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## ON INTEGRATION IN COMPACT METRIC SPACES Zdeněk HEDRLÍN, Praha

Theorem. Let  $\mu$  be a finite Borel measure on a compact metric space X. Then there exist  $x_k \in X$ , k=1,2..., such that, for any continuous function f,

$$\int_{X} f d\mu = \mu(X) \lim_{n \to \infty} \sum_{k=1}^{\infty} f(x_k)$$

<u>Proof.</u> Clearly we may suppose that  $\mu(X) = 1$  and  $\mu(G) > 0$  for any open non-void Gc X. It is easy to prove that there exist finite closed coverings

$$\mathcal{A}_{m} = \left\{A_{1}^{m}, \ldots, A_{r}^{m}\right\}, m = 1, 2, \ldots$$
 such that

- (1)  $A_{m+1}$  refines  $A_m$ ,
- (2)  $d(A_m) \rightarrow 0$  where  $d(A_m)$  denotes the maximum of diameters of  $A_j^m$ ,
  - (3)  $\mu(A_i^m \cap A_j^m) = 0 \text{ for } i \neq j,$
- (4)  $\alpha$  (A<sub>j</sub><sup>m</sup>) > 0 for any m, j; then, evidently,
- (5) for any m, m', m > m', there exists, for every  $i=1,\ldots,r_m$ , exactly one j=j(i) with  $A_i^m\subset A_j^m$ . Moreover, we may suppose that
- Denote by  $B_{\mathbf{j}}^{\mathbf{m}}$ ,  $\mathbf{m}=1,\,2,\,\ldots,\,\mathbf{j}=1,\,\ldots,\,\mathbf{r}_{\mathbf{m}}$ , the closed interval with endpoints  $\mathbf{j}_{\mathbf{i}=1}^{\mathbf{m}}$ ,  $\mathbf{k}_{\mathbf{i}}^{\mathbf{m}}$ ,  $\mathbf{j}_{\mathbf{i}=1}^{\mathbf{m}}$ ,  $\mathbf{k}_{\mathbf{i}}^{\mathbf{m}}$ . For any set M C  $\langle$  0, 1 $\rangle$ , denote by  $\mathcal{X}_{\mathbf{M}}$  the characteristic

function of M; that is,  $\mathcal{X}_{_{ ext{M}}}$  is defined on  $\langle ext{0, 1} 
angle$  , equal to 1 on M, and to 0 on its complement. It is well known that there exists a sequence  $\{\xi_k\}$ ,  $0 \le \xi_k \le 1$ , such that, for any  $0 \le x \le \beta \le 1$ ,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{(\alpha,\beta)} (\xi_k) = \beta - \alpha$$

It is easy to see that  $\xi_k$  may be chosen to be distinct from any endpoint of the intervals  $B_i^m$ .

For a given k , there exists, for any m , exactly one j=j(m) such that  $\left\{k\in B_j^m : since \ A_{j(m+1)}^{m+1} \in B_j^m : since \ A_{j(m+1)}^{m+1} = B_$  $\subset A_{j(m)}^{m}$  (this follows from the above property (6)), the intersection  $C_k$  of all  $A_{j(m)}^m$  is non-void. Now, choose a point  $\mathbf{x}_k$  from every  $\mathbf{C}_k$  . We are going to show that xk possess the required properties.

For any Y c X, put  $u^*(Y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{Y}(x_k)$ 

 $u_*(Y) = \lim_{n \to \infty} \inf \frac{1}{n} \sum_{k=1}^n x_Y(x_k)$ where  $X_{Y}$  is the characteristic function of Y (i.e.  $X_{Y}(x) = 1$  if  $x \in Y$ ,  $X_{Y}(x) = 0$  if  $x \in X - Y$ ).

Denote by & the collection of those Z C X which can be represented, for some m , as union of certain  $A_{j}^{m}$ . Clearly,  $Z \in \mathcal{X}$  implies  $\mu_{*}(Z) \geq \mu_{*}(Z)$ . On the other hand, for any closed F c X and any  $\varepsilon > 0$ , there exists  $Z \in \mathcal{X}$  such that  $F \cap Z = \emptyset$ ,

$$\mu(Z) > 1 - \mu(F) - \varepsilon$$
 . Hence

 $\mu^*(F) \leq 1 - \mu_*(Z) \leq 1 - \mu(Z) < \mu(F) + \epsilon$ Therefore  $u^*(F) \leq u^*(F)$  for any closed F, and

 $\mu_{\star}(z) = \mu_{\star}(z) = \mu(z)$  for  $z \in \mathcal{Z}$ .

Moreover,

 $\mu^*(F) = 0$  whenever F is closed,  $\mu(F) = 0$ . It follows that

$$\int_{\mathbf{x}} \mathbf{g} \, d\mu = \lim_{n \to \infty} \frac{1}{n} \int_{k=1}^{n} \mathbf{g}(\mathbf{x}_k)$$

for any function g which is, for some m, constant in the interior of each  $\mathbb{A}^m_j$ . This concludes the proof since every continuous function can be uniformly approximated by functions of this kind.