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AN ANALOGON OF THE FIXED-POINT THEOREM AND ITS APPLICATION FOR GRAPHS

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§ 1. Introduction

In the § 2 of the present paper the notion of the N - space and the N-map is defined. The reasons for using the new notation for something, what is, in fact, a finite symmetrical graph with the diagonal, are partly technical: In the last paragraph we deal with more general graphs, which fact would force us all the same to some shortified notation of the term "finite symmetrical graph with diagonal" in the medium paragraphs. Another idea leading to our approach can be demonstrated in the following way. Let us say, for example, that two vertices of a simplicial complex are neighbours, if they belong to a common simplex (consequently, if they are different, they belong to a common 1-simplex). This way, we get a reflexive symmetrical relation on the 0-skeleton of the complex. Observe that for many complexes (e.g. for every complex representable as the barycentrical subdivision of another one), the structure of the whole complex is completely defined by the structure of its 0-skeleton with the relation, and that the simplicial mappings of such complexes correspond with the relation-preserving mappings of their 0-skeletons. Consequently, the properties of a polyhedron may be studied from the properties of its adequate discrete subset endowed with an adequate reflexive symmetrical relation. Another point of view: A polyhedron may be, for many purposes, represented by some of its ε -nets provided by the relation, e.g., "to have less than 2ε in distance".

The main results of the paper are the theorems 6.3 , 6.4 and 6.5 . The theorem 6.5 is a generalization of the following, evident, fact: A mapping φ of the set $\{0,1, \ldots, n\}$ into itself, such that $|\varphi(i) - \varphi(i+1)| \leq 1$, need not have a fixed point, but if it has none, it has a fixed set of a type $\{i, i+1\}$.

§ 2. N-spaces

2.1. Definition. An N-space $(X; R)$ is a non-void finite set X endowed with a reflexive symmetrical relation R . An N-map $f: (X; R) \to (Y; S)$ is a mapping $f: X \to Y$ such that $x R x' \Rightarrow f(x) S f(x')$. A subspace of an N-space $(X; R)$ is an N-space $(Y; S)$, where Y is a subset of X , $S = R \cap Y \times Y$. In further, unless otherwise stated, a subset of an N-space will be regarded always as its subspace.

2.2. Convention. 1) We are going to write simply X instead of $(X; R)$, etc., if there is no danger of misunderstanding.

2) We shall write $R(x) = \{y \mid y R x\}$.

3) The elements of N-space will be frequently called points.

2.3. Remarks. 1) N-spaces with N-maps form a category.

2) A mapping $f: X \to Y$ is an N-map of $(X; R)$ into $(Y; S)$, iff, for any $x \in X$, $f(R(x)) \subset S(f(x))$.

2.4. Theorem. A 1-1 N-map onto need not be an isomorphism (see 2.3.1). However, a 1-1 N-map of an N-space onto itself is always an isomorphism.

Proof: Let X be the set $\{0, 1\}$, $R = \{(0,0); (1, 1)\}$, $S = X \times X$. Then $\varphi : (X; R) \to (X; S)$, defined by $\varphi(0) = 0$ $\varphi(1) = 1$, is a 1-1 N-map onto, but it is not an isomorphism.

Now, let $(X; R)$ be an arbitrary N-space, $\varphi : X \to X$ a 1-1 N-map. As φ is 1-1, the mapping $\varphi^* = \varphi \times \varphi$ is 1-1. From the definition of the N-map we get $\varphi^*(R) \subset R$ and therefore $\varphi^*(R) = R$, because of finiteness of R. Hence $\varphi(a) \, R \, \varphi(b)$ implies $a \, R \, b$, q.e.d.

2.5. Definition. An N-space $(X; R)$, where $R = X \times X$, is called simple.

§ 3. Homotopical triviality and retraction

3.1. Definition. The product $(X; R) \times (Y; S)$ of the N-spaces $(X; R)$, $(Y; S)$, is the N-space $(X \times Y; T)$, where

$$T = \{((x, y), (x', y')) \mid x \, R \, x', \; y \, R \, y'\}.$$

3.2. Definition. Let us denote I_k the set $\{0, 1, \ldots, k\}$ with the relation R defined by: $i \, R(i + 1)$, $i \, R \, i$, $(i + 1) \, R \, i$. N-maps $f, g: X \to Y$ are said to be homotopical, if there exist a k and an N-map $h: X \times I_k \to Y$ such that $h(x, 0) = f(x)$, $h(x, k) = g(x)$ for every $x \in X$.

3.3. Definition. An N-space X is said to be homotopically trivial (h.t.), if its identical map is homotopical with a constant map.

3.4. Definition. An N-space Y is said to be a retract of an N-space X if there exist N-maps $j: Y \to X$, $r: X \to Y$ such that $r \circ j$ is the identity map of Y. The N-map r is called a retraction.

3.5. Theorem. Every retract of a h.t. N-space is h.t.

Proof: Let Y be a retract of X, j, r the corresponding maps. Let $h: X \times I_k \to X$ be a homotopy between the identical map and a constant one. Let us denote i the identical map of I_k and define $h' = r \circ h \circ (j \times i)$. We have $h': Y \times I_k \to$ $\to Y$ and $h'(x, 0) = r \circ h \circ (j \times i)(x, 0) = r \circ h(j(x), 0) =$ $= r \circ j(x) = x$, $h'(x, k) = r \circ h(j(x), k) = r(c) = \text{const.}$

§ 4. Contractibility

4.1. Definition. Let X be an N-space, $x, y \in X$. We say that x is contractible to y if $R(x) \subset R(y)$, which fact will be denoted by $x > y$. Elements x and y will be called equivalent (notation $x E y$), if $R(x) = R(y)$ (which is the same as $x > y$ and $y > x$). Obviously $>$ is a reflexive and transitive relation, E reflexive, transitive and symmetrical one. A point $x \in X$, such that there exists a point $y \in X$ with $y \neq x$, $x > y$, is called contractible.

4.2. Theorem. Let $\varphi: (X; R) \to (Y; S)$ be an isomorphism. Then $x > y$ implies $\varphi(x) > \varphi(y)$, $x E y$ implies $\varphi(x) E \varphi(y)$ and the image of a contractible point is contractible.

Proof: Let $R(x) \subset R(y)$, $z \in S(\varphi(x))$. Hence $z S \varphi(x)$ and therefore $\varphi^{-1}(z) R x$. Hence $\varphi^{-1}(z) R y$ and we get $z \in S(\varphi(y))$.

4.3. Theorem. A h.t. N-space X, consisting of more than one point, contains a contractible point.

Proof: Let $h: X \times I_k \to X$ be a homotopy between the identity map and a constant one. Without loss of generality we may assume that there exists a point $x \in X$ such that $y =$ $= h(x, 1) \neq x$. We are going to show that $R(x) \subset R(y)$. Let us denote T the relation in $X \times I_k$. Let $z \in R(x)$.

Because of $(z, 0)\ T\ (x, 1)$, we have $z = h(z, 0)\ R\ h(x, 1) = y$ and hence $z \in R(y)$.

Evidently holds:

4.4. Lemma. Let $A \subset X$, $x, y \in A$, $x > y$ in X. Then $x > y$ in A.

§ 5. Strongly contractible points

5.1. Definition. A point $x \in X$ is said to be strongly contractible, if:

1) There exists a point $y \in X$ such that $x > y$ and it is not $y > x$.

2) $z > x$ implies $x > z$.

5.2. Theorem. Let $\varphi : (X; R) \to (Y; S)$ be an isomorphism. Let $x \in X$ be a strongly contractible point. Then $\varphi(x)$ is strongly contractible.

Proof: There exists a point $y \in X$, $x > y$, $y \not> x$.[x)]
Hence, according to 4.2, $\varphi(x) > \varphi(y)$ and it is not $\varphi(y) > \varphi(x)$.
Let $z > \varphi(x)$. Hence $\varphi^{-1}(z) > x$ and therefore $x > \varphi^{-1}(z)$. Hence finally $\varphi(x) > z$.

5.3. Lemma. Let X be an N-space. Let A be a subset of X such that $a, b \in A$ implies $a > b$. Then the following statements hold:

1) If we choose an $a \in A$ and define $r(x) = a$ for $x \in A$, $r(x) = x$ for $x \notin A$, the mapping r is a retraction.

2) If there is $x > y$ in $Y = (X \setminus A) \cup (a) = r(X)$, it

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x) i.e. y non $> x$.

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is $x > y$ in X.

Proof: 1) It suffices to prove r to be an N-map; as the N-map j from the definition of retraction will be used the embedding of $Y = r(X)$ into X. Let $x, y \in X$, $x R y$. If it is $x, y \in X \setminus Y$, or $x, y \in Y$, we have obviously $r(x) R r(y)$. Now, let $x \in Y$, $y \in X \setminus Y$. As $R(y) = R(a)$, we have $r(x) = x R a = r(y)$.

2) Let $x > y$ in Y, i.e. $R(x) \setminus (A \setminus (a)) \subset R(y) \setminus (A \setminus (a))$. Let $x \notin R(a)$. The existence of $z \in A$, $z \in R(x)$, implies $x \in R(z) = R(a)$. Therefore $R(x) = R(x) \setminus (A \setminus (a)) \subset R(y) \setminus (A \setminus (a)) \subset R(y)$. Let $x \in R(a)$. Hence, $a \in R(x) \setminus (A \setminus (a)) \subset R(y)$ and consequently $y \in R(a)$. Hence, $y \in R(z)$ for every $z \in A$ and therefore $A \subset R(y)$. Finally, $R(x) \subset R(y)$.

5.4. Lemma. Let X be an N-space. Let $E_1, E_2, \ldots, E_m$ be all equivalence classes of the equivalence E and let us choose a point e_i in each E_i. Define a mapping e by: $e(x) = e_i$ iff $x \in E_i$. Then e is a retraction and the following holds:

$$x, y \in e(X), \quad x > y, \ y > x \Longrightarrow x = y .$$

Proof: Le us denote $E_i' = E_i \setminus (e_i)$. Let us form a retraction $r_1 : X \to X_1 = X \setminus E_1'$ by the lemma 5.3. In the X_1 we get the equivalence classes $(e_1), E_2, \ldots, E_m$. (For if $e_1 > x$, $x > e_1$ in X_1, the same holds in X and hence $x \in E_1 \cap X_1 = (e_1)$; as for other equivalence classes, the statement is obvious according to 5.3.)
If we have X_i with equivalence classes $(e_1), \ldots, (e_i)$, $E_{i+1}, \ldots, E_m$, let us form the retraction $r_{i+1} : X_i \to X_{i+1} = X_i \setminus E_{i+1}'$ by 5.3. Obviously, $r_m \circ r_{m-1} \circ \ldots \circ r_1(x) = e(x)$

and therefore e is a retraction. The rest is evident, for all equivalence classes consist of one point.

5.5. Lemma. Let there exists a contractible point in $e(X)$. Then X contains a strongly contractible point.

Proof: First, we are going to prove the existence of a strongly contractible point in $e(X)$. There exist x_0 , $y \in e(X)$, $x_0 > y$, $x_0 \neq y$. The point x_0 is either strongly contractible or there exists a point $x_1 \in e(X)$, $x_1 \neq x_0$, $x_1 > x_0$. If x_1 is not strongly contractible, there is a point $x_2 \in e(X)$, $x_2 \neq x_1$, $x_2 > x_1$, etc. After a finite number of steps we get x_n strongly contractible (x_i never repeats). We are going to show that x_n is strongly contractible in X . Let $z \in X$, $z > x$. Hence $e(z) > x$ and consequently $e(z) = x$ and hence finally $x > z$.

5.6. Theorem. A h.t. N-space contains a strongly contractible point, unless it is simple.

Proof: Let X be a h.t. N-space, which is not simple. Hence, there exist $x, y \in X$ such that it is not $x R y$. Consequently, it is not $x E y$ and hence $e(X)$ has more than one point. Hence, by 5.5 , X contains a strongly contractible point.

5.7. Theorem. The set Y of all points of an N-space X , which are not strongly contractible, is a retract of the N-space X .

Proof: First, let us remark that the set Y is always non-void, because the existence of a strongly contractible point implies the existence of a point, which is not strongly contractible (the point y from the definition). Let us find, for every x strongly contractible, a point $r(x)$ such that $x > r(x)$, $r(x) \neq x$. Obviously, $r(x) \in Y$. We define

$r(x) = x$ for $x \in Y$. We want to prove the implication $x R y \Rightarrow r(x) R r(y)$, which is obvious for $x, y \in Y$. Let $x \in Y$, $y \in X \setminus Y$. Then $r(x) = x \in R(y) \subset R(r(y))$ and hence, $r(x) R r(y)$. Now, let $x, y \in X \setminus Y$. We have $x \in R(y) \subset \subset R(r(y))$ and hence, $r(y) \in R(x) \subset R(r(x))$, and we get finally $r(x) R r(y)$.

§ 6. An analogon of the fixed-point theorem

6.1. Theorem. Let X be a finite set, φ a transformation of X. Then there exists a non-void set $M \subset X$ such that $\varphi(M) = M$.

Proof: Let us denote $X_1 = \varphi(X)$, $X_i = \varphi(X_{i-1})$. Obviously,

$$X \supset X_1 \supset X_2 \supset \dots \supset X_i \supset \dots$$

Because of finiteness of X we have $X_k = X_{k+1}$ for sufficiently large k. Therefore $\varphi(X_k) = X_k$ and we may take $M = X_k$.

6.2. Lemma. Let in 6.1 X be an N-space, φ an N-map. Let M be the X_k in the proof of 6.1. Then M is a retract of X.

Proof: Let us define $\psi : X \rightarrow M$ by $\psi(x) = \varphi^k(x)$. Then (2.4) $\psi' = \psi | M$ is an isomorphism. Let us denote j' the N-map of embedding M into X, $j = j' \circ \psi'^{-1}$. Evidently, $\psi \circ j$ is the identical map of M.

6.3. Theorem. Let X be an N-space. Let us denote $X_0 = X$, $X_i = \{x \in X_{i-1} \mid x$ is not strongly contractible in $X_{i-1}\}$. For a sufficiently large integer k, $X_k = X_{k+1}$ let us call the N-space X_k the centre of the N-space X and let us denote it by $K(X)$. Then the following statements hold:

1) For an arbitrary isomorphism $\varphi : X \rightarrow X$, holds

$\varphi(K(X)) = K(X)$.

2) The centre of a h.t. N-space is simple.

Proof: By 5.2 we have $\varphi(X_i) = X_i$ for every i . The second statement is a consequence of 5.6 , 5.7 and 3.5 .

6.4. Theorem. For every N-map φ of a h.t. N-space X into itself there exists a simple $A \subset X$ such that $\varphi(A) = A$.

Proof: According to 6.1 , 6.2 , 6.3 , 2.4 and 3.5 , the set $A = K(\varphi^n(X))$ for sufficiently large n is simple and invariant under φ .

6.5. Theorem. Let X be an arbitrary N-space. Let φ be an N-map of X into X , homotopical with a constant one. Then there exists a simple $A \subset X$, such that $\varphi(A) = A$, and we may take $A = K(\varphi^n(X))$ for sufficiently large n .

Proof: Let n be such an integer that $\varphi(\varphi^n(X)) = \varphi^n(X)$. Let $h: X \times I_k \to X$ be the homotopy between the N-map φ and a constant one. Let us denote χ the N-map of X onto $\varphi^n(X)$ defined by $\chi(x) = \varphi^n(x)$, and ψ the isomorphism of $\varphi^n(X)$ onto itself, defined by $\psi(x) = \varphi(x)$. Let us define $h' : \varphi^n(X) \times I_k \to \varphi^n(X)$ by $h' = \psi^{-n-1} \circ \chi \circ (h \mid (\varphi^n(X) \times I_k))$. Evidently the N-map h' is a homotopy between the identity map and a constant one. The rest of the proof is obvious.

§ 7. An application for graphs

7.1. Convention. By a graph is meant a finite graph, i.e. some $(X; R)$, where X is a finite set, R some relation on X . By the mapping is always meant a relation-preserving mapping of one graph into another one. The arrow beginning in a and finishing in b is the couple (a, b) such that $a R b$.

7.2. Definition. An N-modification of a graph $(X; R)$ is the N-space $(X; \bar{R})$, where $\bar{R}$ is defined by: $x \bar{R} y \Longleftrightarrow x = y$ or $x R y$ or $y R x$.

7.3. Lemma. 1) A 1-1- mapping of a graph onto itself is an isomorphism, i.e. its inverse is a mapping.

2) Let $(X; R)$, $(Y; S)$ be graphs, φ be a mapping of $(X; R)$ into $(Y; S)$. Then φ is an N-map of $(X; \bar{R})$ into $(Y, \bar{S})$.

Proof: The proof of the first statement was done, in the fact, in the proof of 2.4, where we used neither symmetry nor reflexivity. The proof of the remaining one is trivial.

7.4. Theorem. Let $(X; R)$ be a graph such that its N-modification is h.t. Let φ be a mapping of $(X; R)$ into itself. Then there exists a non-void set $A \subset X$ such that $\varphi(A) = A$ and

1) $x, y \in A$, $x \neq y \Rightarrow x R y$ or $y R x$.

2) If $x \in A$, the number of the arrows beginning in x and finishing in the other elements of A is equal to the number of arrows beginning in the other elements of A and finishing in x.

Proof: The existence of a set A such that $\varphi(A) = A$ and $x, y \in A$, $x \neq y \Rightarrow x R y$ or $y R x$ is an immediate corollary of 6.4 and 7.3. Because of finiteness we can find the A such that it is minimal (i.e. for no proper subset B of A holds $\varphi(B) = B$). Let us denote $n(x)$ the number of the arrows beginning in x and finishing in the other elements of A minus the number of arrows beginning in the other elements of A and finishing in x. Obviously (see 7.3. 1) $n(\varphi(x)) = n(x)$ and hence, according to the minimality, all $n(x)$ are equal to the same number. On the other hand, obviously $\sum_{x \in A} n(x) = 0$ and therefore $n(x) = 0$ for every x.

7.5. Remark. There arises a question of characterizing graphs with the h.t. N-modification without using of the notion of homotopy. Let us return for a moment to the N-spaces. Let us take some N-space X_0 and do the following construction. If an X_i is constructed and if it has no contractible points, we stop the construction. If it has contractible points, let us throw away one of them and denote X_{i+1} the remaining N-space. It is not difficult to see that X_0 is h.t. iff the described construction stops with an X_k consisting of a single point. Therefore, a graph $(X; R)$ has h.t. N-modification iff we can get a sub-graph of X consisting of a single point by the following construction: Throw away an element x such that $\bar{R}(x) \subset \bar{R}(y)$, $x \neq y$; with the remaining graph try to do the same etc. We can define the $\bar{R}(z)$, without using of the modification, as the set consisting of the element z and all the elements which are contained together with the z in an arrow.