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PARTITIONS AND CONGRUENCES IN ALGEBRAS III. COMMUTATIVITY OF CONGRUENCES

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The basic information about the object of this paper is given in secs. $0-0.5$ [12]. Recall that a partition in a set G is a system (eventually empty) of nonempty mutually disjoint subsets in G [1, 2, 3, 4, 6, 7, 10]. On the ground of existence of a 1-1 correspondence between the set $P(G)$ of all partitions in G and the set of all symmetric and transitive relations (ST-relations) in G there is often made no difference between both the notions. The element of the partition is called a *block* and the union $\bigcup A$ of all the blocks of A is called a *domain* of the partition A. By a congruence in an algebra (G, Ω), is meant a stable ST-relation in the algebra (G, Ω). The block $A(0)$ of the congruence A in an Ω-group G containing the zero element 0 of the group G is called a *null block*.

3.0 The objects of our considerations in this paper are problems concerning the commutativity of partitions and congruences "in". The following results demonstrate that if we replace "on" by "in", then some of the known characteristics of the commutativity of partitions or congruences on a set or on an algebra, respectively (see e.g. [10, 11]), will fail. The commutativity of congruences B, C in an algebra G is characterized by any of the following properties (see 3.3) : 1. BC is a partition 2. BC is a symmetric relation, 3. BC is a congruence. The first two properties are the well known characteristics of commutativity of partitions on a set; they also characterize the commutativity of partitions in a set (3.1).

In case of congruences in an Ω-group G we may join to properties 1, 2, 3 characterizing the commutativity of congruences B, C another one: 4. $B(0) \cup C(0) \subseteq \subseteq \bigcup B \cap \bigcup C$ (3.9). The equalities $BC = B \vee_P C$, $BC = B \vee_{\mathscr{K}} C$ are mutually equivalent in any algebra (3.3.1(4)), they imply the commutativity of the congruences B and C, and in case of congruences "on" they characterize the commutativity of B and C (3.3.1(1)); in case of congruences in an Ω-group they characterize the relation $\bigcup B = \bigcup C$ (3.11). The commutativity of the nullblocks $B(0)$ and $C(0)$ is implied by the commutativity of the congruences B and C in an Ω-group, but it does not imply it (3.9.3, 3.9.4). The equality $B \vee_P C = B \vee_{\mathscr{K}} C$ is neither necessary nor sufficient for the commutativity of the congruences B and C in an Ω-group (3.3.1(6)). Some formal weakenings of definition requirement of the commutativity of congruences give other criteria of the commutativity of the congruences B and C in an Ω-group: 5. $(BC)(0) = (CB)(0)$, 6. $\mathsf{U}(BC) = \mathsf{U}(CB)$ (3.8, 3.10). On the other hand, analogous weakenings of the equality $B \vee_P C = BC$ (this equality implies the commutativity of B and C) are no longer sufficient for the commutativity (3.9.5). The above criteria of the commutativity of congruences B and C refer to its product BC. In 3.5.1 the relation BC is described and on this ground there are constructed $\mathsf{U}(BC)$ and $(BC)(0)$ (see Definition 3.5); the first set is equal to $B(0) + \mathsf{U}B \cap \mathsf{U}C$, the second one to $B(0) + \mathsf{U}B \cap C(0)$. Both the assertions are the subject-matter of the main theorem 3.5.5 of this paper. From the fact that $(BCBC\ldots)(0) = (BC)(0)$ (3.5.6), $(B \vee_P C)(0)$ follows as the union of two Ω-subgroups $[B(0) + \mathsf{U}B \cap C(0)] \cup [C(0) + (\mathsf{U}C \cap B(0)]$. In 3.6.3 and 3.7 there are characterized the equalities $(B \vee_P C)(0) = (BC)(0)$ and $(B \vee_{\mathscr{K}} C)(0) = (B \vee_P C)(0)$, in 3.7.5 and 3.7.1 the equalities $\mathsf{U}(B \vee_P C) = \mathsf{U}(BC)$ and $\mathsf{U}(B \vee_{\mathscr{K}} C) = \mathsf{U}(B \vee_P C)$.

3.1 *For partitions B and C in a set G the following conditions are equivalent:*
1. *B and C commute* 2. *BC is a partition in G* 3. *BC is a symmetric relation in G.*

Proof. $2 \Rightarrow 3$ is evident.

$3 \Rightarrow 1$: $xBCy \Leftrightarrow yBCx \Leftrightarrow yBaCx \Leftrightarrow xCaBy \Leftrightarrow xCBy$.
$1 \Rightarrow 2$: The symmetry of the relation BC: $xBCy \Rightarrow xCBy \Rightarrow yBCx$. The transitivity of the relation BC: $xBCyBCz \Rightarrow xBCBCz \Rightarrow xBBCCz \Rightarrow xBCz$ since evidently $BB = B$ and $CC = C$).

3.1.1 Remarks

(1) *For partitions B and C in a set G it holds: B and C commute $\Rightarrow B \vee_P C = BC \cup B \cup C$.*

Proof. From the relation $x(B \vee_P C)\,y$ it follows: $xA_1x_1 \ldots x_{n-1}A_ny$, where $A_i = B$ or $= C$ $(i = 1, 2, \ldots, n)$, $A_i \neq A_{i+1}$ $(i = 1, 2, \ldots, n-1)$ hence we get the following six possibilities: for $n = 1$ either xBy or xCy, and for $n \geqq 2$ the following four ones: $xBx_1 \ldots x_{n-1}By$, or $xBx_1 \ldots x_{n-1}Cy$, or $xBx_1 \ldots x_{n-1}By$, or $xCx_1 \ldots x_{n-1}Cy$. From these four possibilities it follows respectively: $x(BC)^mBy$, or $x(BC)^my$, or $x(CB)^my$, or $x(CB)^mCy$ where the exponent m is a suitable positive integer. From the commutativity of B and C and from the obvious fact that $B^m = B$, $C^m = C$ we get in the first case

$x(BC)^mBy \Rightarrow xB^mC^mBy \Rightarrow xBCBy \Rightarrow xBBCy \Rightarrow xBCy$, in the second one $x(BC)^my \Rightarrow xB^mC^my \Rightarrow xBCy$. Similarly in the cases three and four. So $B \vee_P C \subseteq BC \cup B \cup C$ is proved. The reverse inclusion is evident.

(2) *The reverse implication in* (1) *does not hold in general even for congruences in an* Ω-group.

Examples. Let $G = R \dotplus R$, where R is the additive group of reals (in geometric meaning G is a plane), let the partition B the the set of all points on the axis X, and the partition C the set of all straight lines in G parallel to the axis Y. The partitions B, C do not commute, but $B \vee_P C = BC \cup B \cup C$ is true. Indeed, let $T \in C$, $x \in T \setminus X$, $y \in T \cap X$. Then $xCyBy$ thus $xCBy$, while $xBCy$ is not true since $x \bar{\in} \bigcup B = X$. Thus $BC \neq CB$. Furthermore BC is the set of all ordered pairs of points the first of which lies on the axis X and the second one on the parallel line with the axis Y which passes through the first point. Thus $BC \subseteq C$; as also $B \subseteq C$, it holds $BC \cup B \cup C = C = B \vee_P C$.

(3) *For partitions B and C in a set it holds: $B \vee_P C = BC \Rightarrow B$ and C commute.*

Indeed, from the condition follows that BC is a partition and from 3.1 that B and C commute.

(4) *The reverse implication in* (3) *does not hold in general even for congruences in an Ω-group.*

This will be proved in 3.3.1 (2).

(5) *For partitions B and C on a set there holds the following: B and C commute $\Leftrightarrow BC = B \vee_P C$* ([10, 11]).

The implication $\Leftarrow$ follows from (3), the implication $\Rightarrow$ from (1) and from that partitions on a set are contained in their product.

(6) *For partitions B, C in a set there holds as follows: $B \vee_P C = BC \Rightarrow \bigcup B = \bigcup C$.*

In fact, for $x \in \bigcup C \setminus \bigcup B$ there holds $xB \vee_P Cx$, but $xBCx$ does not hold; thus $\bigcup B \supseteq \bigcup C$. By 3.1.1 (3) the condition is symmetrical with respect to B, C thus $\bigcup B = \bigcup C$.

3.2 *The product of congruences in an algebra G is a binary relation preserving operations in G. The product of commutative congruences in G is a congruence in G.*

Proof. If B and C are congruences in an algebra G, ω an n-ary operation in G and $x_i BC y_i$ $(i = 1, 2, \ldots, n)$, then we get consecutively the following: $x_i B a_i C y_i$, $x_1 \ldots x_n \omega B a_1 \ldots a_n \omega C y_1 \ldots y_n \omega$, $x_1 \ldots x_n \omega BC y_1 \ldots y_n \omega$. If the congruences B and C are permutable, BC is, by 3.1, a partition and hence a congruence in G.

3.3 *For congruences B, C in an algebra G the following conditions are equivalent:* 1. *B and C commute* 2. *BC is a partition in G* 3. *BC is a symmetric relation in G* 4. *BC is a congruence in G.*

Proof. The first three statements are mutually equivalent by 3.1. $4 \Rightarrow 2$ is obvious and $1 \Rightarrow 4$ follows from 3.2.

3.3.1 Remarks

(1) *For congruences B and C on an algebra the following conditions are equivalent.* 1. *B and C commute* 2. *$BC = B \vee_P C$* 3. *$BC = B \vee_{\mathscr{K}} C$.*

Proof. $1 \Rightarrow 3$: BC is a congruence in G by 3.2 and $B, C \leqq B \vee_P C = BC$ holds by 3.1.1 (5) thus $B \vee_{\mathscr{K}} C \leqq BC$. Hence and from the obvious relations $BC \leqq B \vee_P C \leqq B \vee_{\mathscr{K}} C$ there follows $BC = B \vee_P C = B \vee_{\mathscr{K}} C$,

$3 \Rightarrow 2$ follows from the fact that $BC \leqq B \vee_P C \leqq B \vee_{\mathscr{K}} C$.
$2 \Rightarrow 1$ by 3.1.1 (5).

(2) *For congruences B and C in an algebra there holds* $2 \Rightarrow 1$, $3 \Rightarrow 1$ (1, 2, 3 – see (1))
namely because of the fact that BC is a partition in this case and B and C commute by 3.3.

(3) *The implication* $1 \Rightarrow 2$ *does not hold in general even for congruences in an Ω-group.*

Example. Let B and C be the partitions in the plane $G = R \dot{+} R$ (see 3.1.1 (2)) as follows: B the set of all points on the axis X, C the set of all points on the axis Y. Then B and C are congruences in the group G which commute and therefore BC is a congruence in G by 3.2; on the other hand $\mathsf{U}(B \vee_P C) = X \cup Y \neq \{(0, 0)\} = \mathsf{U}(BC)$ thus $B \vee_P C \neq BC$.

(4) *For congruences B and C in an algebra G it holds:* $BC = B \vee_P C \Leftrightarrow BC = B \vee_{\mathscr{K}} C$.

Proof. The implication $\Leftarrow$ is obvious since $BC \leqq B \vee_P C \leqq B \vee_{\mathscr{K}} C$. In proving the reverse implication let us recall that by 3.2 the relation BC preserves operations in G and by assumption it is a partition thus a congruence in G. Hence and from the relations $B, C \leqq B \vee_P C = BC$ there follows $B \vee_{\mathscr{K}} C \leqq BC$. The desired equality follows now from the relations $BC \leqq B \vee_P C \leqq B \vee_{\mathscr{K}} C$.

(5) *The equality $B \vee_P C = B \vee_{\mathscr{K}} C$ holds always for congruences on an algebra* ([12] 0.4).

It does not represent a characteristic of commutativity on an algebra since there exists an algebra G and congruences B, C on G which do not commute (see e.g. the example of noncommutative congruences on a quasi-group in [14]).

(6) *The equality $B \vee_P C = B \vee_{\mathscr{K}} C$* (in contrast to the equalities $BC = B \vee_P C$ or $BC = B \vee_{\mathscr{K}} C$) *is neither* a) *sufficient nor* b) *necessary for the commutativity of congruences in an Ω-group.*

An example for a) is given in 3.1.1 (2) (the partitions B and C do not commute and at the same time $B \vee_P C = B \vee_{\mathscr{K}} C$ since $B \leqq C$), for b) in 3.3.1 (3) (the partitions B and C commute, but $B \vee_P C \neq B \vee_{\mathscr{K}} C$ since $\mathsf{U}(B \vee_P C) = \mathsf{U}B \cup \mathsf{U}C \neq G = \langle \mathsf{U}B, \mathsf{U}C \rangle = \mathsf{U}(B \vee_{\mathscr{K}} C)$.

(7) *Comparable congruences in an Ω-group need not commute.* Also in this case the congruences B and C in 3.1.1 (2) are used as an example.

3.4 *If B and C are congruences in an Ω-group G, Q a subgroup of the additive group G, $Q \subseteq \mathsf{U}B \cap \mathsf{U}C$, then the congruences $B \sqcap Q$ and $C \sqcap Q$ commute.*

(Recall that the partition $B \sqcap Q = \{B^1 \cap Q : B^1 \in B\ B^1 \cap Q \neq \emptyset\}$ is called an intersection of the partition B with the set Q [3] I, 2.3, [4] 2.3, [12] 1.5.1). The statement follows from the fact that $B \sqcap Q$ and $C \sqcap Q$ are congruences on the group Q.

3.4.1 *If B and C are congruences in an algebra G, Q a subalgebra in G, $Q \supseteq \mathsf{U}B \cap \mathsf{U}C$, then $(B \sqcap Q)(C \sqcap Q) = (BC) \sqcap G$.*

Proof. $x[(B \sqcap Q)(C \sqcap Q)]\, y \Rightarrow x(B \sqcap Q)\, a(C \sqcap Q)\, y \Rightarrow x, y,\ a \in Q,\ xBaCy \Rightarrow \Rightarrow xBCy,\ x, y \in Q \Rightarrow x[(BC) \sqcap Q]\, y$, hence $(B \sqcap Q)(C \sqcap Q) \subseteq (BC) \sqcap Q$. Conversely, $x[(BC) \sqcap Q]\, y \Rightarrow x,\ y \in Q,\ xBCy$. From the latter there follows $xBaCy$ with $a \in \mathsf{U}B \cap \mathsf{U}C \subseteq Q$. Hence $x(B \sqcap Q)\, a(C \sqcap Q)\, y$ thus $x(B \sqcap Q)(C \sqcap Q)\, y$. In conclusion $(BC) \sqcap Q \subseteq (B \sqcap Q)(C \sqcap Q)$.

3.4.2 *If $\{A_\alpha\}$ is a system of partitions in a set G, $Q \supseteq \bigcup_{\alpha \neq \beta} (\mathsf{U}A_\alpha \cap \mathsf{U}A_\beta)$, then it holds*

$$\bigvee_{\alpha}{}_P (A_\alpha \sqcap Q) = (\bigvee_{\alpha}{}_P A_\alpha) \sqcap Q.$$

Proof. $x[(\bigvee_{\alpha}{}_P A_\alpha) \sqcap Q]\, y = x, y \in Q, xA_1x_1 \dots x_{n-1}A_ny \Rightarrow x(A_1 \sqcap Q)\, x_1 \dots x_{n-1} (A_n \sqcap Q)\, y \Rightarrow x[\bigvee_{\alpha}{}_P (A_\alpha \sqcap Q)]\, y$.

Conversely, $x[\bigvee_{\alpha}{}_P (A_\alpha \sqcap Q)]\, y = x(A_1 \sqcap Q)\, x_1 \dots x_{n-1}(A_n \sqcap Q)\, y \Rightarrow x_1, \dots x_{n-1} \in \in Q,\ xA_1x_1 \cdots x_{n-1}A_ny \Rightarrow x(\bigvee_{\alpha}{}_P A_\alpha) \sqcap Qy$.

3.4.3 *If $\{A_\alpha\}$ is a system of congruences in an algebra G, Q a subalgebra of G, $Q \subseteq \bigcap_{\alpha} \mathsf{U}A_\alpha$, then $\bigvee_{\alpha}{}_P (A_\alpha \sqcap Q) = \bigvee_{\alpha}{}_{\mathscr{K}} (A_\alpha \sqcap Q)$ holds.*

The proof follows from the fact that $\{A_\alpha \sqcap Q\}$ is a system of congruences on the algebra Q.

3.4.4 *If $\{A_\alpha\}$ is a system of congruences in an algebra G, Q a subalgebra of G, $Q \supseteq \bigcup_{\alpha \neq \beta} (\mathsf{U}A_\alpha \cap \mathsf{U}A_\beta)$, then it holds*

$$(\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha) \sqcap Q = (\bigvee_{\alpha}{}_P A_\alpha) \sqcap Q \Rightarrow (\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha) \sqcap Q = \bigvee_{\alpha}{}_{\mathscr{K}} (A_\alpha \sqcap Q).$$

Proof. By 3.4.2 there holds

$$\bigvee_{\alpha}{}_P (A_\alpha \sqcap Q) = (\bigvee_{\alpha}{}_P A_\alpha) \sqcap Q = (\bigvee_{\alpha}{}_{\mathscr{K}} A_\alpha) \sqcap Q \geqq \bigvee_{\alpha}{}_{\mathscr{K}} (A_\alpha \sqcap Q) \geqq \bigvee_{\alpha}{}_P (A_\alpha \sqcap Q).$$

3.5 Definition. Let A be a binary relation in a set G, $x \in G$. Define

$A(x) = \{y \in G : yAx\}$ (a *row* of A – [5]), $(x)\, A = \{y \in G : xAy\}$

(a *column of* A – [5])

$$\mathsf{U}A = \{y \in G : \exists\, x \in G,\ yAx\} = \bigcup_{x \in G} A(x) \quad (\textit{Vorbereich}, [7] \text{ p. } 191),$$

$$A\mathsf{U} = \{y \in G : \exists\, x \in G,\ xAy\} = \bigcup_{x \in G} (x)\, A \quad (\textit{Nachbereich}, [7] \text{ p. } 191).$$

Remark. *If B, C are symmetric relations in G, then*

$$(BC)(x) = (x)(CB), \qquad \mathsf{U}(BC) = (CB)\,\mathsf{U}.$$

The first equality: $y \in (BC)(x) \Leftrightarrow yBCx \Leftrightarrow xCBy \Leftrightarrow y \in (x)(CB)$: The second equality follows from the first one on the basis of relations

$$\mathsf{U}(BC) = \bigcup_{x \in G} (BC)(x), \qquad (BC)\,\mathsf{U} = \bigcup_{x \in G} (x)(BC).$$

3.5.1 *For congruences B and C in an Ω-group the following conditions are equivalent*

1. $xBCy$
2. $x - y \in B(0) + C(0)$, $x \in \mathsf{U}B$, $y \in \mathsf{U}C$
3. $-x + y \in B(0) + C(0)$, $x \in \mathsf{U}B$, $y \in \mathsf{U}C$
4. $y - x \in C(0) + B(0)$, $x \in \mathsf{U}B$, $y \in \mathsf{U}C$
5. $-y + x \in C(0) + B(0)$, $x \in \mathsf{U}B$, $y \in \mathsf{U}C$.

Proof. $1 \Rightarrow 2$: $xBCy \Rightarrow xBaCy$ (with $a \in \mathsf{U}B \cap \mathsf{U}C) \Rightarrow x - a \in B(0)$, $a - y \in C(0) \Rightarrow x - y = (x - a) + (a - y) \in B(0) + C(0)$. Of course, it also holds: $xBCy \Rightarrow x \in \mathsf{U}B$, $y \in \mathsf{U}C$.

$2 \Rightarrow 1$: $x - y \in B(0) + C(0) \Rightarrow x - y = b + c$ for suitable $b \in B(0)$, $c \in C(0) \Rightarrow -b + x = c + y$. Since, by hypothesis, $x \in \mathsf{U}B$, $y \in \mathsf{U}C$ holds, we have $xB(-b + x) = (c + y)\,Cy$ thus $xBCy$.

The equivalence $1 \Leftrightarrow 3$ is proved in a similar way. The equivalence $2 \Leftrightarrow 4$ follows from the fact that $x - y \in B(0) + C(0) \Leftrightarrow y - x = -(x - y) \in C(0) + B(0)$. Similarly $3 \Leftrightarrow 5$.

3.5.2 *For congruences B and C in an Ω-group there holds*

$$\mathsf{U}(BC) = \mathsf{U}B \cap [B(0) + \mathsf{U}C] = \mathsf{U}B \cap [\mathsf{U}C + B(0)],$$
$$(BC)\,\mathsf{U} = \mathsf{U}C \cap [C(0) + \mathsf{U}B] = \mathsf{U}C \cap [\mathsf{U}B + C(0)].$$

Proof. $x \in \mathsf{U}(BC) \Rightarrow xBCy$ for some $y \in \mathsf{U}C \Rightarrow$ (by 3.5.1 (2)) $x - y \in B(0) + C(0)$, $x \in \mathsf{U}B, y \in \mathsf{U}C \Rightarrow x \in \mathsf{U}B \cap [B(0) + C(0) + y] \subseteq \mathsf{U}B \cap [B(0) + C(0) + \mathsf{U}C] = \mathsf{U}B \cap [B(0) + \mathsf{U}C]$. Conversely, $x \in \mathsf{U}B \cap [B(0) + \mathsf{U}C] \Rightarrow x \in \mathsf{U}B$, $x = b + c$, where $b \in B(0)$, $c \in \mathsf{U}C \Rightarrow -b + x = c \in [B(0) + x] \cap \mathsf{U}B \cap \mathsf{U}C \Rightarrow xB(-b + x) = cCy$ for some $y \in \mathsf{U}C \Rightarrow xBCy \Rightarrow x \in \mathsf{U}(BC)$. Hence the first expression of the set $\mathsf{U}(BC)$. The second one is got in a similar way (this time 3.5.1 (5) is used).

We get the expression of $(BC)\,\mathsf{U}$ from above in consequence of Remark 3.5.

3.5.3 *Let B and C be congruences in an Ω-group G, Q an Ω-sugbroup of G, $Q \supseteq \mathsf{U}B \cap \mathsf{U}C$. Then there holds*

$$(Q \cap \mathsf{U}B) \cap [B(0) + C(0)] = \mathsf{U}B \cap [Q \cap B(0) + Q \cap C(0)].$$

The equality remains preserved if we change $B(0)$ and $C(0)$ on one side of the equation.

Especially for $Q = \mathsf{U}B$ we get

$$\mathsf{U}B \cap [B(0) + C(0)] = B(0) + \mathsf{U}B \cap C(0) = \mathsf{U}B \cap [C(0) + B(0)] = \\ = \mathsf{U}B \cap C(0) + B(0).$$

Proof. By 3.4.1 $(B \sqcap Q)(C \sqcap Q) = (BC) \sqcap Q$ and by 3.5.1 (2) and (5) there holds

$$x[(B \sqcap Q)(C \sqcap Q)]\, 0 \Leftrightarrow x \in (Q \cap \mathsf{U}B) \cap [Q \cap B(0) + Q \cap C(0)]$$
$$(= \mathsf{U}B \cap [Q \cap B(0) + Q \cap C(0)])$$
$$\Leftrightarrow x \in (Q \cap \mathsf{U}B) \cap [Q \cap C(0) + Q \cap B(0)]$$
$$(= \mathsf{U}B \cap [Q \cap C(0) + Q \cap B(0)])$$
$$x[(BC) \sqcap Q]\, 0 \Leftrightarrow x \in 0,\ xBC0 \Leftrightarrow$$
$$\Leftrightarrow x \in Q \cap \mathsf{U}B \cap [B(0) + C(0)] \Leftrightarrow x \in Q \cap \mathsf{U}B \cap [C(0) + B(0)].$$

Hence the statement.

3.5.4 Zassenhaus lemma. *Let $\mathfrak{B}$ and $\mathfrak{L}$ be Ω-subgroups of an Ω-group G, $\mathfrak{B}'$ or $\mathfrak{L}'$ an ideal of $\mathfrak{B}$ or $\mathfrak{L}$, respectively. Then $\mathfrak{B}' + \mathfrak{B} \cap \mathfrak{L}$ is an Ω-subgroup of G and $\mathfrak{B}' + \mathfrak{B} \cap \mathfrak{L}'$ its ideal. $\mathfrak{L}' + \mathfrak{L} \cap \mathfrak{B}$ is an Ω-subgroup of G and $\mathfrak{L}' + \mathfrak{L} \cap \mathfrak{B}'$ its ideal, $\mathfrak{B} \cap \mathfrak{L}' + \mathfrak{L} \cap \mathfrak{B}'$ is an ideal of $\mathfrak{B} \cap \mathfrak{L}$ and there holds*

$$\mathfrak{B}' + \mathfrak{B} \cap \mathfrak{L}/\mathfrak{B}' + \mathfrak{B} \cap \mathfrak{L}' \cong \mathfrak{L}' + \mathfrak{L} \cap \mathfrak{B}/\mathfrak{L}' + \mathfrak{L} \cap \mathfrak{B}' \cong \mathfrak{B} \cap \mathfrak{L}/\mathfrak{B} \cap \mathfrak{L}' + \mathfrak{L} \cap \mathfrak{B}'.$$

(e.g. [9] III 4.3)

3.5.5 *Let B and C be congruences in an Ω-group G. Then*

$$(3.5,1) \qquad \mathsf{U}(BC) = \mathsf{U}B \cap [B(0) + \mathsf{U}C] = B(0) + \mathsf{U}B \cap \mathsf{U}C = \\ = \mathsf{U}B \cap [\mathsf{U}C + B(0)] = \mathsf{U}B \cap \mathsf{U}C + B(0).$$

$$(3.5,2) \qquad (BC)(0) = \mathsf{U}B \cap [B(0) + C(0)] = B(0) + \mathsf{U}B \cap C(0) = \\ = \mathsf{U}B \cap [C(0) + B(0)] = \mathsf{U}B \cap C(0) + B(0),$$

$\mathsf{U}(BC)$ is an Ω-subgroup of G, $(BC)(0)$ its ideal, $\mathsf{U}B \cap C(0) + \mathsf{U}C \cap B(0)$ an ideal of $\mathsf{U}B \cap \mathsf{U}C$ and there holds

$$\mathsf{U}(BC) \mid (BC)(0) \cong \mathsf{U}(CB) \mid (CB)(0) \cong \mathsf{U}B \cap \mathsf{U}C \mid \mathsf{U}B \cap C(0) + \mathsf{U}C \cap B(0).$$

Proof. The equalities (3.5,2) follows from 3.5.1 and from 3.5.3 for $Q = \mathsf{U}B$. The equalities (3.5,1) are got from 3.5.2 and from the statement 3.5.3 applied to the congruences B and $\bar{C} = \mathsf{U}C/\mathsf{U}C$ and to $Q = \mathsf{U}B$. Actually, $\mathsf{U}(BC) = \mathsf{U}B \cap [B(0) + \mathsf{U}C] = \mathsf{U}B \cap [B(0) + \bar{C}(0)] = B(0) + \mathsf{U}B \cap \bar{C}(0) = B(0) + \mathsf{U}B \cap \mathsf{U}C$.

Similarly we can get the other expressions of the set $\mathsf{U}(BC)$. The rest of the statement follows from the Zassenhaus's lemma 3.5.4 by setting $\mathfrak{B} = \mathsf{U}B$, $\mathfrak{B}' = B(0)$, $\mathfrak{L} = \mathsf{U}C$, $\mathfrak{L}' = C(0)$.

3.5.6 *For congruences B and C in an Ω-group there holds $(BCBC\ldots)(0) = (BC)(0)$ provided the product on the left contains any finite number of factors $n \geqq 2$.*

Proof by induction with respect to the number of factors n. It is obvious that the statement holds for two factors. Now we shall prove its validity for three factors. It holds $x(BCB)\,0 \Rightarrow xBCaB0 \Rightarrow$ (by 3.5.1) $-a + x \in C(0) + B(0)$, $x \in \mathsf{U}B$, $a \in \mathsf{U}C \cap B(0) \Rightarrow$ (by 3.5.5) $-a + x \in \mathsf{U}B \cap [C(0) + B(0)] = B(0) + \mathsf{U}B \cap C(0)$, $a \in B(0) \Rightarrow x = a + (-a + x) \in B(0) + \mathsf{U}B \cap C(0) = (BC)(0)$. Hence $(BCB)(0) \subseteq \subseteq (BC)(0)$. The reverse inclusion is evident since $xBC0 \Rightarrow xBC0B0 \Rightarrow xBCB0$. The induction hypothesis: Let $n \geqq 4$ be a positive integer, $(BCBC\ldots)(0) = (BC)(0)$ provided that $2 \leqq p \leqq n - 1$ holds for the number of factors p in the product on the left. Now let $x(BCBC\ldots)\,0$ and let the product contain $n \geqq 4$ factors. Then there exists an element $a \in \mathsf{U}B \cap \mathsf{U}C$ such that $x(BC)\,a(BC\ldots)\,0$ thus by 3.5.1 and 3.5.5 we have $x - a \in \mathsf{U}B \cap [B(0) + C(0)] = (BC)(0)$ and by the induction hypothesis there will be $a \in (BC\ldots)(0) = (BC)(0)$. Hence $x = (x - a) + a \in (BC)(0)$ since $(BC)(0)$ is a subgroup by 3.5.5. So $(BCBC\ldots)(0) \subseteq (BC)(0)$ is proved. The reverse inclusion is obvious since $xBC0 \Rightarrow x(BC)\,0(BC\ldots)\,0 \Rightarrow x(BCBC\ldots)\,0$.

3.5.7 Corollary. *For congruences B and C in an Ω-group there holds*

$$(B \vee_P C)(0) = B(0) \cup (BC)(0) \cup C(0) \cup (CB)(0) =$$
$$= [B(0) + \mathsf{U}B \cap C(0)] \cup [C(0) + \mathsf{U}C \cap B(0)].$$

The member in the first square bracket or in the second one is an ideal in the Ω-group $B(0) + \mathsf{U}B \cap \mathsf{U}C$ or $C(0) + \mathsf{U}C \cap \mathsf{U}B$, respectively. The order of summands (in one or both square brackets) may be changed.

Proof. If the product $BCBC\ldots$ contains n factors ($n \geqq 1$) denote $BCBC\ldots = A_n$. Analogously define $CBCB\ldots = D_n$. Proof follows now from 3.5.6 and 3.5.5 since $(B \vee_P C)(0) = \bigcup_{n=1}^{\infty} A_n(0) \cup \bigcup_{n=1}^{\infty} D_n(0) = B(0) \cup (BC)(0) \cup C(0) \cup (CB)(0) = [B(0) + \mathsf{U}B \cap C(0)] \cup [C(0) + \mathsf{U}C \cap B(0)]$. The rest of the assertion follows from 3.5.5.

3.5.8 *For congruences B and C in an algebra there holds $\mathsf{U}(BCB\ldots) = \mathsf{U}(BC) \subseteq \mathsf{U}B$ (provided that on the left there are at least two factors).*

Proof. $x \in \mathsf{U}(BCB\ldots) \Rightarrow x(BC)\,a(B\ldots)\,y$ for some $a, y \Rightarrow xBCa \Rightarrow x \in \mathsf{U}(BC)$. Conversely, $x \in \mathsf{U}(BC) \Rightarrow xBaCy$ for some a, y. Hence $x \in \mathsf{U}B$ and $xBaCaBa\ldots a$ so $x(BCB\ldots)\,a$ thus $x \in \mathsf{U}(BCB\ldots)$.

3.6 Definition. If Q is a subset of an Ω-group G, denote by $[\![Q]\!]$ the subgroup of the additive group G generated by the subset Q.

3.6.1 Corollary. *For congruences B and C in an Ω-group there holds*

$$(B \vee_P C)(0) \subseteq [B(0) + C(0)] \cup [C(0) + B(0)] \subseteq [B(0) \cup C(0)].$$

This is a corollary to 3.5.7 and 3.6.

3.6.2 *For congruences B and C in an Ω-group there holds*

$$x(B \vee_P C)\,y \Rightarrow x - y, \quad y - x \in [\![B(0) \cup C(0)]\!].$$

Proof. $x(B \vee_P C)\,y \Rightarrow x = x_0 A_1 x_1 \ldots x_{n-1} A_n x_n = y$, where $A_i = B$ or $= C$ ($i = 1, 2, \ldots, n) \Rightarrow x_{i-1} - x_i, x_i - x_{i-1} \in A_i(0)$ $(i = 1, 2, \ldots, n) \Rightarrow x - y = (x_0 - x_1) + (x_1 - x_2) + \ldots + (x_{n-1} - x_n) \in [\![B(0) \cup C(0)]\!]$, $y - x = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots + (x_1 - x_0) \in [\![B(0) \cup C(0)]\!]$.

3.6.3 *For congruences B and C in an Ω-group G the following conditions are equivalent*

1. $(B \vee_P C)(0)$ *is a subgroup of the additive group G*
2. $(B \vee_P C)(0) = B(0) + C(0) = C(0) + B(0)$
3. $(B \vee_P C)(0) = (BC)(0)$ or $= (CB)(0)$
4. $[\![B(0) \cup C(0)]\!] \subseteq \mathsf{U}B \cup \mathsf{U}C$
5. $C(0) \subseteq \mathsf{U}B$ or $B(0) \subseteq \mathsf{U}C$.

Remark. The first (second) alternatives of the conditions 3 and 5 are equivalent.

Proof. $2 \Rightarrow 4$: Since $B(0)$ and $C(0)$ commute so $B(0) + C(0) = [\![B(0) \cup C(0)]\!]$. Then by [12] 1.6 $[\![B(0) \cup C(0)]\!] = (B \vee_P C)(0) \subseteq \mathsf{U}(B \vee_P C) = \mathsf{U}B \cup \mathsf{U}C$.

$4 \Rightarrow 5$: If it were $B(0) \nsubseteq \mathsf{U}C$, $C(0) \nsubseteq \mathsf{U}B$, then for $b \in B(0) \setminus \mathsf{U}C$, $c \in C(0) \setminus \mathsf{U}B$ there would hold $b + c \notin \mathsf{U}B \cup \mathsf{U}C$ while $b + c \in B(0) + C(0) \subseteq [\![B(0) \cup C(0)]\!] \subseteq \mathsf{U}B \cup \mathsf{U}C$ – a contradiction.

$5 \Rightarrow 2$, $5 \Rightarrow 3$: By 3.6.1 we have $(B \vee_P C)(0) \subseteq [\![B(0) \cup C(0)]\!]$. Since $C(0)$ or $B(0)$ is a normal subgroup of the subgroup $\mathsf{U}C$ or $\mathsf{U}B$, respectively, it follows from the condition 5 that $B(0)$ and $C(0)$ are permutable subgroups thus $B(0) + C(0) = C(0) + B(0) = [\![B(0) \cup C(0)]\!]$. Now let e.g. $C(0) \subseteq \mathsf{U}B$. Then by 3.5.5 $(BC)(0) = B(0) + \mathsf{U}B \cap C(0) = B(0) + C(0) = [\![B(0) \cup C(0)]\!]$. Summarizing the got results we have $(B \vee_P C)(0) \subseteq [\![B(0) \cup C(0)]\!] = (BC)(0) \subseteq (B \vee_P C)(0)$, $[\![B(0) \cup C(0)]\!] = B(0) + C(0) = C(0) + B(0)$. Hence 2 and 3: $(B \vee_P C)(0) = B(0) + C(0) = C(0) + B(0) = (BC)(0)$. If we suppose that in 5 the second alternative $B(0) \subseteq \mathsf{U}C$ holds, then the proof is similar.

$2 \Rightarrow 1$ is evident.

$1 \Rightarrow 4$: Since obviously $B(0) \cup C(0) \subseteq (B \vee_P C)(0)$, it will be $[\![B(0) \cup C(0)]\!] \subseteq (B \vee_P C)(0)$ so we have $[\![B(0) \cup C(0)]\!] \subseteq (B \vee_P C)(0) \subseteq \mathsf{U}(B \vee_P C) = \mathsf{U}B \cup \cup C$, i.e., 4 holds.

$3 \Rightarrow 5$: Let $(B \vee_P C)(0) = (BC)(0)$. Then from the relations $B(0) \cup C(0) \subseteq \subseteq (B \vee_P C)(0) = (BC)(0) = \mathsf{U}B \cap [B(0) + C(0)] \subseteq \mathsf{U}B$ (the inclusions are obvious, the second equality by 3.5.5) there follows $C(0) \subseteq \mathsf{U}B$ thus 5. Similarly the second alternative of the condition 5, $B(0) \subseteq \mathsf{U}C$, follows from $(B \vee_P C)(0) = (CB)(0)$. The theorem is proved.

In 3.3.1(6) it was proved that the condition $B \vee_P C = B \vee_{\mathscr{K}} C$ (in contrast to the equalities $BC = B \vee_P C$ or $BC = B \vee_{\mathscr{K}} C$) is neither sufficient nor necessary for the commutativity of congruences B and C in an Ω-group. In the following considerations our attention is paid to this and to the related equalities.

3.7 *For congruences B and C in an Ω-group the following conditions are equivalent.*

1. $(B \vee_{\mathscr{K}} C)(0) = (B \vee_P C)(0)$
2. $(B \vee_{\mathscr{K}} C)(0) = [\![B(0) \cup C(0)]\!] \subseteq \mathsf{U}B \cup \mathsf{U}C$
3. $(B \vee_{\mathscr{K}} C)(0) = (B(0) + C(0) \subseteq \mathsf{U}B$ or $\subseteq \mathsf{U}C$
4. $(B \vee_{\mathscr{K}} C)(0) = (B \vee_P C)(0) = (BC)(0)$ or $= (CB)(0)$.

Remark. The first (second) alternatives in the conditions 3 and 4 are equivalent.

Proof. $1 \Rightarrow 2$: First the subgroup $(B \vee_{\mathscr{K}} C)(0)$ equals $B(0) + C(0)$ by 3.6.3 thus $(B \vee_{\mathscr{K}} C)(0) = [\![B(0) \cup C(0)]\!]$. Second by [12] 1.6 $(B \vee_{\mathscr{K}} C)(0) = (B \vee_P C)(0) \subseteq \subseteq \mathsf{U}(B \vee_P C) = \mathsf{U}B \cup \mathsf{U}C$.

$2 \Rightarrow 3$: By 3.6.3 $C(0) \subseteq \mathsf{U}B$ or $B(0) \subseteq \mathsf{U}C$. Since $C(0)$ or $B(0)$ is a normal subgroup in $\mathsf{U}C$ or $\mathsf{U}B$, respectively, $B(0)$ and $C(0)$ commute thus $(B \vee_{\mathscr{K}} C)(0) = = [\![B(0) \cup C(0)]\!] = B(0) + C(0) \subseteq \mathsf{U}B$ or $\subseteq \mathsf{U}C$.

$3 \Rightarrow 4$: Suppose the first alternative in 3. Then $(B \vee_{\mathscr{K}} C)(0) = \mathsf{U}B \cap [B(0) + + C(0)] = (BC)(0)$ (this follows from 3.5.5). We get the first alternative in 4 from the fact that $(BC)(0) = (B \vee_{\mathscr{K}} C)(0) \supseteq (B \vee_P C)(0) \supseteq (BC)(0)$. The proof is analogous for the other alternatives in 3 and 4.

$4 \Rightarrow 1$: evident.

Proof of the Remark. We have shown ($3 \Rightarrow 4$) that the first (second) alternative of the condition 3 implies the first (second) alternative of the condition 4. If now the first alternative of the condition 4 is valid, then by 3.5.5 $(B \vee_{\mathscr{K}} C)(0) = (BC)(0) = = \mathsf{U}B \cap [B(0) + C(0)] \subseteq \mathsf{U}B$ thus there holds the first alternative of the condition 3. Similarly for the second alternatives.

3.7.1 *For congruences B and C in an Ω-group it holds:*

$$\mathsf{U}(B \vee_{\mathscr{K}} C) = \mathsf{U}(B \vee_P C) \Leftrightarrow \mathsf{U}B \supseteq \mathsf{U}C \text{ or } \mathsf{U}C \supseteq \mathsf{U}B.$$

Proof. Let the first condition hold. There holds $\mathsf{U}C \supseteq \mathsf{U}B$ or $a \in \mathsf{U}B \setminus \mathsf{U}C$ exists. In the second case there will be $(a + \mathsf{U}C) \cap \mathsf{U}C = \emptyset$ and by [12] 1.6 $a + \mathsf{U}C \subseteq$

$\subseteq \langle \mathsf{U}B \cup \mathsf{U}C \rangle = \mathsf{U}(B \vee_{\mathscr{K}} C) = \mathsf{U}(B \vee_P C) = \mathsf{U}B \cup \mathsf{U}C$ thus $a + \mathsf{U}C \subseteq \mathsf{U}B$. Hence $\mathsf{U}C \subseteq -a + \mathsf{U}B = \mathsf{U}B$, i.e. $\mathsf{U}C \subseteq \mathsf{U}B$. The reverse implication follows from [12] 1.6.

3.7.2 *Let B and C be congruences in an Ω-group, $Q = \mathsf{U}B \cap \mathsf{U}C$. Then*

$$Q \cap (B \vee_{\mathscr{K}} C)(0) = Q \cap B(0) + Q \cap C(0) \Leftrightarrow (B \vee_{\mathscr{K}} C) \sqcap Q = (B \vee_P C) \sqcap Q.$$

Remark. 1. If $Q = \mathsf{U}B \cap \mathsf{U}C$, then $Q \cap B(0) + Q \cap C(0) = Q \cap [B(0) + C(0)]$.

2. Thus in an abelian or a hamiltonian group the condition of Theorem 3.7.2 is satified (provided $Q = \mathsf{U}B \cap \mathsf{U}C$) since $Q \cap (B \vee_{\mathscr{K}} C)(0) = Q \cap [B(0) + C(0)]$. Thus the following statement is true:

If B, C are congruences in an abelian or a hamiltonian group, $Q = \mathsf{U}B \cap \mathsf{U}C$, then $(B \vee_{\mathscr{K}} C) \sqcap Q = (B \vee_P C) \sqcap Q$.

Proof to 3.7.2. Let the first condition hold. Let $x[(B \vee_{\mathscr{K}} C) \sqcap Q]\, y$, $x - y = a$. The element a belongs to the set $Q \cap (B \vee_{\mathscr{K}} C)(0)$ and it may be expressed as a sum $a = b + c$, where $b \in B(0)$, $c \in C(0)$. From the relation $a - c = b \in B(0)$ there follows $(a - c)\, B0$; hence and from the fact that $c = -b + a \in \mathsf{U}B$ there follows aBc. Together with the relation $cC0$ we get $aBcC0$, $(x - y)\, BcC0$ and finally (because of $y \in Q = \mathsf{U}B \cap \mathsf{U}C$) $xB(c + y)\, Cy$, i.e. $xBCy$. So $(B \vee_{\mathscr{K}} C) \sqcap Q \leqq BC \sqcap Q$ is proved. This relation together with the evident relations $BC \leqq B \vee_P C \leqq B \vee_{\mathscr{K}} C$ gives the desired equality.

Conversely, if the second condition holds, then by 3.4.4 $(B \sqcap Q) \vee_{\mathscr{K}} (C \sqcap Q) = (B \vee_{\mathscr{K}} C) \sqcap Q = (B \vee_P C) \sqcap Q$. The last partition is equal to $(B \sqcap Q) \vee_P (C \sqcap Q)$ by 3.4.2. Hence we get the equality between $\vee_P$ and $\vee_{\mathscr{K}}$ of the congruences $B \sqcap Q$ and $C \sqcap Q$ so by 3.7 $[(B \vee_{\mathscr{K}} C) \sqcap Q](0) = [(B \sqcap Q) \vee_{\mathscr{K}} (C \sqcap Q)](0) = [(B \sqcap Q) \vee_{\mathscr{K}}) (C \sqcap Q)](0)$. Obviously the first set equals $Q \cap (BC)(0)$ and by 3.5.5 the last set equals $Q \cap B(0) + Q \cap C(0)$. The theorem is proved.

Proof of the remark 1. follows from 3.5.3.

3.7.3 Corollary. *For congruences B and C on an Ω-group there holds $B \vee_{\mathscr{K}} C = B \vee_P C$.*

Indeed, for congruences B and C on the Ω-group G there is $Q = \mathsf{U}B \cap \mathsf{U}C = G$ so that $Q \cap (B \vee_{\mathscr{K}} C)(0) = (B \vee_{\mathscr{K}} C)(0) = \langle\langle B(0) \cup C(0) \rangle\rangle_G = B(0) + C(0) = Q \cap B(0) + Q \cap C(0)$. The second equality follows from [12] 1.6, the third from the fact that the ideal generated by a system of ideals is their sum. Finally, 3.7.2 gives the desired equality.

3.7.4 Remark. *If B and C are congruences in an Ω-group and $Q = \mathsf{U}B \cap \mathsf{U}C$, then*

$$(B \vee_P C) \sqcap Q = (BC) \sqcap Q = (CB) \sqcap Q.$$

Proof. By 3.4.1 and 3.4.2 there holds

$$(3.7.1)\quad (B \vee_P C) \sqcap Q = (B \sqcap Q) \vee_P (C \sqcap Q),\ (BC) \sqcap Q = (B \sqcap Q)(C \sqcap Q),$$
$$(CB) \sqcap Q = (C \sqcap Q)(B \sqcap Q).$$

Since $B \sqcap Q$ and $C \sqcap Q$ are congruences on the Ω-group Q, they commute and by 3.3.1(1) the right sides of the equations (3.7.1) equal one another, so do the left sides.

3.7.5 *For congruences B and C in an Ω-group the following conditions are equivalent*

1. $\mathsf{U}(B \vee_{\mathscr{K}} C) = \mathsf{U}(BC)$
2. $\mathsf{U}(B \vee_P C) = \mathsf{U}(BC)$
3. $B(0) + \mathsf{U}C = \mathsf{U}B$
4. $(B \vee_{\mathscr{K}} C)(0) = (B \vee_P C)(0) = (BC)(0) = B(0) + C(0)$,
   $\mathsf{U}(B \vee_{\mathscr{K}} C) = \mathsf{U}(B \vee_P C) = \mathsf{U}(BC)$.

Each of these conditions implies the condition 5. $B \vee_{\mathscr{K}} C = B \vee_P C$.

Proof. $1 \Rightarrow 2$ evidently.
$2 \Rightarrow 3$: From 3.5.5 and from [12] 1.6 there follows $\mathsf{U}B \supseteq B(0) + \mathsf{U}B \cap \mathsf{U}C = \mathsf{U}(BC) = \mathsf{U}(B \vee_P C) = \mathsf{U}B \cup \mathsf{U}C$. Hence $\mathsf{U}B \supseteq \mathsf{U}C$ thus $B(0) + \mathsf{U}C = \mathsf{U}(BC) = \mathsf{U}(B \vee_P C) = \mathsf{U}B$, i.e. there holds the condition 3.
$3 \Rightarrow 5$: First from the condition 3 there follows obviously $C(0) \subseteq \mathsf{U}C \subseteq \mathsf{U}B$. Further, Teorem 3.5.5 verifies that $(BC)(0) = B(0) + \mathsf{U}B \cap C(0) = B(0) + C(0)$ is an ideal in the Ω-group $\mathsf{U}(BC) = B(0) + \mathsf{U}B \cap \mathsf{U}C = B(0) + \mathsf{U}C = \mathsf{U}B = \langle \mathsf{U}B \cup \mathsf{U}C \rangle$ so that by [12] 1.6 $B(0) + C(0) = \ll B(0) \cup C(0) \gg \cup B \cup \mathsf{U}C = (B \vee_{\mathscr{K}} C)(0)$. This proves $(B \vee_{\mathscr{K}} C)(0) = B(0) + C(0) = (BC)(0)$.

We shall prove $x(B \vee_{\mathscr{K}} C)\, y = x(B \vee_P C)\, y$ and hence the required implication. For this purpose consider that $B(0) + C(0) + \mathsf{U}C = B(0) + \mathsf{U}C = \mathsf{U}B$ holds. It follows that for any $x \in \mathsf{U}B$ there exists $c \in \mathsf{U}C$ such that $B(0) + C(0) + x = B(0) + C(0) + c$. Denote by the symbol b with indices or c with indices elements of $B(0)$ or $C(0)$, respectively; then $x(B \vee_{\mathscr{K}} C)\, y \Rightarrow y - x \in (B \vee_{\mathscr{K}} C)(0) = B(0) + C(0)$, $x, y \in \mathsf{U}B (= \langle \mathsf{U}B \cup \mathsf{U}C \rangle) \Rightarrow y \in B(0) + C(0) + x = B(0) + C(0) + c \Rightarrow$

$$y = b_0 + c_0 = x = b_1 + c_1 + c \tag{3.7.2}$$

(for suitable elements b_0, b_1, c_0 and c_1). From the normality of a subgroup $B(0)$ in $\mathsf{U}B$ we get successively for suitable elements b_2, b_3 and c_2:

$$c_0 + x = -b_0 + b_1 + c_1 + c = c_1 + b_2 + c, \; -c_1 + c_0 + x = \\ = b_2 + c = c + b_3, \; x - b_3 = c_2 + c.$$

Hence $xB(x - b_1) = (c_2 + c)\, CcC(c_1 + c) = (-b_1 + y)\, By$ (the last equality – see (3.7.2)), i.e. $xBCCBy$ thus $x(B \vee_P C)\, y$. So $3 \Rightarrow 5$ is proved.
$3 \Rightarrow 4$: We have just proved $3 \Rightarrow 5$. From the conditions 5 and 3 there follows obviously $\mathsf{U}(B \vee_{\mathscr{K}} C) = \mathsf{U}(B \vee_P C) = \mathsf{U}B \cup \mathsf{U}C = \mathsf{U}B$. Then $\mathsf{U}(BC) = B(0) + \mathsf{U}B \cap \mathsf{U}C = B(0) + \mathsf{U}C = \mathsf{U}B$ by 3.5.5. This verifies the second part of the condition 4

(concerning the domains). In the proof to ($3 \Rightarrow 5$) there was proved $(B \vee_P C)(0) = (B \vee_{\mathscr{K}} C)(0) = B(0) + C(0) = (BC)(0)$. Hence 4.
$4 \Rightarrow 1$ is evident.

3.7.6 Remark. It will be proved in 3.9.5 that the conditions of the preceding theorem do not imply the commutativity of the congruences B and C.

3.8 *Congruences B and C in an Ω-group commute if and only if* $(BC)(0) = (CB)(0)$.

Proof. Let $(BC)(0) = (CB)(0)$ hold. Then $xBCy \Rightarrow xBaCy \Rightarrow xBaCa \Rightarrow (x - a)B0C0 \Rightarrow (x - a)BC0 \Rightarrow (x - a)CB0 \Rightarrow x - a \in \mathsf{U}C$. Since also $a \in \mathsf{U}C$, we have $x = (x - a) + a \in \mathsf{U}C$. Now the symmetry of the relation BC may be proved:

$$xBCy \Rightarrow xBaCy \Rightarrow 0B(a - x)C(y - x) \Rightarrow 0BC(y - x) \Rightarrow (y - x)CB0 \Rightarrow$$
$$\Rightarrow (y - x)BC0 \Rightarrow 0CB(y - x) \Rightarrow 0CbB(y - x) \Rightarrow xC(b + x)By \Rightarrow xCBy \Rightarrow$$
$$\Rightarrow yBCx.$$

By 3.3 the commutativity B and C follows from the symmetry of the relation BC.
Conversely, if B and C commute, i.e. if $BC = CB$ holds, then obviously $(BC)(0) = (CB)(0)$.

3.8.1 *Let B and C be congruences in an Ω-group,* $\mathsf{U}(B \vee_P C) = \mathsf{U}(BC)$. *Then the following conditions are equivalent.*

1. $(B \vee_{\mathscr{K}} C)(0) = (CB)(0)$
2. $(B \vee_P C)(0) = (CB)(0)$
3. $C(0) + B(0) = (CB)(0)$
4. $B(0) + C(0) = (CB)(0)$
5. *B and C commute.*

Proof. First note that the condition $\mathsf{U}(B \vee_P C) = \mathsf{U}(BC)$ implies $B(0) + \mathsf{U}C = \mathsf{U}B$ by 3.7.5(3) thus also $C(0) \subseteq \mathsf{U}C \subseteq \mathsf{U}B$.
$1 \Rightarrow 2$ is evident.
$2 \Rightarrow 3$: By 3.5.7 $(CB)(0) = (B \vee_P C)(0) = [C(0) + B(0)] \cup [C(0) + \mathsf{U}C \cap B(0)] = C(0) + B(0)$.
$3 \Rightarrow 4$: $C(0) + B(0) = (CB)(0) \subseteq \mathsf{U}C$ (by 3.5.5) thus $B(0)$ and $C(0)$ are ideals in $\mathsf{U}C$ and thus they commute.
$4 \Rightarrow 1$: By 3.7.5 the condition $\mathsf{U}(B \vee_P C) = \mathsf{U}(BC)$ implies $(B \vee_{\mathscr{K}} C)(0) = B(0) + C(0)$.
$4 \Rightarrow 5$: The condition $\mathsf{U}(B \vee_P C) = \mathsf{U}(BC)$ implies $(BC)(0) = B(0) + C(0)$ by 3.7.5 thus $(BC)(0) = (CB)(0)$. B and C commute by 3.8.
$5 \Rightarrow 1$: By 3.7.5 the condition $\mathsf{U}(B \vee_P C) = \mathsf{U}(BC)$ implies $(B \vee_{\mathscr{K}} C)(0) = B(0) + C(0)$and $\mathsf{U}B \supseteq \mathsf{U}C$, the condition $BC = CB$ gives $(CB)(0) = (BC)(0) = B(0) + \mathsf{U}B \cap C(0) = B(0) + C(0)$ by 3.5.5. Hence 1.

3.9 *Congruences B and C in an Ω-group commute if and only if* $B(0) \cup C(0) \subseteq \subseteq \mathsf{U}B \cap \mathsf{U}C$.

Proof. Let B and C commute. If $x \in B(0) \setminus \mathsf{U}C$ exists, then $xB0Cy$ for all $y \in C(0)$ thus $xBCy$. By hypothesis $xCBy$, i.e. $xCbBy$ for suitable $b \in \mathsf{U}C \cap \mathsf{U}B$ thus $x \in \mathsf{U}C$, a contradiction. Hence $B(0) \subseteq \mathsf{U}C$. The validity of the inclusion $C(0) \subseteq \mathsf{U}B$ is proved symmetrically.

Let $B(0) \cup C(0) \subseteq \mathsf{U}B \cap \mathsf{U}C$ hold. $B(0)$ and $C(0)$ are ideals in $\mathsf{U}B \cap \mathsf{U}C$, thus permutable subgroups. Hence we get by 3.5.5 $(BC)(0) = \mathsf{U}B \cap [B(0) + C(0)] = = B(0) + C(0) = C(0) + B(0) = \mathsf{U}C \cap [C(0) + B(0)] = (CB)(0)$. The commutativity B and C now follows from 3.8.

3.9.1 Corollary. *Congruences on an Ω-group are permutable.*

3.9.2 Corollary. *For permutable congruences B and C in an Ω-group G there holds* $(B, C)\,M^*$ and $(C, B)\,M^*$ *both with respect to $P(G)$.*

It follows from 3.9 and [13] 2.2.

3.9.3 Corollary. *Null blocks of permutable congruences B and C in an Ω-group are permutable subgroups.*

It follows from 3.9 since $B(0)$ and $C(0)$ are ideals in $\mathsf{U}B \cap \mathsf{U}C$.

3.9.4 *The commutativity of the subgroups* $B(0)$ *and* $C(0)$ *does not imply the commutativity of the congruences B and C in Ω-group*
as the example 3.1.1(2) proves:
The congruences B and C do not commute, but $B(0)$ and $C(0)$ are permutable subgroups in G since the group G is abelian. (The noncommutativity of the congruences B, C follows also easily from 3.9 since $B(0) \cup C(0) = Y \nsubseteq X = \mathsf{U}B \cap \mathsf{U}C$).

3.9.5 The same example as in 3.9.4 proves that
the conditions of Theorem 3.7.5
do not imply the commutativity of congruences B and C in an Ω-group.

Indeed, the congruences B and C do not commute, but at the same time the condition 3.7.5(3) $C(0) + \mathsf{U}B = Y + X = G = \mathsf{U}C$ is satisfied.

3.10 *Congruences B and C in an Ω-group commute if and only if* $\mathsf{U}(BC) = \mathsf{U}(CB)$.

Proof. From the first condition, $BC = CB$, there follows obviously the second one, $\mathsf{U}(BC) = \mathsf{U}(CB)$. Now suppose that $\mathsf{U}(BC) = \mathsf{U}(CB)$. By 3.5.5 $\mathsf{U}B \supseteq B(0) + + \mathsf{U}B \cap \mathsf{U}C = C(0) + \mathsf{U}C \cap \mathsf{U}B \subseteq \mathsf{U}C$, thus $B(0) \cup C(0) \subseteq \mathsf{U}B \cap \mathsf{U}C$. By 3.9 B and C commute.

3.11 *For congruences B and C in an Ω-group the following conditions are equivalent.*
1. $\mathsf{U}B = \mathsf{U}C$ 2. $BC = B \vee_P C$ 3. $BC = B \vee_{\mathscr{X}} C$.

Proof. $2 \Leftrightarrow 3$ – see 3.3.1(4).
$1 \Rightarrow 2$ follows from 3.3.1(1) and from the fact that congruences on group commute.
$2 \Rightarrow 1$ follows from 3.1.1(6).

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