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# ON CONVERGENCE OF THE ITERATIVE METHODS

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A. Birger [1] gives a new method for calculation of characteristic values and characteristic functions, so-called composed iterative method with variable parameter and similar iterative method for solving the linear functional equations, but without any conditions and proofs of convergence and any error bounds of these methods. In this paper the convergence conditions, the error bounds of these methods and some new methods are given. Proofs and other theorems are omitted and will be published in Czech.Math.Journ. and Čas.pěst.mat.

1. Let the equation

(1) $$y - \lambda K y = 0$$

be given, where $K$ is a linear symmetric bounded operator in Hilbert space $H$, $\lambda$ is a real parameter. We use the iterative formulae

$$y_{n+1} = \tilde{\lambda}_{n+1} K y_n ,$$

where $\tilde{\lambda}_{n+1} (n = 0, 1, 2, \ldots)$ are to be determined from the conditions

$$\frac{\partial \| \tilde{\lambda}_{n+1} K y_n - y_n \|^2}{\partial \tilde{\lambda}_{n+1}} = 0, \quad n = 0, 1, 2, \ldots$$

Then

(2) $$\tilde{\lambda}_{n+1} = \frac{(K y_n, y_n)}{\| K y_n \|^2}, \quad y_{n+1} = \frac{(K y_n, y_n)}{\| K y_n \|^2} K y_n .$$

THEOREM 1. Let $K$ be a positive symmetric completely continuous operator in Hilbert space $H$. If an element $y_0 \in H$ is not orthogonal to the space $H_{\lambda 1}$ generated by characteristic functions corresponding to the first characteristic number $\lambda_1$ of (1), then the sequence $\{\tilde{\lambda}_n\}$ converges to $\lambda_1$. The subsequence $\{y_{nk}\}$ is convergent in the norm of the space $H$ to one of characteristic functions corresponding to $\lambda_1$.

Now we introduce a simple method for calculation of eigenvalues and eigenfunctions of the equation

(3) $$Ay - \mu y = 0,$$

where $A$ is a linear bounded operator in $H$, $\mu$ is a real parameter.

THEOREM 2. Let $A$ be a positive symmetric completely continuous operator in Hilbert space $H$. Let $y_0 \in H$ be not orthogonal to the eigenspace corresponding to the first eigenvalue $\mu_1$ of (3). Then the sequence $\{\tilde{\mu}_n\}$ defined by

$$\tilde{\mu}_{n+1} = \frac{(Ay_n, y_n)}{\|y_n\|^2}, \quad y_{n+1} = \frac{1}{\tilde{\mu}_{n+1}} Ay_n$$

converges to $\mu_1$ and the sequence $\{y_n\}$ is convergent in the norm of the space $H$ to one of the eigenfunctions corresponding to $\mu_1$.

This method can be used to non symmetric symmetrizable completely continuous operators.

NUMERICAL EXAMPLE 1. To illustrate the application one of the method we consider the equation

$$y(x) - \lambda \int_0^1 K(x,s) y(s)\, ds = 0,$$

where

(4) $$K(x,s) = \begin{cases} \frac{1}{2} x(2-s), & x \leqq s \\ \frac{1}{2} s(2-x), & x \geqq s. \end{cases}$$

The first characteristic number $\lambda_1$ of this equation is equal with the accuracy on the third decimal place to 4,115. Take $y_0 = x$, then the first step of (2) yields $\lambda_1 \approx 4{,}136$, while Ritz´s method for $n=3$ gives $\lambda_1 \approx 4{,}371$. Kellog´s iterative formulas by the third step yield $\lambda_1 \approx 4{,}998$, $\lambda_1 \approx 4{,}156$. By the second step pf (2) we get $\lambda_1 \approx 4{,}115$.

2. Let the equation

(5) $$Ay = f$$

be given, where $A$ is a linear symmetric bounded operator in $H$, $f \in H$. The composed iterative method may be written in the form:

(6) $$y_{n+1} = y_n + \alpha_n h_n, \quad \alpha_n = \frac{(Ah_n, h_n)}{\|Ah_n\|^2}, \quad h_n = f - Ay_n.$$

THEOREM 3. Let $A$ be a linear self-adjoint operator in $H$. Suppose that $m\|y\|^2 \leqq (Ay, y) \leqq M\|y\|^2$ holds for every $y \in H$, where $m = \inf_{\|y\|=1}(Ay, y)$, $M = \sup_{\|y\|=1}(Ay, y)$, $0 < m \leq M < +\infty$. Then the sequence defined by (6) is convergent in the norm of the space $H$ to the solution of (5) and its error is bounded by

$$\|y - y_n\| \leq k\gamma^n \|f - Ay_0\|,$$

$$\|y - y_n\| \leq k\gamma \|f - Ay_{n-1}\|,$$

$$\|y - y_n\| \leq k \left\{ \|h_{n-1}\|^2 - \frac{(Ah_{n-1}, h_{n-1})^2}{\|Ah_{n-1}\|^2} \right\},$$

where $\gamma = (M^2 - m^2)^{\frac{1}{2}}/M$, $k = \|A^{-1}\| \leq \frac{1}{m}$ and $y_0 \in H$.

THEOREM 4. Let $A$ be a symmetric bounded and positive ~~definite~~ operators ( $(Ay, y) > 0$ for every $y \neq 0$ ) in $H$ having bounded $A^{-1}$. Then the sequence $\{y_n\}$ defined by (6) is convergent in the norm of the space $H$ to the solu - tion $y$ of (5) and its error is bounded by

$$\|y - y_n\| \leq \frac{1}{l}\|f - Ay_n\|,$$

where the number $l > 0$ satisfies the inequality

$$\|Ay\| \geq l\|y\|.$$

We introduce an iterative method for solving the equa - tion (5) which is more simple for computation and makes smaller demands on the operator $A$. The method is based on the following theorem.

THEOREM 5. If $A$ is a linear bounded operator in $H$ and if there exists a real positive number $m$ such that the inequality

$$(Ay, y) \geq m\|y\|^2 \quad \text{for every } y \in H$$

holds, then the sequence $\{y_n\}$ defined by the equations

$$y_n = \sum_{s=0}^{n-1} \beta_s x_s, \qquad x_0 = f,$$

$$x_{s+1} = x_s - \beta_s A x_s,$$

$$\beta_s = \frac{(Ax_s, x_s)}{\|Ax_s\|^2}, \quad s = 0, 1, 2, \ldots$$

converges in the norm of the space $H$ to the solution $y$ of (5). The error $\|y - y_n\|$ of the approximative solution $y_n$ is bounded by the inequality

$$\|y - y_n\| \leq \frac{1}{m}\|f - Ay_n\|.$$

In contradistinction to the method of the steepest descent [2] and the composed iterative method with variable parameter the operator $A$ need not be self-adjoint.

3. Let the equation (5) where $A$ is a linear bounded operator in a real Hilbert space $H$, be given. The so-called similar iterative method may be written in the form

$$y_{n+1} = Pf + \varepsilon_n (I-PA) y_n,$$

where $P$ is a linear bounded operator in $H$, the parameters $\varepsilon_n$ $(n=0,1,2,\dots)$ are determined from the conditions

$$\frac{\partial F(\varepsilon_n y_n)}{\partial \varepsilon_n} = 0$$

, where $F(\varepsilon_n y_n) = \|f - \varepsilon_n A y_n\|^2$ . Then

(7) $$y_{n+1} = Pf + \frac{(f, Ay_n)}{\|Ay_n\|^2}(I-PA) y_n.$$

THEOREM 6. Let $A, P$ be a linear bounded commutative operators in a real Hilbert space $H$ (not necessarily complete) and $P$ is such that $P^{-1}$ exists and $\|I-PA\| = q < 1$

Let one of the following conditions be fulfilled:

(1 $H$ is complete space

(2 Operator $I-PA$ is completely continuous in $H$.

Then the equation (5) has a unique solution $y$ . The iterative process defined by (7) is convergent in the norm of the space $H$ to the solution $y$ of (5) and its error is bounded by

$$\|y - y_n\| \leq k q^n \|f - Ay_0\|,$$

$$\|y - y_n\| \leq k q \|f - Ay_{n-1}\|,$$

$$\|y - y_n\| \leq k q \left\{ \|f\|^2 - \frac{(f, Ay_{n-1})^2}{\|Ay_{n-1}\|^2} \right\}^{\frac{1}{2}},$$

where $k = \|A^{-1}\| \leq \|P\|/(1-q)$ , $y_0$ is an arbitrary element from $H$ .

If the parameters $\sigma_n$ $(n=0,1,2,\dots)$ are to be determined from the conditions

$$\frac{\partial F(y_{n+1})}{\partial \sigma_n} = 0, \quad (n=0,1,2,\dots)$$

where $y_{n+1} = Pf + \sigma_n R y_n$ , $R = I-PA$ ; $P, A$ are commutative operators and $F(y) = \|f - Ay\|^2$ . We obtain the iterative formulae

(8) $$\tilde{y}_{n+1} = Pf + \frac{(Rf, AR\tilde{y}_n)}{\|AR\tilde{y}_n\|^2} R\tilde{y}_n$$

THEOREM 7. Let $A, P$ be a linear bounded commutative operators in a real Hilbert space $H$ (not necessarily complete) and $P$ is such that $P^{-1}$ exists and $\|R\| = q < 1$. Let one of the following conditions be fulfilled:

(1 $H$ is complete space.

(2 $R$ is completely continuous operator in the space $H$.

Then the equation (5) has only one solution $y$. The sequence $\{\tilde{y}_n\}$ defined by (8) converges in the norm of the space $H$ to the solution $y$ of (5) and the error of approximation $\|y - \tilde{y}_n\|$ is bounded by the following inequalities:

$$\|y - \tilde{y}_n\| \leqq k q^n \|f - A\tilde{y}_0\|,$$

$$\|y - \tilde{y}_n\| \leqq k q \|f - A\tilde{y}_{n-1}\|,$$

$$\|y - \tilde{y}_n\| \leqq k q \left\{ \|Rf\|^2 - \frac{(Rf, AR\tilde{y}_{n-1})^2}{\|AR\tilde{y}_{n-1}\|^2} \right\}^{\frac{1}{2}},$$

where $k = \|A^{-1}\| \leqq \|P\|/(1-q)$, $\tilde{y}_0$ is an arbitrary element from $H$.

Let one of the conditions (1, (2 of the theorem 6 be fulfilled. We set $A = I - \lambda K$, where $K$ is a linear bounded operator. Operator $P$ will be now specified to obtain several iterative methods analoguous to methods of Neumann, Wiarda [3], Bückner [4] and Samuelson [5].

I. $P = I$ ($I$ is the identity operator). In this case we obtain Birger´s iterative method.

II. $P = \vartheta I$, where $0 < \vartheta < 1$.

III. $P = \vartheta^2 (I - \lambda K)$, $0 < \vartheta < 1$.

IV. $P = I + J$, where $J$ is a linear bounded operator in $H$.

The convergence conditions for these cases are the following:

I. $\|\lambda K\| < 1$.

II. a) $K$ is a symmetric operator.
b) $-(\lambda K y, y) \geqq 0$ for every $y \in H$.
c) $\vartheta$ satisfies the inequality

(9) $$0 < \vartheta < \frac{1}{1+\|\lambda K\|}$$

III. a) $K$ is a symmetric operator.
b) $\lambda$ is not a characteristic value of $\lambda K$.
c) $\vartheta$ satisfies (9).

IV. a) $J$ is commutative with $K$.
b) The inequality $\|G-J\| \leqq 1/(1+\|\lambda K\|)$ holds, where $G$ is the resolvent operator for $\lambda K$.

NUMERICAL EXAMPLE 2. To illustrate the application of some of the methods and error bounds given, an approximate solution of the integral equation

$$y(x) - \lambda \int_0^1 K(x,s)\, y(s)\, ds = x,$$

where $K(x,s)$ is given by (4), will be sought for some values of $\lambda$ by the similar iterative method. For $\lambda = 1$, $\|\lambda K\| < 1$ and thus method I will be used. For $y_0 = x$ we get $\varepsilon_0 =$ 0,80980 and

$$y_1 = 1{,}26993\, x - 0{,}13497\, x^3$$

with $\|y - y_1\| \leqq$ 0,013, while the method of successive approximations yields $\|y - y_1\| \leqq$ 0,043. For $\lambda = -6$, $\|\lambda K\| > 1$ and the condition for the convergence of method I is not satisfied. However, $-(\lambda K y, y) \geqq 0$ for every $y \in H$, so that method II is applicable. Take $y_0 = x$, $\vartheta = 0{,}40682 < 1/(1+\|\lambda K\|)$, then

$$y_1 = 0{,}31604\, x + 0{,}16751\, x^3,$$
$$y_2 = 0{,}27464\, x + 0{,}22996\, x^3 + 0{,}02063\, x^5$$

with $\|y - y_1\| \leqq$ 0,037, $\|y - y_2\| \leqq$ 0,018. Rall-Wiard's iterative method yields

$$y_1 = 0{,}18636\, x + 0{,}40682\, x^3,$$
$$y_2 = 0{,}21679\, x + 0{,}31713\, x^3 + 0{,}04965\, x^5$$

with $\|y - y_1\| \leqq$ 0,487, $\|y - y_2\| \leqq$ 0,060.

It is seen that the application of the similar iterative method can be more advantageous than the application of the older ones.

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