Erich Barvínek

Fundamental central dispersions in a double system $\langle\langle\mathfrak{G}\rangle\rangle$

*Archivum Mathematicum*, Vol. 7 (1971), No. 4, 145--152

Persistent URL: `http://dml.cz/dmlcz/104748`

# FUNDAMENTAL CENTRAL DISPERSIONS IN A DOUBLE SYSTEM $\langle\langle \mathfrak{G} \rangle\rangle$

E. BARVÍNEK, BRNO

Received November 15, 1970

**1. The double system** $\langle\langle \mathfrak{G} \rangle\rangle$. The fundament of the abstract theory of dispersions is an arbitrary group $\mathfrak{G}$ (with the unit $\iota$) in which, in addition to the fundamental subgroup $\mathfrak{F}$, is still an invariant subgroup $\mathfrak{P}$ of the index 2. Besides the decomposition $\mathfrak{G}/_r\mathfrak{F}$ we have also the decomposition $\mathfrak{P}/_r(\mathfrak{P} \cap \mathfrak{F})$. The one-to-one mapping of all classes $\mathfrak{F}\alpha$, $\alpha \in \mathfrak{G}$ onto the set of all carriers determines also a one-to-one mapping of all classes $(\mathfrak{P} \cap \mathfrak{F})\,\beta$, $\beta \in \mathfrak{P}$, but generally only into the set o fall carriers, see fig. 1.

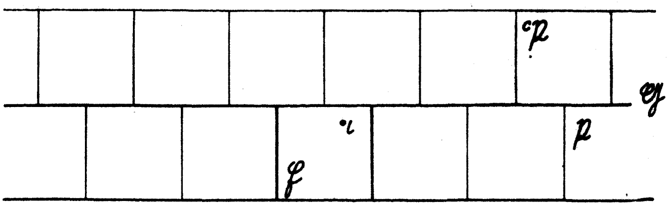


*Fig. 1.*

In every class $\mathfrak{F}\alpha$, $\alpha \in \mathfrak{G}$ there is always contained at most one class $(\mathfrak{P} \cap \mathfrak{F})\,\beta$, $\beta \in \mathfrak{P}$, namely $\mathfrak{P} \cap \mathfrak{F}\alpha$, since $(\mathfrak{P} \cap \mathfrak{F})\,\beta \subseteq \mathfrak{F}\alpha$ gives $\beta \in \mathfrak{F}\alpha$ and thus $\mathfrak{F}\beta = \mathfrak{F}\beta$ and, with regard to the circumstance of $\beta \in \mathfrak{P}$, we have $(\mathfrak{P} \cap \mathfrak{F})\,\beta = \mathfrak{P} \cap \mathfrak{F}\beta = \mathfrak{P} \cap \mathfrak{F}\alpha$.

Iff for any $\alpha \in \mathfrak{G}$ there is $\mathfrak{P} \cap \mathfrak{F}\alpha \neq \emptyset$, every class $\mathfrak{F}\alpha$, $\alpha \in \mathfrak{G}$ will contain just one class $(\mathfrak{P} \cap \mathfrak{F})\,\beta$, $\beta \in \mathfrak{P}$ since for arbitrary $\alpha \in \mathfrak{G}$ there exists $\beta \in \mathfrak{P} \cap \mathfrak{F}\alpha$ and thus $(\mathfrak{P} \cap \mathfrak{F})\,\beta \subseteq \mathfrak{F}\alpha$, see Fig. 2.

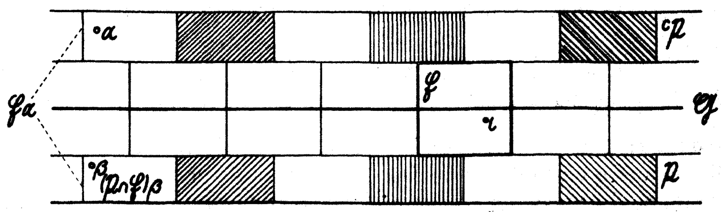


*Fig. 2.*

For an arbitrary subgroup $\mathfrak{A} \subseteq \mathfrak{G}$ let us denote ${}^c\mathfrak{A} = \mathfrak{G} \setminus \mathfrak{A}$ the complement, ${}^n\mathfrak{A} = \{x \in \mathfrak{G};\ x\mathfrak{A} = \mathfrak{A}x\}$ the normalizator, ${}^z\mathfrak{A} = \{x \in \mathfrak{G};\ xa = ax$ for all $a \in \mathfrak{G}\}$ the centralizator, ${}^i\mathfrak{A} = \{x \in \mathfrak{G};\ xa = a^{-1}x$ for all $a \in \mathfrak{A}\}$ the invertor, and ${}^3\mathfrak{A} = \{x \in \mathfrak{A};\ xa = ax$ for all $a \in \mathfrak{A}\}$ the centre of $\mathfrak{A}$.

**1.1. Lemma.** Have arbitrary subgroups $\mathfrak{P}$ and $\mathfrak{F}$ in an arbitrary group $\mathfrak{G}$. Then the following statements are equivalent

a) for any $\alpha \in \mathfrak{G}$ there holds $\mathfrak{P} \cap \mathfrak{F}\alpha \neq \emptyset$,

b) for any $\alpha \in \mathfrak{G}$ there holds $\mathfrak{F} \cap \mathfrak{P}\alpha \neq \emptyset$.

If, moreover, $\mathfrak{P}$ has the index 2, then the mentioned statements are equivalent with any of the statements

c) there holds $\mathfrak{F} \cap {}^c\mathfrak{P} \neq \emptyset$,

d) for any $\alpha \in \mathfrak{G}$ there holds ${}^c\mathfrak{P} \cap \mathfrak{F}\alpha \neq \emptyset$.

Proof. The equivalence of the statements a), b) follows from the equality $\mathfrak{P} \cap \mathfrak{F}\alpha = (\mathfrak{P}\alpha^{-1} \cap \mathfrak{F})\,\alpha$. Their equivalence with the statement c) follows from that ${}^c\mathfrak{P} = \mathfrak{P}\gamma$ for arbitrary $\gamma \in {}^c\mathfrak{P}$ so that then $\mathfrak{F} \cap {}^c\mathfrak{P} \neq \emptyset \equiv \mathfrak{F} \cap \mathfrak{P}\gamma \neq \emptyset$ for any $\gamma \in {}^c\mathfrak{P}$, whereas for any $\beta \in \mathfrak{P}$ there is $\mathfrak{P}\beta = \mathfrak{P}$ and thus $\mathfrak{F} \cap \mathfrak{P}\beta = \mathfrak{F} \cap \mathfrak{P} \neq \emptyset$ automatically. The equivalence of the statement d) with b) follows from the equality ${}^c\mathfrak{P} \cap \mathfrak{F}\alpha = ({}^c\mathfrak{P}\alpha^{-1} \cap \mathfrak{F})\,\alpha = (\mathfrak{P}\gamma\alpha^{-1} \cap \mathfrak{F})\,\alpha$ for arbitrary $\gamma \in {}^c\mathfrak{P}$.

**1.2. Definition.** A double system $\langle\langle\mathfrak{G}\rangle\rangle$ will be called an arbitrary group $\mathfrak{G}$ (with the unit $\iota$) in which a so-called fundamental subgroup $\mathfrak{F}$ and an invarinat subgroup $\mathfrak{P}$ of the index 2 are given, where the centre $\mathfrak{Z}$ of the subgroup $\mathfrak{P} \cap \mathfrak{F}$ is an infinite cyclic group (with a generator $\varepsilon$) whereas the centre of the fundamental subgroup $\mathfrak{F}$ is trivial.

**1.3. Corollary.** $\mathfrak{F} \cap {}^c\mathfrak{P} \neq \emptyset$.

Proof. There holds $\mathfrak{F} \cap {}^c\mathfrak{P} = \emptyset \equiv \mathfrak{F} \subseteq \mathfrak{P} \equiv \mathfrak{F} \cap \mathfrak{P} = \mathfrak{F} \Rightarrow {}^{\mathfrak{Z}}(\mathfrak{P} \cap \mathfrak{F}) = {}^{\mathfrak{Z}}\mathfrak{F}$ and this is a contradiction.

**1.4. Lemma.** For arbitrary $\alpha \in \mathfrak{G}$ there holds ${}^{\mathfrak{Z}}(\alpha^{-1}(\mathfrak{P} \cap \mathfrak{F})\,\alpha) = \alpha^{-1}\mathfrak{Z}\alpha$.

Proof. The function $y = \alpha^{-1}x\alpha$, $x \in \mathfrak{F}$ is an isomorphism of $\mathfrak{F}$ onto $\alpha^{-1}\mathfrak{F}\alpha$ where $\mathfrak{P} \cap \mathfrak{F}$ is mapped on $\alpha^{-1}(\mathfrak{P} \cap \mathfrak{F})\,\alpha$ and $\mathfrak{Z}$ on $\alpha^{-1}\mathfrak{Z}\alpha$. We have $\gamma \in \mathfrak{Z} \equiv x\gamma = \gamma x$ for all $x \in \mathfrak{F} \cap \mathfrak{P} \equiv y(\alpha^{-1}\gamma\alpha) = (\alpha^{-1}\gamma\alpha)\,y$ for all $y \in \alpha^{-1}(\mathfrak{F} \cap \mathfrak{P})\,\alpha \equiv \alpha^{-1}\gamma\alpha \in$ $\in {}^{\mathfrak{Z}}(\alpha^{-1}(\mathfrak{F} \cap \mathfrak{P})\,\alpha)$.

**1.5. Corollary.** For all $\alpha \in \mathfrak{F}$ there holds $\alpha^{-1}\mathfrak{Z}\alpha = \mathfrak{Z}$, and thus $\mathfrak{Z}$ is an invariant subgroup in $\mathfrak{F}$.

**1.6. Theorem.** For any $\alpha \in \mathfrak{F} \cap {}^c\mathfrak{P}$ there holds $\alpha\varepsilon = \varepsilon^{-1}\alpha$.

Proof. Since $\alpha$ transforms $\mathfrak{Z}$ on $\mathfrak{Z}$, it transforms $\varepsilon$ to $\varepsilon$ or to $\varepsilon^{-1}$, and thus there holds $\alpha^{-1}\varepsilon\alpha = \varepsilon^{\pm 1}$ with a suitable sign. Because of $\alpha \in \mathfrak{F} \cap {}^c\mathfrak{P}$ and $\mathfrak{P}$ having the index 2, there is $\tilde{\alpha} = \alpha\beta$, $\beta \in \mathfrak{F} \cap \mathfrak{P}$ a one-to-one mapping of $\mathfrak{F} \cap \mathfrak{P}$ onto $\mathfrak{F} \cap {}^c\mathfrak{P}$. If $\alpha^{-1}\varepsilon\alpha = \varepsilon$, then for all $\tilde{\alpha}$ we should have $\tilde{\alpha}^{-1}\varepsilon\tilde{\alpha} = \beta^{-1}\alpha^{-1}\varepsilon\alpha\beta = \beta^{-1}\varepsilon\beta = \varepsilon$ and thus $\varepsilon \in {}^{\mathfrak{Z}}\mathfrak{F}$, which is a contradiction. Therefore it necessarily holds $\alpha^{-1}\varepsilon\alpha = \varepsilon^{-1}$.

**1.7. Remark.** In the system $\langle\langle\mathfrak{G}\rangle\rangle$ we have not only a one-to-one correspondence between all carriers $q$ and all classes $\mathfrak{F}\alpha$, $\alpha \in \mathfrak{G}$, but also between all carriers $q$ and all classes $(\mathfrak{P} \cap \mathfrak{F})\,\beta$, $\beta \in \mathfrak{P}$ owing to the condition $\mathfrak{F} \cap {}^c\mathfrak{P} \neq \emptyset$. Every class $(\mathfrak{F} \cap \mathfrak{P})\,\beta$, $\beta \in \mathfrak{P}$ is of the form $(e, q) \cap \mathfrak{P}$ for just one carrier $q$ where $\beta \in (e, q)$.

The condition $\mathfrak{F} \cap {}^c\mathfrak{P} \neq \emptyset$, or its equivalent $\mathfrak{P} \cap \mathfrak{F}\alpha \neq \emptyset$ for all $\alpha \in \mathfrak{G}$ resp., guarantees that for any complex $\alpha^{-1}\mathfrak{F}A$, $\alpha, A \in \mathfrak{G}$ there exist phases $\beta, B \in \mathfrak{P}$ such that $\alpha^{-1}\mathfrak{F}A = \beta^{-1}\mathfrak{F}B$ because for arbitrary $\beta \in \mathfrak{P} \cap \mathfrak{F}\alpha$, $B \in \mathfrak{P} \cap \mathfrak{F}A$ we have $\alpha^{-1}\mathfrak{F}A = (\alpha^{-1}\mathfrak{F})(\mathfrak{F}A) = (\beta^{-1}\mathfrak{F})(\mathfrak{F}B) = \beta^{-1}\mathfrak{F}B$. Hence it further follows that $\mathfrak{P} \cap$ $\cap\, \alpha^{-1}\mathfrak{F}A = \mathfrak{P} \cap \beta^{-1}\mathfrak{F}B = \beta^{-1}(\mathfrak{P} \cap \mathfrak{F})\,B$ so that between all complexes $\alpha^{-1}\mathfrak{F}A$ in $\mathfrak{G}$ and all complexes $\beta^{-1}(\mathfrak{P} \cap \mathfrak{F})\,B$ in $\mathfrak{P}$ we have a one-to-one correspondence on the basis of the equation $\mathfrak{P} \cap \alpha^{-1}\mathfrak{F}A = \beta^{-1}(\mathfrak{P} \cap \mathfrak{F}) B$. I. e. every complex $\beta^{-1}(\mathfrak{F}\cap\mathfrak{P})\beta$ in $\mathfrak{P}$ is of the form $(q, Q) \cap \mathfrak{P}$ where $\beta \in (e, Q)$, $B \in (e, Q)$.

**1.8. Lemma.** For $\alpha, \beta \in (e, q)$ there holds $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon\beta$ iff, either $\alpha, \beta \in \mathfrak{P} \cap (e, q)$ or $\alpha, \beta \in {}^c\mathfrak{P} \cap (e, q)$.

Proof. For $\alpha, \beta \in (e, q)$ we have $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon\beta \equiv \beta\alpha^{-1}\varepsilon = \varepsilon\beta\alpha^{-1} \equiv \beta\alpha^{-1} \in \mathfrak{F}\cap\mathfrak{P} \equiv$ $\equiv \beta \in (e, q) \cap \mathfrak{P}\alpha \equiv \alpha, \beta \in \mathfrak{P} \cap (e, q)$ or $\alpha, \beta \in {}^c\mathfrak{P} \cap (e, q)$.

**1.9. Corollary.** All $\alpha \in (e, q) \cap \mathfrak{P}$ transform $\varepsilon$ to the only and same element $\varphi$, whereas all $\beta \in {}^c\mathfrak{P} \cap (e, q)$ transform $\varepsilon$ to the element $\varphi^{-1}$ or $\varepsilon^{-1}$ to $\varphi$.

**1.10. Remark.** In comparison with $\langle\mathfrak{G}\rangle$ the inclusion $\mathfrak{F} \subseteq {}^n\mathfrak{Z}$ holds in $\langle\langle\mathfrak{G}\rangle\rangle$, but it does not hold an inclusion like $\mathfrak{F} \subseteq {}^z\mathfrak{Z}$. Further in $\langle\langle\mathfrak{G}\rangle\rangle$, for $\alpha, \beta \in \mathfrak{G}$, we have $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon\beta$ iff $\beta\alpha^{-1} \in {}^z\mathfrak{Z}$. According to 1.6., moreover, we have here that ${}^z\mathfrak{Z} \cap (\mathfrak{F} \cap {}^c\mathfrak{P}) = \emptyset$, and evidently $\mathfrak{F} \cap \mathfrak{P} \subseteq {}^z\mathfrak{Z}$. Hence $\mathfrak{F} \cap {}^z\mathfrak{Z} \subseteq \mathfrak{P}$ and thus $\mathfrak{F} \cap {}^z\mathfrak{Z} \subseteq \mathfrak{P} \cap \mathfrak{F} \subseteq {}^z\mathfrak{Z} \cap \mathfrak{F}$ so that there holds $\mathfrak{F} \cap {}^z\mathfrak{Z} = \mathfrak{P} \cap \mathfrak{F}$. As a matter of fact, it is another proof of the lemma 1.8.

Further, in $\langle\langle\mathfrak{G}\rangle\rangle$ for $\alpha, \beta \in \mathfrak{G}$ we have $\alpha^{-1}\mathfrak{Z}\alpha = \beta^{-1}\mathfrak{Z}\beta$ iff $\beta\alpha^{-1} \in {}^n\mathfrak{Z}$.

**1.11. Theorem.** For an arbitrary carrier $q$ the centre ${}^3(\mathfrak{P} \cap (q, q))$ of the subgroup $\mathfrak{P} \cap (q, q)$ is an infinite cyclic group. For all $\alpha \in (e, q)$ there holds ${}^3(\mathfrak{P} \cap (q, q)) =$ $= \alpha^{-1}\mathfrak{Z}\alpha$. One of its generators is $\alpha^{-1}\varepsilon\alpha$, independently on the choice of $\alpha$ in $\mathfrak{P} \cap (e, q)$, the second is $\beta^{-1}\varepsilon\beta$, independently on the choice of $\beta$ in ${}^c\mathfrak{P} \cap (e, q)$, being necessarily $(\beta^{-1}\varepsilon\beta)^{-1} = \alpha^{-1}\varepsilon\alpha$.

Proof. We are going to link up with 1.4. Every subgroup $\alpha^{-1}(\mathfrak{P} \cap \mathfrak{F})\,\alpha$, conjugated with $\mathfrak{P} \cap \mathfrak{F}$ by means of the element $\alpha \in \mathfrak{G}$ is of the form $\mathfrak{P} \cap (q, q)$ for a suitable carrier $q$, where $\alpha \in (e, q)$, see 1.7., and vice versa, for every carrier there is $\mathfrak{P} \cap (q, q) = \alpha^{-1}(\mathfrak{P} \cap \mathfrak{F})\,\alpha$ where $\alpha \in (e, q)$ being arbitrary.

According to 1.4. it is evident that for all $\alpha \in (e, q)$ ${}^3(\mathfrak{P} \cap (q, q)) = \alpha^{-1}\mathfrak{Z}\alpha$ is an infinite cyclic group with generators $\alpha^{-1}\varepsilon^{\pm 1}\alpha$. For all $\alpha \in \mathfrak{P} \cap (e, q)$, according to 1.8., $\alpha^{-1}\varepsilon\alpha$ is always the same generator, whereas for all $\beta \in {}^c\mathfrak{P} \cap (e, q)$ is $\beta^{-1}\varepsilon\beta$ the other generator. At the same time $\beta\alpha^{-1} \in {}^c\mathfrak{P} \cap \mathfrak{F}$ and thus $\beta\alpha^{-1}\varepsilon = \varepsilon^{-1}\beta\alpha^{-1}$ or $\beta^{-1}\varepsilon^{-1}\beta =$ $= \alpha^{-1}\varepsilon\alpha$ or $(\beta^{-1}\varepsilon\beta)^{-1} = \alpha^{-1}\varepsilon\alpha$.

**1.12. Definition.** Put $\varphi_q = \beta^{-1}\varepsilon\beta$ for every carrier $q$, where $\beta \in \mathfrak{P} \cap (e, q)$. Iff for all $\alpha \in \mathfrak{P}$ there holds $\{q;\ {}^3(\mathfrak{P} \cap (q, q)) = \alpha^{-1}\mathfrak{Z}\alpha\} = \{q;\ \varphi_q = \alpha^{-1}\varepsilon\alpha\}$, then $\varphi_q$ is called the central dispersion of the carrier $q$, and $\langle\langle\mathfrak{G}\rangle\rangle$ is called the system with fundamental central dispersions.

**1.13. Remark.** In the system $\langle\langle\mathfrak{G}\rangle\rangle$ with fundamental central dispersions the same centres have the same central dispersion without regard to which carriers they belong.

**1.14. Theorem.** If $\varphi$ is a fundamental central dispersion, then $\varphi^{-1}$ is not a fundamental central dispersion for any carrier.

Proof. If $\varphi^{-1} = \alpha^{-1}\varepsilon\alpha$ were for some $\alpha \in (e, q) \cap \mathfrak{P}$, then it would be $\varphi^{-1} \in {}^3(\mathfrak{P} \cap (q, q))$ and also $\varphi \in {}^3(\mathfrak{P} \cap (q, q))$, where $\varphi = \alpha^{-1}\varepsilon\alpha$ owing to $\varphi$ being a fundamental central dispersion. Hence $\varphi^{-1} = \alpha^{-1}\varepsilon^{-1}\alpha$ and thus $\varepsilon = \varepsilon^{-1}$, which is a contradiction.

**1.15. Lemma.** If for one $\alpha \in \mathfrak{P}$ there is $\{q;\ {}^3(\mathfrak{P} \cap (q, q)) = \alpha^{-1}\mathfrak{Z}\alpha\} = \{q;\ \varphi_q =$ $= \alpha^{-1}\varepsilon\alpha\}$, then it holds ${}^n\mathfrak{Z} \cap \mathfrak{P} = {}^z\mathfrak{Z} \cap \mathfrak{P}$ and ${}^n\mathfrak{Z} \cap {}^c\mathfrak{P} = {}^1\mathfrak{Z} \cap {}^c\mathfrak{P}$.

Proof. Put $N = \{q;\ {}^{3}(\mathfrak{P} \cap (q, q)) = \alpha^{-1}\mathfrak{Z}\alpha\}$. Let us mention that $\alpha \in \mathfrak{P}$.

I. There holds $\bigcup_{q \in N} (e, q) \cap \mathfrak{P} = ({}^{n}\mathfrak{Z} \cap \mathfrak{P})\alpha$ because $\beta \in \bigcup_{q \in N} (e, q) \cap \mathfrak{P} \equiv \beta \in (e, q) \cap \cap \mathfrak{P},\ q \in N \equiv \beta \in (e, q) \cap \mathfrak{P},\ {}^{3}\mathfrak{P} \cap (q, q)) = \beta^{-1}\mathfrak{Z}\beta = \alpha^{-1}\mathfrak{Z}\alpha \equiv \beta^{-1}\mathfrak{Z}\beta = \alpha^{-1}\mathfrak{Z}\alpha$, $\beta \in \mathfrak{P} \equiv \beta\alpha^{-1} \in {}^{n}\mathfrak{Z} \cap \mathfrak{P} \equiv \beta \in ({}^{n}\mathfrak{Z} \cap \mathfrak{P})\, \alpha$.

II. There holds $\bigcup_{q \in N} (e, q) \cap {}^{c}\mathfrak{P} = ({}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P})\alpha$ because $\beta \in \bigcup_{q \in N} (e, q) \cap {}^{c}\mathfrak{P} \equiv \equiv \beta \in (e, q) \cap {}^{c}\mathfrak{P},\ q \in N \equiv \beta \in (e, q) \cap {}^{c}\mathfrak{P},\ {}^{3}(\mathfrak{P} \cap (q, q)) = \beta^{-1}\mathfrak{Z}\beta = \alpha^{-1}\mathfrak{Z}\alpha \equiv \equiv \beta^{-1}\mathfrak{Z}\beta = \alpha^{-1}\mathfrak{Z}\alpha,\ \beta \in {}^{c}\mathfrak{P} \equiv \beta\alpha^{-1} \in {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} \equiv \beta \in ({}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P})\, \alpha$.

III. Suppose that $N = \{q;\ \varphi_q = \alpha^{-1}\varepsilon\alpha\}$. We are going to show that ${}^{n}\mathfrak{Z} \cap \mathfrak{P} \subseteq \subseteq {}^{z}\mathfrak{Z} \cap \mathfrak{P},\ {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} \subseteq {}^{i}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$. For $\gamma \in {}^{n}\mathfrak{Z} \cap \mathfrak{P}$ put $\beta = \gamma\alpha$. Then $\beta \in ({}^{n}\mathfrak{Z} \cap \mathfrak{P})\alpha$ and thus $\beta \in \bigcup_{q \in N} (e, q) \cap \mathfrak{P}$, and consequently $\beta^{-1}\varepsilon\beta = \alpha^{-1}\varepsilon\alpha$ so that $\beta\alpha^{-1} \in {}^{z}\mathfrak{Z}$ and therefore $\gamma \in {}^{z}\mathfrak{Z} \cap \mathfrak{P}$. Similarly put $\beta = \gamma\alpha$ for $\gamma \in {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$. Then $\beta \in ({}^{n}\mathfrak{Z} \cap \mathfrak{P})\, \alpha$ and accordingly $\beta \in \bigcup_{q \in N} (e, q) \cap {}^{c}\mathfrak{P}$ so that, according to 1.9., we have $\beta^{-1}\varepsilon^{-1}\beta = = \alpha^{-1}\varepsilon\alpha$ and thus $\beta\alpha^{-1} \in {}^{i}\mathfrak{Z}$ and consequently $\gamma \in {}^{i}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$.

IV. In the system $\langle\langle\mathfrak{G}\rangle\rangle$ always holds ${}^{n}\mathfrak{Z} = {}^{z}\mathfrak{Z} \cup {}^{i}\mathfrak{Z}$ and thus ${}^{z}\mathfrak{Z} \cap \mathfrak{P} = \subseteq {}^{n}\mathfrak{Z} \cap \mathfrak{P},\ {}^{i}\mathfrak{Z} \cap {}^{c}\mathfrak{P} \subseteq {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$. Hence with regard to III. the assertion follows.

**1.16. Lemma.** In the system $\langle\langle\mathfrak{G}\rangle\rangle$ the statements are equivalent

a) ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z} \cap \mathfrak{P},\ {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} = {}^{i}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$,

b) ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}, \qquad {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} = {}^{i}\mathfrak{Z}$,

c) ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}$,

d) ${}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} = {}^{i}\mathfrak{Z}$,

e) for arbitrary $q$ and arbitrary $\alpha \in (e, q)$ in the denotation $\varphi_q = \tilde{\alpha}^{-1}\varepsilon\tilde{\alpha}$ for $\tilde{\alpha} \in \mathfrak{P} \cap (e, q)$ there holds $\beta \in \mathfrak{P} \cap ({}^{n}\mathfrak{Z})\, \alpha \equiv \beta\varphi_q = \varepsilon\beta$,

f) for arbitrary $q$ and arbitrary $\alpha \in (e, q)$ in the denotation $\varphi_q = \tilde{\alpha}^{-1}\varepsilon\tilde{\alpha}$ for $\tilde{\alpha} \in \mathfrak{P} \cap (e, q)$ there holds $\beta \in {}^{c}\mathfrak{P} \cap ({}^{n}\mathfrak{Z})\, \alpha \equiv \beta\varphi_q = \varepsilon^{-1}\beta$,

g) for arbitrary $\alpha \in {}^{n}\mathfrak{Z} \cap \mathfrak{P}$ and arbitrary $\beta \in {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$ there holds $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon^{-1}\beta$,

h) for arbitrary $\alpha, \beta \in \mathfrak{G}$ there holds $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon^{-1}\beta$ iff $\alpha, \beta$ are in the same class of the decomposition $\mathfrak{G}/_{r}{}^{n}\mathfrak{Z}$ and in the opposite classes of the factor group $\mathfrak{G}/\mathfrak{P}$.

Proof. I. Evidently b) $\Rightarrow$ a). Let a) hold. Then ${}^{z}\mathfrak{Z} \cap {}^{c}\mathfrak{P} = ({}^{z}\mathfrak{Z} \cap {}^{c}\mathfrak{P}) \cap \cap ({}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}) = {}^{z}\mathfrak{Z} \cap {}^{i}\mathfrak{P} \cap {}^{c}\mathfrak{P} = \emptyset$ and thus ${}^{z}\mathfrak{Z} \subseteq \mathfrak{P}$ so that ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}$. Similarly ${}^{i}\mathfrak{Z} \cap \mathfrak{P} = ({}^{i}\mathfrak{Z} \cap \mathfrak{P}) \cap ({}^{n}\mathfrak{Z} \cap \mathfrak{P}) = {}^{i}\mathfrak{Z} \cap {}^{z}\mathfrak{Z} \cap \mathfrak{P} = \emptyset$ and consequently ${}^{i}\mathfrak{Z} \subseteq {}^{c}\mathfrak{P}$ so that ${}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} = {}^{i}\mathfrak{Z}$. We have proved that a) $\Rightarrow$ b).

II. Evidently b) $\Rightarrow$ c), d) Let c) hold. Then ${}^{i}\mathfrak{Z} = {}^{n}\mathfrak{Z} \setminus {}^{z}\mathfrak{Z} = {}^{n}\mathfrak{Z} \cap {}^{cz}\mathfrak{Z} = {}^{n}\mathfrak{Z} \cap \cap {}^{c}({}^{n}\mathfrak{Z} \cap \mathfrak{P}) = {}^{n}\mathfrak{Z} \cap ({}^{cn}\mathfrak{Z} \cup {}^{c}\mathfrak{P}) = {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$. We can see that c) $\Rightarrow$ d), b). Similarly d) $\Rightarrow$ c), b).

III. Let $q$ be an arbitrary carrier. Put $\varphi_g = \tilde{\alpha}^{-1}\varepsilon\tilde{\alpha}$ for $\tilde{\alpha} \in \mathfrak{P} \cap (e, q)$. For arbitrary $\alpha \in (e, q)$ there is $({}^{n}\mathfrak{Z})\, \alpha = ({}^{n}\mathfrak{Z})\, \tilde{\alpha}$ because $\tilde{\alpha} \in \mathfrak{J}\alpha$. There holds $\beta\varphi_g = \varepsilon\beta \equiv \beta^{-1}\varepsilon\beta = = \tilde{\alpha}^{-1}\varepsilon\tilde{\alpha} \equiv \beta\tilde{\alpha}^{-1} \in {}^{z}\mathfrak{Z}$. From the other side $\beta\tilde{\alpha}^{-1} \in {}^{n}\mathfrak{Z} \cap \mathfrak{P} \equiv \beta \in \mathfrak{P}\tilde{\alpha} \cap ({}^{n}\mathfrak{Z})\, \alpha = = \mathfrak{P} \cap ({}^{n}\mathfrak{Z})\, \alpha$. We can see that, iff c) holds, then $\beta\varphi_g = \varepsilon\beta \equiv \beta \in \mathfrak{P} \cap ({}^{n}\mathfrak{Z})\alpha$ holds for arbitrary $\alpha \in (e, q)$ adn consequently e).

Similarly $\beta\varphi_g = \varepsilon^{-1}\beta \equiv \beta^{-1}\varepsilon^{-1}\beta = \tilde{\alpha}^{-1}\varepsilon\tilde{\alpha} \equiv \beta\tilde{\alpha}^{-1} \in {}^{i}\mathfrak{Z}$. On the other hand $\beta\tilde{\alpha}^{-1} \in \in {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} \equiv \beta \in {}^{c}\mathfrak{P}\tilde{\alpha} \cap ({}^{n}\mathfrak{Z})\, \tilde{\alpha} = {}^{c}\mathfrak{P} \cap ({}^{n}\mathfrak{Z})\, \alpha$. We can see that, iff d) holds, then $\beta\varphi_g = \varepsilon^{-1}\beta \equiv \beta \in {}^{c}\mathfrak{P} \cap ({}^{n}\mathfrak{Z})\, \alpha$ holds for arbitrary $\alpha \in (e, q)$ and thus f).

IV. Evidently b) $\Rightarrow$ g). Let g) hold. Put $\alpha^{-1}\varepsilon\alpha = \varphi$ for arbitrary $\alpha \in {}^{n}\mathfrak{Z} \cap \mathfrak{P}$. Then $\beta^{-1}\varepsilon^{-1}\beta = \varphi$ holds for arbitrary $\beta \in {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$. As an arbitrary $\gamma \in {}^{n}\mathfrak{Z}$ transforms

$\mathfrak{Z}$ to $\mathfrak{Z}$, there is $\varphi = \varepsilon^{\pm 1}$. As $\varphi$ does not depend on the choice of $\beta$ in ${}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$, we can choose $\tilde{\beta} \in \mathfrak{F} \cap {}^{c}\mathfrak{P}$. As $\varphi$ depends neither on the choice of $\alpha$ in ${}^{n}\mathfrak{Z} \cap \mathfrak{P}$, we can choose $\tilde{\alpha} \in \mathfrak{F} \cap \mathfrak{P}$. Hence $\varphi = \tilde{\alpha}^{-1}\varepsilon\tilde{\alpha} = \tilde{\beta}^{-1}\varepsilon^{-1}\tilde{\beta} = \varepsilon$. We get $\alpha^{-1}\varepsilon\alpha = \varepsilon = \beta^{-1}\varepsilon^{-1}\beta$ and accordingly $\alpha \in {}^{z}\mathfrak{Z} \cap \mathfrak{P}$, $\beta \in {}^{i}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$. By this it is proved that g) $\Rightarrow$ a) and thus b), as well.

V. For $\alpha, \beta \in \mathfrak{G}$ there holds $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon^{-1}\beta \equiv \beta\alpha^{-1} \in {}^{i}\mathfrak{Z}$ and likewise there holds $\beta\alpha^{-1} \in {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} \equiv \beta \in ({}^{n}\mathfrak{Z})\,\alpha \cap ({}^{c}\mathfrak{P})\,\alpha$. Iff there holds d), there is $\beta\alpha^{-1} \in {}^{i}\mathfrak{Z} \equiv$ $\equiv \beta\alpha^{-1} \in {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$ and consequently h) holds.

**1.17. Lemma.** If ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}$ holds in a system $\langle\langle\mathfrak{G}\rangle\rangle$, then $\langle\langle\mathfrak{G}\rangle\rangle$ is a system with fundamental central dispersions.

Proof. Take arbitrary $\alpha \in \mathfrak{P}$. Evidently there always holds $\{q;\ \varphi_q = \alpha^{-1}\varepsilon\alpha\} \subseteq$ $\subseteq \{q;\ {}^{z}(\mathfrak{P} \cap (q, q)) = \alpha^{-1}\mathfrak{Z}\alpha\}$. Denote by $N = \{q;\ {}^{z}(\mathfrak{P} \cap (q, q)) = \alpha^{-1}\mathfrak{Z}\alpha\}$. Under the supposition of b) we have—see the proof of 1.15.—that $\bigcup_{q \in N} (e, q) \cap \mathfrak{P} =$ $= ({}^{z}\mathfrak{Z})\,\alpha$. For arbitrary $q \in N$ and arbitrary $\beta \in (e, q) \cap \mathfrak{P}$ we have then $\beta \in ({}^{z}\mathfrak{Z})\,\alpha$ so that $\varphi_q = \beta^{-1}\varepsilon\beta = \alpha^{-1}\varepsilon\alpha$ and consequently $q \in \{q;\ \varphi_q = \alpha^{-1}\varepsilon\alpha\}$ so that $N = \{q;\ \varphi_q = \alpha^{-1}\varepsilon\alpha\}$ also holds. According to the definition 1.12., $\langle\langle\mathfrak{G}\rangle\rangle$ is then a system with fundamental central dispersions.

**1.18. Theorem.** Iff ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}$, then $\langle\langle\mathfrak{G}\rangle\rangle$ is a system with fundamental central dispersions.

Proof. The consequence of 1.17., 1.16., and 1.15.

**1.19. Theorem.** In a system $\langle\langle\mathfrak{G}\rangle\rangle$ with fundamental central dispersions for every carrier $q$, arbitrary phase $\alpha \in (e, q)$ together with fundamental central dispersions $\varphi \in {}^{3}(\mathfrak{P} \cap (q, q))$ and $\varepsilon \in {}^{3}(\mathfrak{P} \cap \mathfrak{F})$ fulfils the Abelian relations

(1) $$\alpha\varphi = \varepsilon\alpha \quad \text{for} \quad \alpha \in (e, q) \cap \mathfrak{P}$$

(2) $$\alpha\varphi = \varepsilon^{-1}\alpha \quad \text{for} \quad \alpha \in (e, q) \cap {}^{c}\mathfrak{P}$$

Proof. The fundamental central dispersion $\varphi$ of the carrier $q$ is defined by the relation $\varphi = \alpha^{-1}\varepsilon\alpha$ for arbitrary $\alpha \in (e, q) \cap \mathfrak{P}$. So we have the relation (1). According to 1.9. we have the relation $\alpha^{-1}\varepsilon^{-1}\alpha = \varphi$ for $\alpha \in (e, q) \cap {}^{c}\mathfrak{P}$, which is the relation (2).

**2.** In an arbitrary system $\langle\langle\mathfrak{G}\rangle\rangle$ always $\mathfrak{P} \cap {}^{n}\mathfrak{Z} \neq \emptyset$ and ${}^{c}\mathfrak{P} \cap {}^{n}\mathfrak{Z} \neq \emptyset$, since $\mathfrak{F} \subseteq {}^{n}\mathfrak{Z}$, $\mathfrak{P} \cap \mathfrak{F} \neq \emptyset$ and ${}^{c}\mathfrak{P} \cap \mathfrak{F} \neq \emptyset$. So for any $\alpha \in \mathfrak{G}$ there is also $\mathfrak{P} \cap ({}^{n}\mathfrak{Z})\alpha =$ $= (\mathfrak{P}\alpha^{-1} \cap {}^{n}\mathfrak{Z})\,\alpha \neq \emptyset$ and ${}^{c}\mathfrak{P} \cap ({}^{n}\mathfrak{Z})\,\alpha = ({}^{c}\mathfrak{P}\alpha^{-1} \cap {}^{n}\mathfrak{Z})\,\alpha \neq \emptyset$.

In an arbitrary system $\langle\langle\mathfrak{G}\rangle\rangle$ for a given $x \in \mathfrak{G}$ there is $\{y \in \mathfrak{G};\ y^{-1}\varepsilon^{-1}y =$ $= x^{-1}\varepsilon x\} = \{y \in \mathfrak{G};\ yx^{-1} \in {}^{i}\mathfrak{Z}\} = ({}^{i}\mathfrak{Z})\,x \neq \emptyset$ since $\mathfrak{F} \cap {}^{c}\mathfrak{P} \subseteq {}^{i}\mathfrak{Z}$.

**2.1. Definition.** A binary relation $<$ on the group $\mathfrak{G}$ will be called a pseudo-order of the system $\langle\langle\mathfrak{G}\rangle\rangle$ if

a) $\alpha < \beta \Rightarrow \alpha \neq \beta, \quad \beta \nless \alpha$

b) $\alpha < \beta \Rightarrow x\alpha < x\beta$ for all $x \in \mathfrak{P}$
$\alpha < \beta \Rightarrow x\alpha > x\beta$ for all $x \in {}^{c}\mathfrak{P}$

c) $\alpha < \beta \Rightarrow \alpha x < \beta x$ for all $x \in \mathfrak{G}$

d) the generator $\varepsilon$ of the centre $\mathfrak{Z}$ of the subgroup $\mathfrak{P} \cap \mathfrak{F}$ fulfils $\iota < \varepsilon$.

**2.2. Theorem.** In a pseudo-ordered system $\langle\langle\mathfrak{G}\rangle\rangle$ hold

$$\alpha, \beta \in \mathfrak{P}, \qquad \alpha < \beta \Rightarrow \alpha^{-1} > \beta^{-1}$$
$$\alpha, \beta \in {}^{c}\mathfrak{P}, \qquad \alpha < \beta \Rightarrow \alpha^{-1} < \beta^{-1}$$

Proof. a) Let $\alpha, \beta \in \mathfrak{P}$, $\alpha < \beta$. Then $\iota = \alpha^{-1}\alpha < \alpha^{-1}\beta$, $\beta^{-1} = \iota\beta^{-1} < \alpha^{-1}$.

b) Let $\alpha, \beta \in {}^{c}\mathfrak{P}$, $\alpha < \beta$. Then $\iota = \alpha^{-1}\alpha > \alpha^{-1}\beta$, $\beta^{-1} = \iota\beta^{-1} > \alpha^{-1}$.

**2.3. Theorem.** In a pseudo-ordered system $\langle\langle\mathfrak{G}\rangle\rangle$ it follows from the relation $\alpha < \beta$ that $\alpha, \beta$ lie in the same class of the factor group $\mathfrak{G}/\mathfrak{P}$.

Proof. Admit that $\alpha < \beta$, $\alpha \in \mathfrak{P}$, $\beta \in {}^{c}\mathfrak{P}$. The other case $\alpha \in {}^{c}\mathfrak{P}$, $\beta \in \mathfrak{P}$, by the multiplication from the right side by an arbitrary element $\gamma \in {}^{c}\mathfrak{P}$, gives $\alpha\gamma < \beta\gamma$, $\alpha\gamma \in \mathfrak{P}$, $\beta\gamma \in {}^{c}\mathfrak{P}$ so that, without any loss of generality, the first case can be considered. Then the multiplication from the left side gives $\iota < \alpha^{-1}\beta$, $\beta^{-1}\alpha > \iota$. By the multiplication from the right side of the first relation by the element $\beta^{-1}\alpha$ we get $\beta^{-1}\alpha < \iota$ which is a contradiction.

**2.4. Theorem.** In a pseudo-ordered system $\langle\langle\mathfrak{G}\rangle\rangle$ there holds

$${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}, \qquad {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} = {}^{i}\mathfrak{Z}.$$

Proof. Let $x \in {}^{n}\mathfrak{Z} \cap \mathfrak{P}$. Then $x\varepsilon = \varepsilon^{\pm 1}x$ and under the influence of $\iota < \varepsilon$ we have $x < x\varepsilon$, $\varepsilon^{-1}x < x$ and consequently it cannot hold $x\varepsilon = \varepsilon^{-1}x$. Therefore it necessarily holds $x\varepsilon = \varepsilon x$ and thus $x \in {}^{z}\mathfrak{Z}$. Then we have ${}^{n}\mathfrak{Z} \cap \mathfrak{P} \subseteq {}^{z}\mathfrak{Z}$.

Let $x \in {}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P}$. Then $x\varepsilon = \varepsilon^{\pm 1}x$ and under the influence of $\iota < \varepsilon$ we have $x > x\varepsilon$, $x < \varepsilon x$ and consequently it cannot hold $x\varepsilon = \varepsilon x$. Therefore it needs hold $x\varepsilon = \varepsilon^{-1}x$ and thus $x \in {}^{i}\mathfrak{Z}$. We have the relation ${}^{n}\mathfrak{Z} \cap {}^{c}\mathfrak{P} \subseteq {}^{i}\mathfrak{Z}$.

Now we have ${}^{n}\mathfrak{Z} \cap \mathfrak{P} \subseteq {}^{z}\mathfrak{Z} = {}^{n}\mathfrak{Z} \setminus {}^{i}\mathfrak{Z} = {}^{n}\mathfrak{Z} \cap {}^{ci}\mathfrak{Z} \subseteq {}^{n}\mathfrak{Z} \cap ({}^{cn}\mathfrak{Z} \cup \mathfrak{P}) = {}^{n}\mathfrak{Z} \cap \mathfrak{P}$ and thus equality holds everywhere.

**2.5. Corollary.** A pseudo-ordered system $\langle\langle\mathfrak{G}\rangle\rangle$ is a system with fundamental central dispersions. For any fundamental central dispersion $\varphi$ there holds $\varphi > \iota$, since $\varphi = \alpha^{-1}\varepsilon\alpha$ for $\alpha \in \mathfrak{P}$. For arbitrary $\mu < \nu \in \mathbf{Z}$ there holds $\varphi^{\mu} < \varphi^{\nu}$ and therefore every centre $\{\varphi^{\nu}\}_{\nu} \in \mathbf{Z}$ is completely ordered by the relation $<$.

**2.6. Theorem.** Let ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}$ hold in a system $\langle\mathfrak{G}\rangle$. Then the relation $<$ betwen the elements $\alpha, \beta \in \mathfrak{G}$ defined by

$$\alpha < \beta \equiv \beta\alpha^{-1} = x^{-1}\varepsilon x \quad \text{for some} \quad x \in \mathfrak{P} \tag{3}$$

is a pseudo-order of the system $\langle\langle\mathfrak{G}\rangle\rangle$.

Proof. Let $\alpha < \beta$ so that for some $\gamma \in \mathfrak{P}$ we have $\beta\alpha^{-1} = \gamma^{-1}\varepsilon\gamma$.

a) If it were $\alpha = \beta$, we should have $\iota = \gamma^{-1}\varepsilon\gamma$ and thus $\gamma = \varepsilon\gamma$ and consequently $\iota = \varepsilon$, which does not hold. If it were $\beta < \alpha$, we should have for some $y \in \mathfrak{P}$ the relation $\alpha\beta^{-1} = y^{-1}\varepsilon y$ or $\beta\alpha^{-1} = y^{-1}\varepsilon^{-1}y$ and thus $\gamma^{-1}\varepsilon\gamma = y^{-1}\varepsilon^{-1}y$ or $y\gamma^{-1} \in {}^{i}\mathfrak{Z} \subseteq {}^{c}\mathfrak{P}$, which is a contradiction, since $y\gamma^{-1} \in \mathfrak{P}$.

b) Choose $x \in \mathfrak{G}$. Multiplicating from the left side by $x$ and from the right side by $x^{-1}$ we get $(x\beta)(x\alpha)^{-1} = x\beta\alpha^{-1}x^{-1} = x\gamma^{-1}\varepsilon\gamma x^{-1} = (\gamma x^{-1})^{-1}\varepsilon(\gamma x^{-1})$. For $x \in \mathfrak{P}$ we have $\gamma x^{-1} \in \mathfrak{P}$ and thus $x\alpha < x\beta$. For $x \in {}^{c}\mathfrak{P}$ there is $\gamma x^{-1} \in {}^{c}\mathfrak{P}$. According to the beginning of paragraph 2 there exists $y \in ({}^{i}\mathfrak{Z})\,\gamma x^{-1}$ such that $y^{-1}\varepsilon^{-1}y = (\gamma x^{-1})^{-1}\varepsilon(\gamma x^{-1}) = (x\beta)(x\alpha)^{-1}$ accordingly $(x\alpha)(x\beta)^{-1} = y^{-1}\varepsilon y$. At the same time $y \in \mathfrak{P}$ because ${}^{i}\mathfrak{Z} \subseteq {}^{c}\mathfrak{P}$. We get $x\beta < x\alpha$.

c) Choose $x \in \mathfrak{G}$. Then $(\beta x)(\alpha x)^{-1} = \beta\alpha^{-1} = \gamma^{-1}\varepsilon\gamma$ and thus there holds $\alpha x < \beta x$.

d) Since $\iota \in \mathfrak{P}$ and it holds $\varepsilon\iota^{-1} = \iota^{-1}\varepsilon\iota$, we have $\iota < \varepsilon$.

**2.7. Corollary.** For any system $\langle\langle\mathfrak{G}\rangle\rangle$ the following statements are equivalent:
a) ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}$,
b) in $\langle\langle\mathfrak{G}\rangle\rangle$ a pseudo-order may be defined,
c) $\langle\langle\mathfrak{G}\rangle\rangle$ is a system with fundamental central dispersions.

**2.8. Remark.** Let $\langle\langle\mathfrak{G}\rangle\rangle$ be a pseudo-ordered system. Then every $\varphi > \iota$ fulfils $\varphi \in \mathfrak{P}$ according to 2.3. Further, $\varphi$ generates an infinite cyclic group $\{\varphi^{\nu}\}_{\nu} \in \mathbf{Z}$ because there holds

$$\ldots < \varphi^{-2} < \varphi^{-1} < \iota < \varphi < \varphi^{2} < \ldots$$

**2.9. Definition.** The pseudo-order from the definition 2.1. will be called the pseudo-order with regard to $\varepsilon$. Similarly it is possible to define the pseudo-order with regard to $\varepsilon^{-1}$.

**2.10. Theorem.** Let $<$ be a pseudo-order with regard to $\varepsilon$. Then the relation $\alpha < \beta$ defined by the relation $\alpha > \beta$ is not a pseudo-order with regard to $\varepsilon$, but it is a pseudo-order with regard to $\varepsilon^{-1}$.

**2.11. Definition.** The pseudo-order (3) of the system $\langle\langle\mathfrak{G}\rangle\rangle$ will be called canonical (with regard to $\varepsilon$).

**2.12. Theorem.** In the canonical pseudo-order there is $\iota < \varphi$ iff $\varphi$ is a fundamental central dispersion.

Proof. $\iota < \varphi \equiv \varphi = x^{-1}\varepsilon x$ for some $x \in \mathfrak{P} \equiv \varphi$ is a fundamental central dispersion.

**2.13. Remark.** The canonical pseudo-order of the system $\langle\langle\mathfrak{G}\rangle\rangle$ (with regard to $\varepsilon$) is unique. An arbitrary pseudo-ordered system $\langle\langle\mathfrak{G}\rangle\rangle$ fulfis the condition ${}^{n}\mathfrak{Z} \cap \mathfrak{P} = {}^{z}\mathfrak{Z}$ and therefore it is possible to be ordered canonically (with regard to $\varepsilon$).

**2.14. Theorem.** Let $<$ be an arbitrary pseudo-order of the system $\langle\langle\mathfrak{G}\rangle\rangle$. If any $\varphi > \iota$ is a fundamental central dispersion, then $<$ is a canonical pseudo-order (with regard to $\varepsilon$). I.e. that the canonical pseudo-order (with regard to $\varepsilon$) is characterized by the property $\varphi > \iota$ iff $\varphi$ is a fundamental central dispersion.

Proof. I. Let $\alpha < \beta$. Then $\beta\alpha^{-1} = x^{-1}\varepsilon x$ for some $x \in \mathfrak{P}$. Then in the acnonical pseudo-order $<_1$ there holds $\iota <_1 \beta\alpha^{-1}$ and therefore $\alpha <_1 \beta$.

II. Let $\alpha <_1 \beta$ in the canonical pseudo-order. Then $\beta\alpha^{-1} = x^{-1}\varepsilon x$ for some $x \in \mathfrak{P}$ and therefore $\iota < \beta\alpha^{-1}$ or $\alpha < \beta$.

We can see both pseudo-order relations $<$ and $<_1$ to be identical.

**2.15. Lemma.** The pseudo-order $<$ of the system $\langle\langle\mathfrak{G}\rangle\rangle$ defines in $\mathfrak{G}$ the order relation $\leq$ iff the relation $<$ is transitive.

Proof. According to 2.1. a) the relation $\leq$ is reflexive and antisymmetric. The transitivity of $<$ is then a necessary and sufficient condition for the transitivity of $\leq$.

**2.16. Theorem.** For the canonical pseudo-order $<$ of the system $\langle\mathfrak{G}\rangle$ the relation $\leq$ is and order relation in $\mathfrak{G}$ iff the composition of each two fundamental central dispersions is again a fundamental central dispersion.

Proof. I. Let $<$ be transitive. Let $\varphi$, $\psi$ be fundamental central dispersions. Then we have $\iota < \varphi$, $\varphi < \psi\varphi$ and thus $\iota < \psi\varphi$. According to 2.12., $\psi\varphi$ is a fundamental central dispersion.

II. Let the composition of each two fundamental central dispersions be again a fundamental central dispersion. Let $\alpha < \beta$, $\beta < \gamma$. Then $\beta\alpha^{-1}$, $\gamma\beta^{-1}$ are fundamental central dispersions according to 2.12., and consequently $\gamma\alpha^{-1} = (\gamma\beta^{-1})(\beta\alpha^{-1})$ is also a fundamental central dispersion so that $\gamma\alpha^{-1} = x^{-1}\varepsilon x$ for some $x \in \mathfrak{P}$. According to the definition of the canonical pseudo-order is then $\alpha < \gamma$ so that the relation $<$ is a transitive one.

## REFERENCES


[1] Borůvka, O., *Lineare differentialtransformationen 2. Ordnung*, Deutscher Verlag der Wissenschaften, Berlin 1967

[2] Borůvka, O., *Über dine Charakterisierung der allgemeinen Dispersionen linearer Differentialgleichungen 2. Ordnung*, Math. Nachr. Bd. 38 (1968), H. 5/6, 261—266

[3] Borůvka, O., *Algebraic elements in the transformation theory of 2nd order linear oscillatory differential equations*, Acta Univ. Com. Bratislava, Math. XVII, 1967, 27—36

[4] Borůvka, O., *Sur quelques propriétés de structure du groupe de phases des équations différentielles linéaires du deuxième ordre*, Revue Roumaine de Mathématiques pures et appliquées, XV (1970); pp. 1345—1356

[5] Barvínek, E., *Fundamental central dispersion in a simple system* Arch. Math. 1, VII: 1—8, 1971



*E. Barvínek*
*Department of Mathematics*
*J. E. Purkyně University Brno*
*Czechoslovakia*