\Gamma_j x_m$). Therefore, $x_k \in (Z \cap w_jB) - \pi(x_0, \ldots, x_k)$ (namely, if $x_k \in \pi(x_0, \ldots, x_k)$, then we can obtain a contradiction by means of (*1*), the condition (ii) in the definition of π and the supposition $\bigwedge_{m<k} x_{m+1}\Gamma_j x_m$, cf. above), $\pi(x_0, \ldots, x_k) \in [\mathfrak{K}_j]_P$, $x_{k+1} \in \in \boldsymbol{\sigma}(x_0, \ldots, x_k) = \sigma_{\pi(x_0, \ldots, x_k) \cap \chi(x_0, \ldots, x_k)} x_k = \sigma_{\pi(x_0, \ldots, x_k)} x_k$, hence $x_{k+1} \in w_jB$, $x_{k+1}\Gamma_j x_k$ (see (L3)).
Consequently, V_{k+1} holds.

We have proved $\mathrm{V}_k \Rightarrow \mathrm{V}_{k+1}$ for each k, but V_0 holds trivially, hence V_k holds for each k. From this it follows that there holds:

$$l(\mathbf{x}) = \omega_0; \tag{2}$$

$$\bigwedge_k x_{k+1}\Gamma_j x_k; \tag{3}$$

$$\bigwedge_k x_k \in w_jB. \tag{4}$$

The above considerations (cf. (ii) in 16/3, (3) etc.) imply

$$(5) \qquad \bigwedge_k x_k \notin \pi(x_0, \ldots, x_k);$$

$$(6) \qquad \bigwedge_k x_{k+1} \in \sigma_{\pi(x_0,\ldots,x_k)} x_k;$$

the supposition $\mathbf{x} \notin \mathbf{A}_j$ and the properties (1), (2) and (3) imply by means of the property (iii) (in the definition of π) that

(7) there exists n such that $\bigwedge_{k>n} x_k \notin \pi(x_0, \ldots, x_n)$.

Let $W_k := (\pi(x_0, \ldots, x_k) = \pi(x_0, \ldots, x_n))$. W_n is valid. If W_r holds for some $r \geqslant n$, then $x_r, x_{r+1} \notin \pi(x_0, \ldots, x_n) = \pi(x_0, \ldots, x_r)$ (see (5), (7), W_r), hence (see (1), (2) and the condition (i) in the definition of π) $\pi(x_0, \ldots, x_{r+1}) = \pi(x_0, \ldots, x_r) = \pi(x_0, \ldots, x_n)$, thus W_{r+1} holds. By induction, W_k holds for each $k \geqslant n$, i.e. there holds

$$(8) \qquad \bigwedge_{k \geqslant n} \pi(x_0, \ldots, x_k) = \pi(x_0, \ldots, x_n).$$

Therefore (see (6) and (8)) $\mathbf{x}^{[n]} \in s(x_n, \sigma_{\pi(x_0,\ldots,x_n)}) \subset \mathbf{B}_{\pi(x_0,\ldots,x_n)}$ (namely, $\pi(x_0, \ldots, x_n) \in [\mathfrak{K}_j]_P$, hence (see (4)) $x_n \in w_j B = B_{\pi(x_0,\ldots,x_n)}$, see (L1)). But $\mathbf{x}^{[n]} \in \mathbf{B}_{\pi(x_0,\ldots,x_n)}$ implies that there exists $r \geqslant n$ such that $x_r \in \pi(x_0, \ldots, x_n) \geqq \pi(x_0, \ldots, x_r)$ (cf. (8)), which is a contradiction (see (5)).

In such a way we have proved that $s(x, \boldsymbol{\sigma}) \subset \mathbf{A}_j$ for each $x \in w_j B$, and, therefore, for each $x \in B$ ($\subset w_j B$).

⟨***the property* (I)**⟩ := lemma 15

⟨***the remainder of the proof of* 22.3/α**⟩ := Let $K \in \mathfrak{K}_i$ be fixed in the following part of the proof, let $\mathbf{C} := \mathbf{q}_\triangle(\mathbf{L}_{K \cup A}, \Gamma_i)$, $C := \mathring{u}_i\mathbf{C}$. Let σ be some plainly $\{\mathbf{C}\}$-absolute u_i-strategy (see 25/1 (iii) or 22.2/1). Let σ^* be that plain u_i-strategy for which $\sigma^* \mid Z \cap A = \sigma_i \mid Z \cap A$, $\sigma^* \mid Z - A = \sigma \mid Z - A$.

Let $x \in C$, $\mathbf{x} = (x_k) \in s(x, \sigma^*)$. Then there occurs just one of the following cases (α), (β):

(α) $\bigwedge x_k \notin A$. Then $\mathbf{x} \in s(x, \sigma) \subset \mathbf{C}$ (namely: $\sigma^* \mid Z - A$, σ is plainly $\{\mathbf{C}\}$-absolute, $x \in C = \mathring{u}_i\mathbf{C}$), hence (see 8 (i/1)) there holds

$$(\bigwedge_k x_k \in K \cup A) \vee \bigvee_k (x_{k+1}\Gamma_i x_k \wedge \bigwedge_{m<k} x_m \in K \cup A);$$

this and $\bigwedge_k x_k \notin A$ imply

$$(\bigwedge_k x_k \in K) \vee \bigvee_k (x_{k+1}\Gamma_i x_k \wedge \bigwedge_{m<k} x_m \in K),$$

i.e., $\mathbf{x} \in \mathbf{q}_\triangle(\mathbf{L}_K, \Gamma_i) \subset \mathbf{A}_i$.

(β) $\neg \bigwedge_k x_k \notin A$. Then there exists $n := \min\{k \mid x_k \in A\}$, hence

$$(1) \qquad \bigwedge_{k<n} x_k \notin A$$

and $\mathbf{x}^{[n]} \in s(x_n, \sigma^*) \subset \mathbf{A}_i$ (namely: $x_n \in A = \mathring{u}_i\mathbf{A}_i$, σ^* is plainly $\{\mathbf{A}_i\}$-absolute (22.2/3)), hence there exists $K_1 \in \mathfrak{K}_i$ such that $\mathbf{x}^{[n]} \in \mathbf{q}_\triangle(\mathbf{I}_{K_1}, \Gamma_i)$ (16/3, 13.2). There exists $K_2 \in \mathfrak{K}_i$ such that $K_1 \cup K \subset K_2$ (see 16/3, 13.1). Consequently, $\mathbf{x}^{[n]} \in \mathbf{q}_\triangle(K_2, \Gamma_i)$ (§ 6.14):

$$(2) \qquad (\bigwedge_{k \geqslant n} x_k \in K_2) \vee \bigvee_{k \geqslant n} (x_{k+1}\Gamma_i x_k \wedge \bigwedge_{m<k} x_m \in K_2),$$

(8 (i/1)). Further, (1) implies $\bigwedge_{k<n} x_{k+1} \in \sigma x_k$ (as, clearly, $n \leqslant l(\mathbf{x})$), thus there exists $\mathbf{y} = (y_k) \in s(x, \sigma) \subset \mathbf{B}$ (cf. above) such that

(3) $$\bigwedge_{k \leqslant n} x_k = x_k;$$

$\mathbf{y} \in \mathbf{C}$ together with $K \subset K_2$ implies

(4) $$(\bigwedge_k y_k \in K_2 \cup A) \vee \bigvee_k (y_{k+1}\Gamma_i y_k \wedge \bigwedge_{m \leqslant k} y_m \in K_2 \cup A).$$

There holds

(5) $$(\bigwedge_{k \leqslant n} x_k \in K_2 \cup A) \Rightarrow \mathbf{x} \in \mathbf{q}_\triangle(\mathbf{L}_{K_2}, \Gamma_i).$$

[*Proof*. If $\bigwedge_{k \leqslant n} x_k \in K_2 \cup A$, then $\bigwedge_{k<n} x_k \in K_2$ (see (1)), which together with (2) gives $(\bigwedge_k x_k \in K_2) \vee \bigvee_k (x_{k+1}\Gamma_i x_k \wedge \bigwedge_{m \leqslant k} x_m \in K_2)$, i.e., $\mathbf{x} \in \mathbf{q}_\triangle(\mathbf{L}_{K_2}, \Gamma_i)$ (cf. 8 (i/1).]

(4) shows that there occurs some of the following cases ($\beta1$) — ($\beta3$):

($\beta1$) $\bigwedge_k y_k \in K_2 \cup A$. Then $\bigwedge_{k \leqslant n} x_k \in K_2 \cup A$, $\mathbf{x} \in \mathbf{q}_\triangle(\mathbf{L}_{K_2}, \Gamma_i)$ (see (3), (5)).

($\beta2$) $\bigvee_{k \geqslant n} (y_{k+1}\Gamma_i y_k \wedge \bigwedge_{m \leqslant k} y_m \in K_2 \cup A)$. Then $\bigwedge_{k \leqslant n} x_k \in K_2 \cup A$, $\mathbf{x} \in \mathbf{q}_\triangle(\mathbf{L}_{K_2}, \Gamma_i)$ (see (3),(5)).

($\beta3$) $\bigvee_{k<n} (y_{k+1}\Gamma_i y_k \wedge \bigwedge_{m \leqslant k} y_m \in K_2 \cup A)$. Then by using this, (3), and (1) we get $\bigvee_{k<n} (x_{k+1}\Gamma_i x_k \wedge \bigwedge_{m \leqslant k} x_m \in K_2)$, hence $\mathbf{x} \in \mathbf{q}_\triangle(\mathbf{L}_{K_2}, \Gamma_i)$ (8 (i/1)).

Consequently, always $\mathbf{x} \in \mathbf{q}_\triangle(\mathbf{L}_{K_2}, \Gamma_i) \subset \mathbf{A}_i$.

Thus we have proved that $s(x, \sigma^*) \subset \mathbf{A}_i$ for each $x \in C$. Therefore, $\mathring{\mathbf{u}}_i \mathbf{q}_\triangle(\mathbf{L}_{K \cup A}, \Gamma_i) \subset \subset \mathring{\mathbf{u}}_i \mathbf{A}_i = A$ for each $K \in \mathfrak{K}_i$, hence (cf. 17.0/3) $w_i A \subset A$.

⟨***the definition of*** σ_0⟩ := Let η be a non-limit ordinal number such that $0 < \eta \leqslant \leqslant \xi_0$. Let $K \in \mathfrak{K}_i$. In the remainder of § 6c/3 we shall denote $\mathbf{C}_K^\eta := \mathbf{q}_\triangle(\mathbf{L}_{K \cup A^{\eta-1}}, \Gamma_i)$, $A_K^\eta := \mathring{\mathbf{u}}_i \mathbf{C}_K^\eta$; of course, $A^\eta = \bigcup_{K \in \mathfrak{K}_i} A_K^\eta$. Let there be chosen $\sigma_K^\eta \in \mathring{S}(u_i)$ (to any of those η, K) such that $s(x, \sigma_K^\eta) \subset \mathbf{C}_K^\eta$ for each $x \in A_K^\eta$ (such σ_K exists, cf. 25/1 (iii)). Further, let $\leqslant_\eta$ be a well order of $\mathfrak{K}_i$. For $x \in A^\infty$ we put

$$K(x) := \min_{\eta(x)}\{K \mid K \in \mathfrak{K}_i, x \in A_K^{\eta(x)}\},$$

where $\min_{\eta(x)}$ is taken under $\leqslant_{\eta(x)}$. (This is a correct definition, cf. **23.1/3b**).) Let $\sigma_0 \in S(u_i)$ be such that $\sigma_0 z = \sigma_{K(z)}^{\eta(z)}$ for each $z \in Z \cap A^\infty$.

⟨***the proof of* 23.3/**α⟩ := We shall use the following auxiliary assertions:

(A1) Let η be a non-limit ordinal number, $0 < \eta \leqslant \xi_0$, let $K \in \mathfrak{K}_i$. Let $y \in A_K^\eta$, $y' \begin{cases} = y \\ \in \sigma_K^\eta y \end{cases}$ if $y \in \begin{cases} P_0 \\ Z \end{cases}$. Then $y \in K \cup A^{\eta-1} \wedge (y' \, \Gamma_i y \vee y' \in A_K^\eta)$.

(A2) If $y \in A^\infty$, then $y \in (A_{K(y)}^{\eta(y)} \cap K(y)) - A^{\eta(y)-1}$.

[*Proofs*. Let the suppositions of (A1) be satisfied. If $y \notin K \cup A^{\eta-1}$, then $\mathbf{y} \notin \mathbf{C}_K^\eta$ for any $\mathbf{y} = (y_k)$ with $y_0 = y$, hence $y \notin A_K$. Therefore, $y \in K \cup A^{\eta-1}$. Let neither $y'\Gamma_i y$ nor $y' \in A_K^\eta$ hold. Then there exists $\mathbf{y}' = (y_k') \in s(y', \sigma_K^\eta) - \mathbf{C}_K^\eta$. Let $\mathbf{y} = (y_k) := := (y, y_0', y_1', y_2', \ldots)$; clearly, $\mathbf{y} \in s(y, \sigma_K^\eta) \subset \mathbf{C}_K$ ($y_1 = y_0' = y'$; $y \in A_K^\eta$), but it is easy to see that $\neg y'\Gamma_i y$ (i.e., $\neg y_1 \Gamma_i y_0$) together with ($\mathbf{y}^{[1]} =$) $\mathbf{y}' \notin \mathbf{C}_K^\eta$ implies $\mathbf{y} \notin \mathbf{C}_K^\eta$ (cf. 8 (i/1) etc.), which is a contradiction. Thus (A1) is proved. (A2) follows from (A1) and 23.1/3b).]

Let $x \in A^\infty$, $\mathbf{x} = (x_k) \in \mathrm{s}(x, \sigma_0)$.

a) If k_0 is such that $x_{k_0+1}\Gamma_i x_{k_0} \wedge \bigwedge_{k \leqslant k_0} x_k \in A^\infty$, then there exists $K \in \mathfrak{K}_i$ such that $K(x_0) \cup \ldots \cup K(x_{k_0}) \subset K$ (as $\mathfrak{K}_i$ is $\cup$-directed), hence $x_{k_0+1}\Gamma_i x_{k_0} \wedge \bigwedge_{k \leqslant k_0} x_k \in K$ (see (A2)), thus $\mathbf{x} \in \mathbf{q}_\Delta(\mathbf{L}_K, \Gamma_i) \subset \mathbf{A}_i$.

b) If $\neg \bigwedge_k x_k \in A^\infty$, then there exists m such that $x_m \notin A^\infty \wedge \bigwedge_{k<m} x_k \in A^\infty$. Then $0 < m$ ($x_0 = x \in A^\infty$). We put $k_0 := m - 1$ and obtain $x_{k_0+1}\Gamma_i x_{k_0}$ from (A1) [by choosing $K := K(x_{k_0})$, $\eta := \eta(x_{k_0})$, $y := x_{k_0}$ $y' := x_{k_0+1}$ in (A1) and using $x_{k_0+1} = x_m \notin A^\infty$]. Thus the case a) has occurred, hence $\mathbf{x} \in \mathbf{A}_i$.

c) Let $\bigwedge_k x_k \in A^\infty$. If $\bigvee_k x_{k+1}\Gamma_i x_k$, then again there occurs the case a) and hence $\mathbf{x} \in \mathbf{A}_i$. Let $\bigwedge_k \neg x_{k+1}\Gamma_i x_k$. Then (A1) gives $\bigwedge_k x_{k+1} \in A^{\eta(x_k)}_{K(x_k)} \subset A^{\eta(x_k)}$ [by choosing $y := x_k$, $y' := x_{k+1}$, $K := K(x_k)$, $\eta := \eta(x_k)$ and using $\neg x_{k+1}\Gamma_i x_k$]. Therefore, $(\eta(x_k))_{k \geqslant 0}$ is a nonincreasing sequence of ordinal numbers, thus there exists k_1 such that $\bigwedge_{k \geqslant k_1} \eta(x_k) = \eta(x_{k_1})$; let $\eta_0 := \eta(x_{k_1})$. Hence $\bigwedge_{k \geqslant k_1} x_{k+1} \in A^{\eta_0}_{K(x_k)}$. Therefore, $(K(x_k))_{k \geqslant k_1}$ is a nonincreasing sequence at $\leqslant_{\eta_0}$ as the well order, thus there exists $k_2 \geqslant k_1$ such that $\bigwedge_{k \geqslant k_2} K(x_k) = K(x_{k_2})$; let $K_0 := K(x_{k_2})$. Clearly, $\mathbf{x}^{[k_2]} \in \mathrm{s}(x_{k_2}, \sigma^{\eta_0}_{K_0}) \subset \mathbf{C}^{\eta_0}_{K_0}$, hence $\bigwedge_{k \geqslant k_2} x_k \in A^{\eta_0 - 1} \cup K_0$ (as $\bigwedge_k \neg x_{k+1}\Gamma_i x_k$, see 8 (i/1)), but $\bigwedge_{k \geqslant k_2} x_k \notin A^{\eta(x_k)-1} = A^{\eta_0 - 1}$ (23.1/3b)), consequently, $\bigwedge_{k \geqslant k_2} x_k \in K_0$. The $\cup$-directness of $\mathfrak{K}_i$ guarantees the existence of $K \in \mathfrak{K}_i$ such that $K(x_0) \cup \ldots \cup K(x_{k_2}) \subset K$. Therefore, $\bigwedge_k x_k \in K$, $\mathbf{x} \in \mathbf{q}_\Delta(\mathbf{L}_K, \Gamma_i) \subset \mathbf{A}_i$.

The considerations a), b) and c) have shown that always $\mathbf{x} \in \mathbf{A}_i$. Consequently, $\mathrm{s}(x, \sigma_0) \subset \mathbf{A}_i$ for each $x \in A^\infty$.

(To be continued)

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