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CERTAIN HIGHER MONOTONICITY PROPERTIES OF i-th DERIVATIVES OF SOLUTIONS OF $y'' + a(t)y' + b(t)y = 0$

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1. INTRODUCTION

In this paper there are derived certain higher monotonicity properties of the sequences $\{R_k^{(i)}\}_{k=0}^{\infty}$ denoted simply by $\{R_k^{(i)}\}$, where the terms $R_k^{(i)}$ are defined, for fixed $\lambda > -1$, by

$$R_k^{(i)} \equiv R_k^{(i)}(W, \lambda) = \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} W(t) \left| \exp\left[\frac{1}{2} \int a_i(t)\,\mathrm{d}t\right] y^{(i)}(t) \right|^{\lambda} \mathrm{d}t,$$

where $y(t)$ is an arbitrary (non trivial) solution of

$$y'' + a(t)\,y' + b(t)\,y = 0, \tag{1.1}$$

and $t_0^{(i)}, t_1^{(i)}, \ldots$ [2]) is any sequence of consecutive zeros of i-th derivative ($i = 0, 1, 2, \ldots$) of any solution $z(t)$ of (1.2) which may or may not be linearly independent of $y(t)$. The functions $a_i(t)$ will be defined later. The indefinite integral denotes here as well as throughout this paper any primitive function. The condition $\lambda > -1$ is required to assure convergence of the integral defining $R_k^{(i)}$ and the function $W(t)$ is any sufficiently monotonic function taken subject to the same restriction. The equation (1.1) together with everything else in this paper is considered in the real domain.

The function $\varphi(t)$ is said to be *n-times monotonic* or *monotonic of order n* on an interval (a, b) if

$$(-1)^j\, \varphi^{(j)}(t) \geqq 0 \qquad (j = 0, 1, \ldots, n;\ t \in (a, b)). \tag{1.2}$$

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[1]) The main part of this work was done during the author's stay at the Istituto Matematico "Ulisse Dini", Universitá degli Studi, Firenze and was published there as a preprint No. 1972/17. The principal result of this paper was presented on EQUADIFF III, Czechoslovak Conference on Differential Equations and Their Applications, held in Brno in August 1972.

[2]) It is convenient to use the notation $\{t_k\}$, $\{t_k'\}$, $\{t_k''\}$ for $n = 0, 1, 2$.

For such a function we write $\varphi(t) \in M_n(a, b)$. If (1.2) holds for $n = \infty, f(t)$ is said to be *completely monotonic* and $M_n \equiv M_n(0. \infty)$.[3])

A sequence $\{t_k\}$ is said to be *n-times monotonic* if

$$(1.3) \qquad (-1)^j \Delta^i t_k \geqq 0 \qquad (j = 0, 1, \ldots,; k = 0, 1, \ldots).$$

Here $\Delta^0 t_k = t_k$, $\Delta t_k = t_{k+1} - t_k, \ldots, \Delta^n t_k = \Delta^{n-1} t_{k+1} - \Delta^{n-1} t_k$. If (1.3) holds for $n = \infty$, $\{t_k\}$ is called *completely monotonic.*

In 1963 L. Lorch and P. Szego ([3], [4]) found out a simple sufficient condition for monotonicity of order n of the sequence of differences $\{\Delta t_k\}_{k=0}^{\infty}$ in case of $a(t) \equiv 0$ $(b(t) \equiv q(t))$, i.e. for the differential equation

$$(1.4) \qquad u'' + q(t)\, u = 0.$$

They have proved that the conditions $q' \in M_n$, $q(\infty) > 0$ are sufficient for the monotonicity of order n of $\{\Delta t_k\}$ and $\{M_k\}$, where

$$M_k = \int_{t_k}^{t_{k+1}} |\, y(t)\,|^\lambda \,\mathrm{d}t, \qquad \lambda > -1, k = 0, 1, 2, \ldots$$

In connection with the above mentioned results, L. Lorch pointed out the problem of finding some sufficient conditions for higher monotonicity of sequences $\{\Delta t_k^{(i)}\}_{k=0}^{\infty}$ for some integer $i \geqq 1$. In the paper [9] it is shown that the above mentioned conditions $q' \in M_n$, $q(\infty) > 0$ are sufficient for monotonicity of order $n - 2$ of the sequences $\{\Delta t'_k\}$ and $\{M'_k\}$, where

$$M'_k = \int_{t_k}^{t_{k+1}} |\, q^{-\frac{1}{4}}(t)\, y'(t)\,|^\lambda \,\mathrm{d}t, \qquad \lambda > -1, k = 0, 1, 2, \ldots$$

The paper [9] gives, however, the solution of Lorch's problem for the equation (1.5) in case of $i = 1$. The sequences $\{t'_k\}$ or $\{M'_k\}$, respectively, are studied in [9] as sequences $\{T_k\}$, $\{M_k\}$, respectively, of consecutive zeros or corresponding terms, respectively, of any solution $Y(t) = [q(t)]^{-\frac{1}{4}}\, y(t)$ of

$$Y'' + Q(t)\, Y = 0,$$

where

$$Q(t) = q - \frac{3}{4}\frac{q'^2}{q^2} + \frac{1}{2}\frac{q''}{q}.$$

This way is, however, not possible to be used for higher derivatives.

[3]) This is a usual definition of higher monotonicity. A little different one (n-th and $(n-1)$-th derivatives are not required) is given, e.g. in [8].

Later it was noticed in [5] by L. Lorch, P. Szego and M. Muldoon that one may modify M_k and M'_k by including an arbitrary function $W(t)$ provided that $W(t)$ is sufficiently monotonic. This paper contains the solution of the mentioned problem for the more general equation (1.1) and any nonnegative integer i. The terms M_k and M'_k are included as special cases in the general term $R_k^{(i)}$.

It proved to be fruitful to study the sequences $\{R_k^{(i)}\}$ directly for the more general equation (1.1) since the derivative of any solution of (1.1) satisfies certain equation of the same type. Some properties of $\{R_k^{(i)}\}$ can be obtained by suitable transformation of equation (1.1) into (1.4).

By special choice of $W(t)$ and λ in the integral defining the terms $R_k^{(i)}$, we can obtain $R_k^{(i)}(W, \lambda)$ having different geometrical (or other) meaning.

In case of $W(t) \equiv 1$, $\lambda = 0$ we have

$$R_k^{(i)} \equiv R_k^{(i)}(1, 0) = t_{k+1}^{(i)} - t_k^{(i)} = \Delta t_k^{(i)}. \tag{1.5}$$

By other special choice we have, e.g.

$$R'_k(W_1, 2) = -\Delta y^2(t'_k) \qquad (k = 0, 1, 2, \ldots) \tag{1.6}$$

or

$$R_k^{(i)}(W_i, 2) = -\Delta[y^{(i-1)}(t_k^{(i)})]^2 \qquad (i = 1, 2, \ldots;\ k = 0, 1, 2, \ldots) \tag{1.7}$$

or

$$\Delta R_k^{(i)}(W_1^*, 1) = \Delta| y^{(i-1)} t_{2k}^{(i)} | \qquad (i = 1, 2, \ldots,\ k = 0, 1, 2, \ldots). \tag{1.8}$$

Some properties of (1.6), (1.7) and (1.8) are given in Sections 7, 8 as conclusions of the principal Theorem 5.1. In Section 3 there are some former results concerning the Sturm-Liouville differential equation (1.4) extended to the more general equation (1.1).

Certain applications of the derived results for Bessel functions are given in [11].

All main results are for simplicity formulated for $t \in (0, \infty)$ but there are no difficulties to replace the mentioned interval by (a, ∞) where $a \in (-\infty, \infty)$ is an arbitrary number. All lemmas and theorems are stated as to be applied to n-times monotonic functions and sequences. For the case of complete monotonicity it suffices to put $n = \infty$.

2. PRELIMINARY RESULTS

Let $a_0(t) \equiv a(t)$, $b_0(t) \equiv b(t)$ be continuous and sufficiently differentiable functions on $(0, \infty)$. Let $a_i(t)$, $b_i(t)$ be defined recurrently for $i = 1, 2, 3, \ldots$ by formulas

$$a_i(t) = a_{i-1}b'_{i-1}/b_{i-1} \qquad b_i(t) = b_{i-1} + a'_{i-1} - a_{i-1}b'_{i-1}/b_{i-1}. \tag{2.1$_i$}$$

Suppose that $b_i(t) \neq 0$ for $t \in (0, \infty)$ and all needed i. Let the function $f_i(t)$ be defined by

$$(2.2_i) \qquad f_i(t) = b_i - \frac{1}{2} a_i' - \frac{1}{4} a_i^2 \qquad (i = 0, 1, 2, \ldots).$$

Lemma 2.1. *Let $i \geqq 1$ be an arbitrary but fixed integer and $a_0(t)$ functions such that $a_j(t)$, $b_j(t)$ defined by* (2.1) *are differentiable $b_j(t) \neq 0$ for $j = 0, 1, \ldots, i$ and $t \in (0, \infty)$. Let $y = x(t)$, $\circ(t)$ be non-trivial linearly independent solutions of*

$$(1.1_0) \qquad y'' + a_0(t)\, y' + b_0(t)\, y = 0.$$

Then $y = x^{(i)}(t)$, $z^{(i)}(t)$ are non-trivial linearly indepedent solutions of

$$(1.1_i) \qquad y'' + a_i(t)\, y' + b_i(t)\, y = 0.$$

Proof. We use the induction with respect to i. For $i = 0$ the lemma is trivial. Let us suppose that the assertion holds for some j, $0 \leqq j < i$, consequently the pair $x^{(j)}(t)$, $z^{(j)}(t)$ is a fundamental system of solutions of the equation (1.1_j).

Any solution $y(t)$ of (1.1_j) is at the same time a solution of

$$y''' + a_j y'' + (b_j + a_j')\, y' + b_j' y = 0$$

and hence by (1.1_j) a solution of

$$y''' + (a_j - b_j'/_j b)\, y'' + (b_j + a_j' - a_j b_j'/b_j)\, y' = 0,$$

i.e. of the equation (1.1_{j+1}), where a_{j+1}, b_{j+1} are defined by (2.1_{j+1}). Thus the functions $x^{(j+1)}(t)$, $z^{(j+1)}(t)$ satisfy the equation (1.1_{j+1}).

It remains to show that $x^{(j+1)}(t)$, $z^{(j+1)}(t)$ are linearly independent. Let us suppose the contrary. Then there exists a number $k_1 \neq 0$ such that $x^{(j+1)}(t) = k_1 z^{(j+1)}(t)$ and thus $x^{(j)}(t) = k_1 z^{(j)}(t) + k_2$. In case of $k_2 = 0$ the functions $x^{(j)}(t)$, $z^{(j)}(t)$ are not linearly independent which is a contradiction with the assumptions. If $k_2 \neq 0$, then (1.1_j) must have a solution $y = k_2$. This is, however, possible only if $b_j(t) \equiv 0$ and we have again the contradiction. Hence the lemma is proved.

Lemma 2.2. *Let $a_i(t)$, $b_i(t)$ be defined by* (2.1) *and suppose that $a_i(t) \in C_1$, $b_i(t) \in C_0$ for $t > 0$ and an arbitrary but fixed integer i. The transformation*

$$(2.3_i) \qquad y(t) = u(t) \exp\left\{-\frac{1}{2} \int a_i(t)\, \mathrm{d}t\right\},$$

transforms (1.1_i) into the differential equation

$$(2.4_i) \qquad u'' + f_i(t)\, u = 0,$$

where $f_i(t)$ is defined by (2.2_i).

Proof. For proof in case of $i = 0$ see, e.g., [1] p. 7. For $i = 1, 2, \ldots$ the proof is the same.

Lemma 2.3. *Let $\alpha(t)$, $\beta(t)$, $g(t)$ and $\varphi(t)$, respectively, be n-times differentiable functions on an interval I and $g(I)$, respectively.*

i) *If $k_1 \geqq 0$, $k_2 \geqq 0$, $k_1 + k_2 > 0$ are arbitrary constants and*

$$(2.5) \qquad (-1)^i \alpha^{(i)} \geqq 0, \ (-1)^i \beta^{(i)}(t) \geqq 0 \qquad (i = 0, 1, \ldots, n; \ t \in I),$$

then

$$(2.6) \qquad (-1)^i D_t^i[k_1\alpha(t) + k_2\beta(t)] \geqq 0, \quad (-1)^i D_t^i[\alpha(t)\beta(t)] \geqq 0 \quad (i = 0, 1, \ldots, n; \ t \in I).$$

If, in addition, strict inequality holds throughout (2.5) *at least for $\alpha(t)$ and $\beta(t) \neq 0$, then strict inequality holds throughout* (2.6).

ii) *If*

$$(2.7) \qquad (-1)^{i+1} g^{(i)}(t) \geqq 0 \qquad (i = 1, 2, \ldots, n; \ t \in I)$$

and

$$(2.8) \qquad (-1)^i \varphi^{(i)}(t) \geqq 0 \qquad (i = 1, 2, \ldots, n; \ t \in g(I)),$$

then

$$(2.9) \qquad (-1)^i D_t^i \varphi[g(t)] \geqq 0 \qquad (i = 1, 2, \ldots, n; \ t \in I).$$

If, in addition, $g'(t) > 0$ and strict inequality holds throughout (2.8), *or if $\varphi'(t) < 0$ and strict inequality holds throughout* (2.7), *then strict inequality holds throughout* (2.9).

In particular, if $g(t)$ satisfies (2.7) *and $g'(t) > 0$ for $t \in I = (a, b)$, $\infty < a < b \leqq \infty$, then*

$$(2.10) \qquad (-1)^i D_t^i e^{-g(t)} > 0 \qquad (i = 0, 1, \ldots, n; \ t \in (a, b)).$$

If, in addition, $g(t) > 0$, then

$$(2.11) \qquad (-1)^i D_t^i [g(t)]^{-1} > 0 \qquad (i = 0, 1, \ldots, n; \ t \in (a, b)),$$

Proof. The proof of the first part of (2.6) is trivial. The second part of (2.6) follows from the formula

$$(2.12) \qquad D_t^i[\alpha(t)\beta(t)] = \sum_{k=0}^{i} \binom{i}{k} \alpha^{(k)}(t)\,\beta^{(i-k)}(t).$$

If strict inequality holds throughout (2.5) for $\alpha(t)$, we find that the right-hand side of (2.12) includes the nonzero term $\alpha^{(i)}(t)\,\beta(t)$ for any $i = 0, 1, \ldots, n$ and the modified form of i) holds, too.

For the proof of ii) see, e.g. [5] p. 1241. (2.10) and (2.11) follows from the modified form of (2.9) since

$$(-1)^i D_t^i e^{-t} > 0 \qquad (i = 0, 1, \ldots;\ t \in (-\infty, \infty))$$

and

$$(-1)^i D_t^i t^{-1} > 0 \qquad (i = 0, 1, \ldots;\ t \in (0, \infty)).$$

Lemma 2.4. *If $\varphi(t) \in M_n$ $(n \geqq 1)$ and $\varphi(t)$ is not eventually constant $(\varphi'(t) > 0)$ for $t \in (0, \infty)$, then*

$$(2.13) \qquad (-1)^i \varphi^{(i)}(t) > 0 \qquad (i = 0, 1, \ldots, n-1).$$

Proof. For proof see [10] p. 40 (Lemma 0.3).

3. HIGHER MONOTONICITY OF $\{R_k\}_{k=0}^{\infty}$

In this section some former results concerning the Sturm-Liouville differential equation (1.2) are extended to the more general equation

$$(3.1) \qquad y'' + a(t)\,y' + b(t)\,y = 0,$$

where $a(t) \in C^1(0, \infty)$. There are studied quantities R_k associated with the equation (3.1) which is a certain generalization of the similar quantities M_k studied by L. Lorch, P. Szego, and M. E. Muldoon in [5].

For fixed $\lambda > -1$ and suitable, sufficiently monotonic $W(t)$ we define the quantities R_k by

$$(3.2)\ R_k = R_k(W, \lambda) = \int_{t_k}^{t_{k+1}} W(t) \left| y(t) \exp\left\{\frac{1}{2}\int a(t)\,\mathrm{d}t\right\}\right|^{\lambda} \mathrm{d}t, \qquad (k = 0, 1, 2, \ldots),$$

where $y(t)$ is an arbitrary non-trivial solution of (3.1) and $\{t_k\}_{k=0}^{\infty}$ any sequence of consecutive zeros of any non-trivial solution $z(t)$ of (3.1) which may or may not be linearly independent of $y(t)$.

Theorem 3.1. *Let $n \geqq 2$ be an integer and $W(t) > 0$ any function of class M_n. For the coefficients $a(t)$, $b(t)$ in equation (3.1) suppose that the function*

$$(3.3) \qquad f(t) = b(t) - \frac{1}{2}a'(t) - \frac{1}{4}a^2(t)$$

satisfies

$$(3.4) \qquad f'(t) \in M_n,\ f'(t) > 0 \text{ for } t \in (0, \infty),\ f(\infty) > 0.$$

Then for R_k defined by (3.2) *there holds*

$$(3.5) \qquad (-1)^j \Delta^j R_k > 0 \qquad (j = 0, 1, \ldots, n;\ k = 0, 1, 2, \ldots)$$

If, in adition,

$$(3.6) \qquad (-1)^j W^{(j)}(t) > 0 \qquad (j = 1, 2, \ldots, n;\ 0 < t < \infty),$$

then the hypothesis $f'(t) > 0$ may be weakened to $f'(t) \geqq 0$ and $n = 1$ is allowed.

Finally, if both $f'(t) > 0$ and (3.6) *are weakened to $f'(t) \geqq 0$, $(-1)^j W^{(j)}(t) \geqq 0$ $(j = 1, 2, \ldots, n;\ 0 < t < \infty)$, then the symbol "$>$" in* (3.5) *must be replaced by "$\geqq$" and $n = 1$ is allowed, too.*

Corollary 3.1. *Let $n \geqq 1$ be an integer and $\lambda > 0$ any number. For $a(t)$, $b(t)$ ad $f(t)$ in* (3.1) *and* (3.3) *suppose that*

$$(3.7) \qquad a(t) \in M_{n-1}, a(t) > 0 \text{ for } t \in (0, \infty),$$

$$(3.8) \qquad f'(t) \in M_n,\ f(\infty) > 0.$$

Then for P_k defined by

$$(3.9) \qquad P_k = P_k(\lambda) = \int_{t_k}^{t_{k+1}} |\, y(t)\, |^\lambda \,\mathrm{d}t \qquad (k = 0, 1, \ldots),$$

where $\{t_k\}$ and $y(t)$ have the same meaning as in (3.2), *there holds*

$$(3.10) \qquad (-1)^j \Delta^j P_k > 0 \qquad (j = 0, 1, \ldots, n;\ k = 0, 1, 2, \ldots).$$

In particular the sequence $\{P_k(1)\}_{k=0}^{\infty}$ of the areas bounded by the successive arches or waves (having consecutive zeros as end-points) of the graph of any solution of (3.1) *is monotonic of order n.*

Modifications similar to those in Theorem 3.1 *are possible, too.*

4. PROOFS OF THEOREM 3.1 AND COROLLARY 3.1.

Suppose first that $f'(t) > 0$ for $t \in (0, \infty)$. The conditions of Theorem 3.1 are sufficient for the validity of Lemma 1 p. 41 in [10] and the equation

$$(4.1) \qquad y' + f(t)\, y = 0.$$

By this Lemma, equation (4.1) possesses a pair of solution $u = r(t)$, $s(t)$ such that

$$p(t) = r^2(t) + s^2(t)$$

satisfies for $t \in (0, \infty)$

$$(4.2) \qquad (-1)^j p^{(j)}(t) > 0 \qquad (j = 0, 1, \ldots, n-1),\ (-1)^n p^{(n)}(t) \geqq 0.$$

It follows from [5] p. 1244, (Theorem 3.1) that under the above mentioned conditions we have

$$(4.3) \qquad (-1)^j \Delta^j M_k > 0 \qquad (j = 0, 1, \ldots, n;\ k = 0, 1, 2, \ldots)$$

for

$$M_k = M_k(W, \lambda) = \int_{t_k}^{t_{k+1}} W(t)\,|\,u(t)\,|^{\lambda}\,\mathrm{d}t, \tag{4.4}$$

where $u(t)$ is an arbitrary (non-trivial) solution of (4.1) and $\{t_k\}$ any sequence of consecutive zeros of any non-trivial solution $v(t)$ of (4.1) which may or may not be linearly independent of $u(t)$.

Since the transformation (2.3) preserves zeros t_k of solutions $u(t)$, $y(t)$, respectively, it follows from Lemma 2.2 that

$$R_k \equiv R_k(W, \lambda) = \int_{t_k}^{t_{k+1}} W(t) \left| y(t) \exp\left\{\frac{1}{2} \int a(t)\,\mathrm{d}t\right\} \right|^{\lambda} \mathrm{d}t =$$

$$= \int_{t_k}^{t_{k+1}} W(t)\,|\,u(t)\,|^{\lambda}\,\mathrm{d}t = M_k(W, \lambda) = M_k$$

and the assertion (3.5) follows from (4.4).

To prove the modified form of Theorem 3.1 suppose now $f'(t) \geqq 0$ and validity of (3.6). $f'(t) \in M_n$ implies (see [2] Theorems 18.1_n, 20.1_n) the existence of a function $p(t)$ in (4.2) such that

$$p(t) > 0,\ (-1)^j p^{(j)}(t) \geqq 0 \qquad (j = 1, 2, \dots, n;\ 0 < t < \infty). \tag{4.5}$$

The assertion (3.5) follows from the modified form of Theorem 3.1 in [5], p. 1244 in the same way as in the first part of this proof.

The weakened assertion in case of $f'(t) \geqq 0$, $(-1)^j W^j(t) \geqq 0$ $(j = 1, 2, \dots, n;$ $0 < t < \infty)$ follows immediately from the proof of Theorem 3.1 in [5] in the same way as mentioned above.

To prove the Corollary 3.1 we put in Theorem 3.1

$$W(t) = W_0(t) = \mathrm{e}^{-g(t)}, \qquad \text{where} \qquad g(t) = \frac{1}{2}\lambda \int a(t)\,\mathrm{d}t.$$

We have

$$g'(t) = \frac{1}{2}\lambda a(t) > 0, \qquad (-1)^{j+1}\, g^{(j)}(t) \geqq 0 \qquad (j = 2, \dots, n; 0 < t < \infty).$$

The second part of Lemma 2.3 implies that

$$(-1)^j\, W_0^{(j)}(t) > 0 \qquad (j = 0, 1, \dots, n;\ 0 < t < \infty),$$

and the assertion (3.10) follows from the modified form of Theorem 3.1 since

$$R_k(W_0, \lambda) = \int_{t_k}^{t_{k+1}} W_0(t) \, . \exp\left\{\frac{1}{2}\lambda \int a(t)\,\mathrm{d}t\right\} |\, y(t)\,|^\lambda \,\mathrm{d}t = P_k.$$

The second part of Corollary 3.1 can be received by putting $\lambda = 1$ and $z(t) \equiv y(t)$ in (3.9). The last sentence is obvious.

5. HIGHER MONOTONICITY OF $\{R_k^{(i)}\}_{k=0}^\infty$

This section includes the principal result of the present paper. Theorem 3.1 is included in Theorem 5.1 as a special case if we admit $i = 0$. Only a little change in formulations would be needed.

In this section we shall study sequences $\{R_k^{(i)}\}_{k=0}^\infty$, where $R_k^{(i)}$ is defined for fixed $\lambda > -1$ by

$$R_k^{(i)} \equiv R_k^{(i)}(W, \lambda) = \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} W(t) \, . \exp\left\{\frac{1}{2}\lambda \int a_i(t)\,\mathrm{d}t\right\} |\, y^{(i)}(t)\,|^\lambda \,\mathrm{d}t, \tag{5.1}$$

where $y(t)$ is an arbitrary (non-trivial) solution of (3.1), $t_0^{(i)}$, $t_1^{(i)}, \ldots$ denotes any sequence of consecutive zeros of i-th derivative ($i = 1, 2, \ldots$) of any non-trivial solution $z(t)$ of (3.1) which may or may not be linearly independent of $y(t)$. The function $a_i(t)$ is defined recurrently by (2.1) on the base of the coefficients $a(t)$, $b(t)$ of (3.1).

Theorem 5.1. *Let $n \geqq 2$, $i \geqq 1$ be arbitrary but fixed integers, $W(t) > 0$ any function of class M_n and $t \in (0, \infty)$. Let the coefficients $a(t) \equiv a_0(t)$, $b(t) \equiv b_0(t)$ of* (3.1) $\equiv$ (1.1$_0$) *be such that $a_j(t)$ ($j = 0, 1, \ldots, i$), $b_j(t) \neq 0$ ($j = 0, 1, \ldots, i - 1$) defined by* (2.1) *are differentiable. For the function $f_i(t)$ defined by* (2.2) *suppose that*

$$f_i'(t) \in M_n,\ f_i' > 0 \quad \textit{for} \quad t \in (0, \infty),\ f_i(\infty) > 0. \tag{5.2}$$

Then for $R_k^{(i)}$ defined by (5.1) *there holds*

$$(-1)^j \, \Delta^j R_k^{(i)} > 0 \qquad (j = 0, 1, \ldots, n;\ k = 0, 1, \ldots). \tag{5.3}$$

If, in addition (3.6) *holds, then the hypothesis $f_i'(t) > 0$ may be weakened to $f_i'(t) \geqq 0$ and $n = 1$ is allowed. If the sharp inequality $f_i > 0$ is weakened, then the symbol "$>$" in* (5.3) *must be replaced by "$\geqq$" and $n = 1$ is allowed, too.*

Proof. If $z_0(t)$ denotes some solution of equation (3.1) $\equiv$ (1.1$_0$), then on the base of Lemma 2.1 under above mentioned assumptions $z_0^{(i)}(t)$ complies with the equation (1.1$_i$) and conversely, i.e. to any solution $z_i(t)$ of (1.1$_i$) then there exists a suitable solution $z_0(t)$ of (1.1$_0$) such that $z_0^{(i)}(t) \equiv z_i(t)$ for $t \in (0, \infty)$.

If $\{t_k^{(i)}\}_{k=0}^{\infty}$ denotes the sequence of consecutive zeros of the i-th derivative of above mentioned solution $z_0(t)$ of (1.1_0), then this sequence represents the sequence of consecutive zeros of suitable solution of (1.1_i), explicitly $z_i(t) \equiv z_0^{(i)}(t)$.

Theorem 5.1 follows now from Theorem 3.1 if we replace equation (3.1) by (1.1_i).

Corollary 5.1. *Let the conditions of Theorem* 5.1 *be satisfied. Then*

$$(-1)^{j+1} \Delta^j t_k^{(i)} > 0 \qquad (j = 0, 1, \ldots, n + 1;\ k = 0, 1, 2, \ldots). \tag{5.4}$$

Consequently the sequence of differences of consecutive zeros of the i-th derivative of any solution of (1.1) *is monotonic of order n.*

Proof. To prove the corollary, it suffices to put $W(t) \equiv 1$ and $\lambda = 0$ in (5.3). The validity of (5.4) for $j = 0$ is obvious.

6. REMARKS

(i) Theorem 5.1 has a quite general character. In the case of concrete equation, however, it may be difficult to verify the higher monotonicity of $f_i'(t)$, especially for great i. But it is possible to deduce some conditions for higher monotonicity of $f_i'(t)$, involving only the coefficients $a(t)$, $b(t)$ of (1.1). Some simple ones are mentioned below.

(ii) The explicit form of $f_1(t)$ is

$$f_1(t) = b + \frac{1}{2} b''/b - \frac{3}{4} b'^2/b^2 + \frac{1}{2} a' - \frac{1}{4} a^2 - \frac{1}{2} ab'/b. \tag{6.1}$$

In case of $a(t) \equiv 0$ $(b(t) \equiv q(t))$, i.e. for the equation (1.2), it is proved in [9] p. 107 (Lemma 3) that $q'(t) \in M_{n+2}$, $q(t) > 0$ implies

$$Q'(t) \in M_n, \quad Q(\infty) = q(\infty), \tag{6.2}$$

where

$$Q(t) = q - \frac{3}{4} q'^2/q^2 + \frac{1}{2} q''/q.$$

The mentioned lemma constitutes, however, a criterion for higher monotonicity of $f_1(t) \equiv Q(t)$ in this special case.

(iii) We can easy verify that

$$\begin{gathered} a(t) \in M_{n+2},\ b'(t) \in M_{n+2},\ b(t) > 0 \qquad (t \in (0, \infty)) \\ b(\infty) - \frac{1}{4} a^2(\infty) > 0, \end{gathered} \tag{6.3}$$

imply

(6.4) $\qquad (-1)^j f_1^{(j+1)}(t) \geqq 0,\ f_1(\infty) > 0 \qquad (j = 0, 1, \ldots, n;\ t \in (0, \infty)).$

If, in addition, $b'(t) > 0$, then the symbol "$\geqq$" in (6.4) may be replaced by "$>$".

From this follows that in case of $i = 1$ the basic assumptions (5.2) in Theorem 5.1 may be replaced by (6.3) and $b'(t) > 0$.

To prove the above, we use Lemma 2.3. It implies that under conditions (6.3) there holds

$$-b''/b \in M_{n+1},\ b'/b \in M_{n+2},\ ab'/b \in M_{n+2}$$

and hence

(6.5) $\qquad (b''/b)' \in M_n,\ (-b'^2/b^2)' \in M_{n+1},\ (-ab'/b)' \in M_{n+1}.$

Since all members of the derivative of (6.1) are of class M_n, we have $f_1'(t) \in M_n$.

As it was shown in [9] p. 108 there holds

$$\lim_{t\to\infty}\left[b + \frac{1}{2}b''/b - \frac{3}{4}b'^2/b^2\right] = b(\infty).$$

In the same way we can prove that

$$\lim_{t\to\infty}\left[\frac{1}{2}a' - \frac{1}{4}a^2 - \frac{1}{2}ab'/b\right] = -\frac{1}{4}a^2(\infty),$$

and the proof of (6.4) is complete.

To prove the sharpened assertion we consider the explicit form (6.1) of $f_1(t)$. (6.1) includes the term $-\frac{3}{4}b'^2/b^2$. From (6.5) it follows $(b'^2/b^2) \in M_{n+2}$. Since $(b'/b)' < 0$, it holds that $(b'^2/b^2)' < 0$, and Lemma 2.4 implies

$$(-1)^j D_t^j(b'^2/b^2) > 0 \qquad (j = 0, 1, \ldots, n + 1,\ t \in (0, \infty)).$$

This implies that the expression $(-1)^j D_t^j f_1'(t)$ includes the strictly positive term for $j = 0, 1, \ldots, n$ and therefore (6.4) holds with sharpened sign "$>$".

(iv) The conditions (and also assertions) of the above mentioned theorems have a more simple form in case of completely monotonic functions. If $\varphi \in M_\infty$, i.e. $(-1)^j \varphi^{(j)}(t) \geqq 0$ for $j = 0, 1, \ldots$, then sharp inequality holds for any j, unless $\varphi(t)$ is eventually constant (See Lemma 2.4). From the same lemma it follows that the conditions (3.6) for the function $W(t)$ may be replaced by $W(t) \in M_{n+1}$, $W'(t) > 0$.

7. HIGHER MONOTONICITY OF $\{y^2(t_k')\}_{k=0}^{\infty}$, $\{[y^{(i-1)}(t_k^{(i)})]^2\}_{k=0}^{\infty}$

As an application of the preceding theorems, it is possible to derive some higher monotonicity properties of the above mentioned sequences which seem to be some analoques of Sonin's theorem.

Theorem 7.1. *Let $n \geqq 1$ be an arbitrary but fixed integer. Suppose that the conditions* (6.3) *are satisfied and that*

$$a(t) > 0 \text{ or } a(t) \equiv 0,\ b'(t) > 0 \text{ for } t \in (0, \infty). \tag{7.1}$$

Then

$$(-1)^j \Delta^j y^2(t'_k) > 0 \qquad (j = 0, 1, \ldots, n + 1;\ k = 0, 1, \ldots), \tag{7.2}$$

where $y(t)$ denotes any solution of (1.1) *and $\{t'_k\}_{k=0}^{\infty}$ any sequence of consecutive zeros of its derivative. Hence the sequence $\{y^2(t_k)\}_{k=0}^{\infty}$ is monotonic of order $n + 1$.*

Proof. Consider the expression

$$P'_k = \int_{t'_k}^{t'_{k+1}} y'^2 \exp\left[2 \int a\,\mathrm{d}t\right] \frac{\mathrm{d}}{\mathrm{d}t}\left\{b^{-1} \exp\left[-2 \int a\,\mathrm{d}t\right]\right\} \mathrm{d}t. \tag{7.3}$$

After integration by parts we have

$$P'_k = [y'^2 b^{-1}]_{t'_k}^{t'_{k+1}} - 2 \int_{t'_k}^{t'_{k+1}} y'(ay' + y'')\, b^{-1}\,\mathrm{d}t = \tag{7.4}$$

$$= 2 \int_{t'_k}^{t'_{k+1}} yy'\,\mathrm{d}t = \Delta y^2(t'_k).$$

because of vanishing of the first term. On the other hand

$$P'_k = - \int_{t'_k}^{t'_{k+1}} b^{-1}(b'/b + 2a)\, y'^2\,\mathrm{d}t = \tag{7.5}$$

$$= - \int_{t'_k}^{t'_{k+1}} \exp\left[- \int a\,\mathrm{d}t\right](b'/b + 2a) \,.\, b^{-1} \exp\left[\left[\int a\,\mathrm{d}t\right]\right] y'^2\,\mathrm{d}t =$$

$$= - \int_{t'_k}^{t'_{k+1}} \left\{(b'/b + 2a) \exp\left[-\int a\,\mathrm{d}t\right]\right\} . \left\{\exp\left[\left[\int (a - b'/b)\,\mathrm{d}t\right]\right] y'^2\right\} \mathrm{d}t =$$

$$= - \int_{t'_k}^{t'_{k+1}} W_1 \,.\, \left\{\exp\left[\frac{1}{2} \int a_1\,\mathrm{d}t\right] y'\right\}^2 \mathrm{d}t = -R'_k(W_1, 2),$$

where

$$(7.6)\qquad W_1 \equiv W_1(t) = (b'/b + 2a) \exp [- \textstyle\int a \, dt].$$

By comparison (7.4) and (7.5) we obtain

$$(7.7)\qquad R_k'(W_1, 2) = -\Delta y^2(t_k').$$

It will be shown now that Theorem 5.1 is applicable in case of $i = 1$, $n \geqq 1$. Suppose first that $a(t) > 0$. (2.10) implies that

$$(-1)^j D_t^j \exp [- \textstyle\int a \, dt] > 0 \qquad (j = 0, 1, \dots, n + 3, t \in (0, \infty)).$$

Since $b'/b \in M_{n+2}$ and $0 < a \in M_{n+2}$, we have $0 < (b'/b + 2a) \in M_{n+2}$. For the function $W_1(t)$ defined by (7.6), it follows from the first part of Lemma 2.3 that

$$(7.8)\qquad (-1)^j W_1^{(j)}(t) > 0 \qquad (j = 0, 1, \dots, n + 2; t \in (0, \infty)).$$

On the other hand Remark (iii) in Section 6 shows that under conditions (6.3) the function $f_1(t)$ satisfies $f_1'(t) \in M_n, f_1(\infty) > 0$. Thus, the conditions of the modified form of Theorem 5.1 are, in this case, fulfilled. Hence and from (7.7) it follows

$$(7.9)\quad (-1)^j \Delta^j R_k' = (-1)^{j+1} \Delta^{j+1} y^2(t_k') > 0 \qquad (j = 0, 1, \dots, n; k = 0, 1, \dots).$$

Because of $y^2(t_k') > 0$, (7.2) follows directly from (7.9).

Suppose now that $a(t) \equiv 0$, $b'(t) > 0$. Using Lemma 2.3 we get $b'/b \in M_{n+2}$. Lemma 2.4 implies that $(-1)^j D^j[b'/b] > 0$ for $j = 0, 1, \dots n + 1$. Since in this case $\exp [- \int a \, dt] = \text{const}$ we can see that (7.8) holds again with nonessential exeption for $j = n + 2$. This completes the proof.

Remark 7.2. The conclusion (7.2) remains true if the hypothesis (6.3) are replaced by the slightly more general ones

$$(7.10)\qquad f_1' \in M_n, f_1(\infty) > 0, (b'/b + 2a) \exp [-2 \textstyle\int a \, dt] \in M_n.$$

It follows directly from the proof of Theorem 7.1.

Remark 7.2. During compiling the final version of this work there was published the paper [6] including the similar sufficient conditions implying the validity of (7.2). They are formulated for the selfadjoint equation

$$(g(t) \, y')' + f(t) \, y = 0.$$

Theorem 7.2. *Let n, $i \geqq 1$ be arbitrary but fixed integers and $t \in (0, \infty)$. Let the coefficients $a(t) = a_0(t)$, $b(t) \equiv b_0(t)$ of* (1.1) *be such that $a_j(t)$ $(j = 0, 1, \dots, i - 1)$, $b_j(t) \neq 0$ $(j = 0, 1, \dots, i - 2)$, defined by* (2.1), *are differentiable. Suppose that*

$$a_{i-1}(t) \in M_{n+i+1}, b_{i-1}'(t) \in M_{n+i+1}, b_{i-1}'(t) > 0 \qquad \textit{for} \qquad t \in (0, \infty),$$

(7.11) $$b_{i-1}(\infty) - \frac{1}{4} a_{i-1}^2(\infty) > 0$$

and

$$a_{i-1}(t) > 0 \quad or \quad a_{i-1}(t) \equiv 0, b'_{i-1}(t) > 0 \quad for \quad t \in (0, \infty).$$

Then

(7.13) $$(-1)^j \Delta^j [y^{(i-1)}(t_k^{(i)})]^2 > 0 \quad (j = 0, 1, \ldots, n+1; k = 0, 1, \ldots),$$

where $y(t)$ denotes any solution of (1.1) and $\{t_k^{(i)}\}_{k=0}^{\infty}$ any sequence of consecutive zeros of its i-th derivative. Consequently the sequence $\{[y^{(i-1)}(t_k^{(i)})]^2\}_{k=0}^{\infty}$ is monotonic of order $n + 1$.

Proof. Proof of Theorem 7.2 is similar to the proof of the preceding one. We consider

(7.14) $$P_k^{(i)} = -\Delta[y^{(i-1)}t_k^{(i)}]^2 =$$

$$= \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} \left[y^{(i)} \exp\left\{\int a_{i-1} \, dt\right\}\right]^2 \frac{d}{dt}\left[b_{i-1}^{-1} \exp\left\{-2 \int a_{i-1} \, dt\right\}\right] dt =$$

$$= R_k^{(i)}(W_i, 2),$$

where

(7.15) $$W_i = W_i(t) = (b'_{i-1}/b_{i-1} + 2a_{i-1}) \exp\{-\int a_{i-1} \, dt\}$$

instead of P'_k and (1.1$_i$) instead of (1.1).

Remark 7.2. The conclusion of Theorem 7.2 remains true if the hypotheses (7.11) are replaced by the slightly more general ones

(7.16) $$f'_i \in M_n, f_i(\infty) > 0, (b'_{i-1}/b_{i-1} + 2a_{i-1} \exp\{-\int a_{i-1} \, dt\} \in M_n.$$

It is immediately seen from the proofs of Theorems 7.1 and 7.2.

Remark 7.3. As a very simple example for Theorem 7.2 in case of any i, it is possible to consider the equation

(7.17) $$y'' + ay' + by = 0,$$

where a, b are constants such that $a > 0$, $b - \frac{1}{4}a^2 > 0$. In this case $a_i(t) = a$, $b_i(t) = b$ and $W_i(t) = e^{-at}$ for any i. (7.13) holds therefore for any solution of (7.17) and any i.

In this case, however, we do not obtain from Theorem 7.2 any other results than from Theorem 7.1 since the equation (1.1$_i$) is the same for any i.

8. HIGHER MONOTONICITY OF $\{|y(t_i')|\}_{k=0}^{\infty}$, $\{|y^{(i-1)}(t_k^{(i)})|\}_{k=0}^{\infty}$

As another applications of Theorems 3.1 and 5.1 we give below some higher monotonicity properties of above mentioned sequences.

Theorem 8.1. *Let $n \geqq 1$, $i \geqq 1$ be arbitrary but fixed integers and $t \in (0, \infty)$. Let the coefficients $a(t) \equiv a_0(t)$, $b(t) \equiv b_0(t)$ of (1.1) be such that $a_j(t)$ $(j = 0, 1, \ldots, i)$, $0 \neq b_j(t)$ $(j = 0, 1, \ldots, i-1)$ defined by (2.1) are differentiable. For the function $a_i(t)$ and $f_i(t)$ defined by (2.2) suppose that*

$$(8.1) \qquad 0 < a_i(t) \in M_n$$

$$(8.2) \qquad f_i'(t) \in M_n,\ f_i(\infty) > 0.$$

Then

$$(8.3)\quad (-1)^j \Delta^j[|y^{(i-1)}(t_{k+1}^{(i)})| + |y^{(i-1)}(t_k^{(i)})|] > 0 \qquad (j = 0, 1, \ldots n;\ k = 0, 1, \ldots),$$

and

$$(8.4)\qquad (-1)^j \Delta^j |y^{(i-1)}(t_{\varepsilon+2k}^{(i)})| > 0 \qquad (j = 0, 1, \ldots n;\ k = 0, 1, \ldots),$$

where $\varepsilon = 0$ or 1, $y(t)$ denotes any non-trivial solution of (1.1) and $\{t_k^{(i)}\}_{k=0}^{\infty}$ is any sequence of consecutive zeros of its i-th derivative.

Proof. Theorem 8.1 follows from Theorem 5.1 by putting

$$(8.5)\qquad \lambda = 1, \qquad W_i(t) = \exp\left\{-\frac{1}{2}\int a_i(t)\,dt\right\}.$$

We have

$$(8.6)\qquad R_k^{(i)}(W_i, 1) = \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} W_i \left|\exp\left\{\frac{1}{2}\int a_i\,dt\right\} y^{(i)}\right| dt =$$

$$= \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} |y^{(i)}|\,dt = |y^{(i-1)}(t_{k+1}^{(i)})| + |y^{(i-1)}(t_k^{(i)})|.$$

Using Lemmas 2.3 and 2.4, we obtain (for $W_i(t)$ defined by (8.5))

$$(-1)^j W_i^{(j)}(t) > 0 \qquad (j = 0, 1, \ldots, n;\ t \in (0, \infty)).$$

We can easy verify that the other conditions of the modified form of Theorem 5.1 are satisfied, too. (8.3) follows then from (5.3) and (8.6).

(8.4) follows from (8.3) since

$$\Delta[|y^{(i-1)}(t_{k+1}^{(i)})| + |y^{(i-1)}(t_k)|] = |y^{(i-1)}(t_{k+2}^{(i)})| - |y^{(i-1)}(t_k^{(i)})| \qquad (k = 0, 1, \ldots),$$

and

$$| y^{(i-1)} (t^{(i)}_{\varepsilon+2k}) | > 0 \qquad (k = 0, 1, \dots; \varepsilon = 0 \text{ or } 1).$$

Corollary 8.1. *Let us suppose that* $n \geqq 1$ *that* (6.3) *and* (7.1) *hold for* $a(t)$, $b(t)$ *in* (1.1), *and that*

$$0 < a_1(t) \equiv (a - b'/b) \in M_n. \tag{8.7}$$

Then

$$(-1)^j \Delta^j | y(t'_{\varepsilon+2k}) | > 0 \qquad (j = 0, 1, \dots, n; k = 0, 1, \dots), \tag{8.8}$$

where $\varepsilon = 0$ *or* 1, $y(t)$ *denotes any solution of* (1.1) *and* $\{t_k\}$ *any sequence of consecutive zeros of its derivative.*

Proof. (6.3) and (7.1) imply (8.2) (See Remark (ii) in § 6). Then (8.8) follows from (8.4) by putting $i = 1$.

Remark 8.1. A typical equation satisfying the conditions of most of the obtained theorems is the Bessel equation. In the paper [11] there are derived some higher monotonicity properties of the Bessel functions.

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