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STRUCTURE OF DYNAMICAL SYSTEMS

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Summary: Every local semi-dynamical system τ with unicity on a topological space P may be immersed within a global dyn. system on a topological space $P^\wedge \supset P$. If P is a compact n-manifold, then τ may be extended to a (global) local dyn. system on P itself. There follow results on the local structure, near non-critical points, of local semi-dynamical systems with unicity on 2-manifolds.

The motivation of abstract dynamical systems ("global") in the present paper) is well known. Restricting these to open or to +invariant subsets, there result local dyn. systems and semi dyn.systems respectively (cf. [3], [5]; the latter were named unilateral in [3]). Another motivation for these derived concepts is that local dyn. systems arise naturally from autonomous systems of differential equations satisfying local existence and unicity conditions, but without prolongability of solutions; and that for semi-dyn. systems, rather weak conditions for existence of critical points have been obtained [3]. The present paper is devoted to the study of relations between these different types of dynamical systems. The basic result here is that unicity (see definition) is a necessary and sufficient condition for a local semi-dyn. system on a topological space P to be extendable to a global dynamical system on a larger space, in which P appears as the intersection of an open and a $+$invariant subset (assertions 11,14,17). If, then, P is an n-manifold, it is open in the extension, which is also an n-manifold (theorem 25).

1. Definitions; first consequences.

We shall consider several related structures on an abstract set P. In each case the structure will be termed a d-system, and consists of a partial binary operator from $P \times E^1$ to P, i.e. of a mapping, say τ, from a subset of $P \times E^1$ into P whose value at $(x, \theta) \in P \times E^1$ is denoted by $x \tau \theta$. The definitions to follow concern the

initial value property: $x \tau 0 = x$;

group property: $(x \tau \theta_1) \tau \theta_2 = x \tau (\theta_1 + \theta_2)$;

unicity property: $x \tau \theta = x' \tau \theta'$ *implies* $x = x' \tau (\theta' - \theta)$.

Definition 1

τ is a local dynamical system (ld system) on P if

(i) for every $x \in P$ there are α_x, β_x with $-\infty \leq \beta_x < 0 < \alpha_x \leq +\infty$ such that $(x, \theta) \in$ domain τ iff $\beta_x < \theta < \alpha_x$, and

(ii) the initial value and group properties hold for all x and all θ_j such that both (x, θ_1) and one of $(x \tau \theta_1, \theta_2)$, $(x, \theta_1 + \theta_2)$ are in domain τ.

τ is a global dynamical system (gd system) on P if it is an ld system with $\beta_x \equiv -\infty$, $\alpha_x \equiv +\infty$

τ is a local semi-dynamical system (lsd system) on P if

(iii) for every $x \in P$ there is an α_x, $0 \leq \alpha_x < +\infty$, such that $(x, \theta) \in$ domain τ iff $0 \leq \theta < \alpha_x$, and (ii) holds.

τ is a global semi-dynamical system (gsd system) on P

if it is an $\ell s d$ system with $\alpha_x \equiv +\infty$.

(Cf.[6], chap. V;[3] ,[5].)

Generic names (d system, local system, global system, semi-system) will be used with the obvious meaning. The relation $(x, \theta) \in$ domain T will be expressed, rather ineffectually, by " $x T \theta$ is defined". As usual, E^1 denotes euclidean 1-space, E^1_+ and E^1_- the subspaces consisting of non-negative and non-positive reals, respectively.

Given a d system T on P , a subset $X \subset P$ with the property that

$$x T \theta \in X \quad \text{if} \quad x \in X, \theta \geq 0, x T \theta \quad \text{is defined,}$$

will be termed $+$ invariant (in P, T); similarly for $-$ invariant sets and invariant sets ($\theta \geq 0$ above replaced by $\theta \leq 0$ and $\theta \in E^1$ respectively).

In a gd system, the group property obviously implies the unicity property; and similarly in an ℓd system, at least for arguments such that $x' T (\theta' - \theta)$ is defined. An $\ell s d$ system will be said to possess unicity if the unicity property obtains for all values of arguments indicated with $\theta' - \theta \geq 0$.

Example 2. There exist gsd systems without unicity. E.g., let P consist of all complex numbers x with $\operatorname{Arg} x = \frac{2}{3} k \pi$, $k = 0, \pm 1$ or with $x = 0$; let T describe motion along P with the real coordinate increasing uniformly.

In each case of definition 1, the initial value property implies that T maps onto P (indeed, this may replace the initial value property in definition 1,[3, lemma 1]). Thus, for instance, the existence of a continuous $\ell s d$ system with unicity on a dendrite is apparently a serious restriction on the possible topological structure of the dendrite.

The conditions for the group property to hold, definition 1 ii, may be formulated in terms of the α_x, β_x as follows .

Lemma 3. If T is an ld system then

$$\alpha_{xT\theta} = \alpha_x - \theta, \ \beta_{xT\theta} = \beta_x - \theta \quad \text{for } \beta_x < \theta < \alpha_x ;$$

and similarly for lsd systems.

The proof is straightforward. Hence, immediately,

Lemma 4. If T is an ld system and x has period $\lambda \neq 0$ (i.e., $xT\lambda = x$), then $\beta_x = -\infty$, $\alpha_x = +\infty$. Similarly, for lsd systems: $xT\lambda = x$ with $\lambda > 0$ implies $\alpha_x = +\infty$.

Lemma 5. If T is a d system on P, and $Q \subset P$ is + invariant, then the restriction of T to Q (more precisely, to domain $T \cap Q \times E^1$) is a semi-system on Q, global if T is global. Furthermore, if T is ld, then the restricted semi-system possesses unicity.

Obviously, if Q is invariant then the restricted system is of the same type as T . Next, consider methods of obtaining, from global systems, local systems on subsets. This leads to the problem of choosing new α_x, β_x for all x in the subset, in such a manner as to preserve the formulae of lemma 3 and $\beta_x < 0 < \alpha_x$. It seems hardly reasonable to do this directly; however, in the more special situation described in def. 6, we have the convenient method described in lemma 10.

Definition 6. Let T be a d system on P, and let τ be a topology on P. The structures τ and T are compatible if (i) domain T is open in $P \times E^1$ (in $P \times E^1_+$ for semi-systems), and

(ii) T is continuous.

In this case T may be termed a d system on P, τ or merely on the topological space P (if τ is "understood").

To sketch the background of this definition, there are the obvious corresponding notions of a d system compatible with, e. g., a uniform structure, a metric function, or a measure. Roughly speaking, these correspond to, respectively, a uniformly continuous T, a distance preserving T, a measurability and measure preserving T (comp. integral stability, parallelisable systems, systems with invariant measure; [6], chap. V, VI). It should be remarked, however, that probably the most interesting situations arise if there is given a uniform structure $\mathcal{U}$ on P, and a d system T compatible with the topology induced by $\mathcal{U}$, but not necessarily with $\mathcal{U}$ itself.

In a similar fashion one may consider d systems on P compatible with the structure of a differential or analytic manifold on P. The corresponding semi-systems then define, in the obvious manner, an associated field of directions on P (in differential-geometric language, this is a scalar field on the manifold).

It remains to verify that "natural" d systems do satisfy definition 6. To see this, let

$$\frac{dx}{d\theta} = f(x)$$

be an autonomous system of differential equations in E^n, with $f : E^n \to E^n$ continuous, and postulate local unicity of solutions. For $(x, \theta) \in E^n \times E^1$, define $x \, T \, \theta$ as the value at θ of that solution which has value x at $\theta = 0$. By classical theorems, this defines an ld system in E^n. From, e.g., [2, chap. II, 4.1] it follows that T is compatible with the usual topology of E^n.

The presence of a topology compatible with a d system T has consequences on the properties of T; one of these is exhibited in theorem 8. For this we first need

Lemma 7. If T is an ℓsd system on a topological space P, then the function $\alpha : P \to E^1$ of definition 1 iii is lower semicontinuous. If, then, $\alpha_x < +\infty$ for some $x \in P$ and $\theta_n \to \alpha_x$ in E^1, $0 \leq \theta_n < \alpha_x$, then $x T \theta_n$ does not converge in P.

(Proof.) Consider the sets

$$A_\lambda = \{x \in P : \alpha_x > \lambda\}.$$

If $\lambda \leq 0$, obviously $A_\lambda = P$ is open. For $\lambda > 0$, consider the set

$$\{(x, \theta) : x \in P, \lambda < \theta < \alpha_x\} = \text{domain } T \cap (P \times (\lambda, +\infty)).$$

From definition 5 and $\lambda > 0$, this set is open in $P \times E^1$; hence its projection A_λ is open in P. This proves that α is lower semicontinuous.

Now assume $\theta_n \to \alpha_x$ in E^1 with $0 \leq \theta_n < \alpha_x$, and $x T \theta_n \to y$ in P. Using semicontinuity and lemma 3,

$$0 < \alpha_y \leq \liminf_{z \to y} \alpha_z \leq \liminf_n \alpha_{x T \theta_n} = \liminf_n (\alpha_x - \theta_n) = 0,$$

contradiction. This completes the proof.

Remarks. Obviously, if T is an ℓd system, then similar conclusions obtain for the second function $\beta : P \to E^1$ (definition 1 i). This result shows that for local systems, "limit sets" of trajectories do not have properties analoguous to those of global systems.

If, in lemma 7, T also has the property that α_x is continuous, then we have a stronger conclusion: if $\alpha_x < +\infty$ and $x_i \to x$ in P, $0 \leq \theta_i < \alpha_{x_i}$ and $\theta_i \to \alpha_x$ in E^1, then $x_i T \theta_i$ does not converge in P. However, the postulated property of T is rather artificial:

Example 8. Let

$$P = \{(x, y) \in E^2 : x < 0 \text{ or } y > 0\}$$

and define

$$(x, y)\,\tau\,\theta = (x + \theta, y) \cdot$$

Then τ is an ld system in P, $\alpha_{(x,y)} = +\infty$ for $y > 0$, $\alpha_{(x,y)} = |x|$ for $y \leq 0$.

A direct consequence of lemma 7 is

Theorem 9. Every local system on a countably compact space is global.

The obvious interpretation of this result is that, on countably compact spaces, one cannot have a non-global local system. However, another possible application may be suggested: The differential equation in one unknown

$$\frac{dx}{d\theta} = f(\theta, x)$$

with $f: E^2 \to E^1$ continuous and periodic in both variables, and with unicity of solutions, defines an ℓsd system on a torus [2,chap.XVII]. Theorem 9 states that solutions are prolongable over the entire real axis. In the present case, this also follows from boundedness of f and familiar theorems on prolongability.

Lemma 10. If τ is a local system on a topological space P, and $G \subset P$ is open, then the restriction of τ to G is a local system on G (of the same type). If τ has unicity then so does the restricted system.

This is quite obvious; the α_x's of the restricted system may be determined, for $x \in G$, as

$$\alpha_x = \sup\{\theta : 0 \leq \theta' \leq \theta \text{ implies } x\,\tau\,\theta' \in G\} \cdot$$

Lemma 7 yields another view of this construction. If the original system is global, then the restricted system may well be local non-global. Collecting parts of lemmas 5 and 10 we obtain

Proposition 11. Let τ be a gd system on a topological space P. Let G be open, Q + invariant in P. Then the restriction of τ to $G \cap Q$ is an lsd system with unicity.

2. Construction of gd extensions.

The construction to follow shows that every lsd system with unicity may be obtained by restricting some gd system on a larger carrier set. In the topological case there results an assertion converse to proposition 11.

Construction 12. Assume given, a d system τ with unicity on an abstract set P. We proceed to define the following: a relation $\sim$, a set $P^\wedge$, a binary operator $\hat{\tau}$ and two sets P^+, P^-. It may be noticed that the construction of $P^\wedge$ is a close analogue of the method used in elementary number theory to obtain the set of all integers from the positive integers.

On $P \times E^1$, let $\sim$ be obtained by symmetrising the relation between

$$(x, \theta) \text{ and } (x \tau \varepsilon, \theta - \varepsilon) \text{ for } 0 \leq \varepsilon < \alpha_x .$$

It is readily verified that $\sim$ is an equivalence relation on $P \times E^1$; the unicity property is used to establish transitivity. Define $P^\wedge$ as $P \times E^1 \; mod \sim$

Next, define a binary operator $\hat{\tau}$ from $P^\wedge \times E^1$ to P^1 by first setting

(1) $$(x, \theta) \hat{\tau} \lambda = (x, \theta + \lambda)$$

and then passing to equivalence classes. Obviously, $\hat{\tau}$ is a gd system on $P^\wedge$.

Define a map $p : P \to P^\wedge$ by taking for $p(x)$ the equivalence class containing $(x, 0)$:

$$(x, 0) \in \mu(x) \in P^{\wedge} .$$

It is easily shown that μ is $1-1$ and that

(2) $$\mu(x \, T \, \lambda) = \mu(x) \, \hat{T} \, \lambda$$

for $0 \leq \lambda < \alpha_x$; if the given T is an ld system, then (2) also holds for $\beta_x < \lambda < \alpha_x$. Thus we may and shall identify P with $\mu(P)$, thus obtaining $P \subset P^{\wedge}$; an $x \in P$ is identified with the equivalence class containing $(x, 0)$. From (2), T is obtained by restriction of $\hat{T}$. Since $(x, \theta) = (x, 0) \, \hat{T} \, \theta$ from (1), P generates $P^{\wedge}$ in the sense that $P^{\wedge}$ is the least invariant subset of $P^{\wedge}$, $\hat{T}$ containing P.

Finally, define a subset

$$P^{+} = P \, \hat{T} \, E^1_{+} ,$$

the least $+$ invariant subset of $P^{\wedge}$ containing P ; and similarly

$$P^{-} = P \, \hat{T} \, E^1_{-} .$$

Next we shall exhibit some important properties of $P^{\wedge}, \hat{T}$. One of these is that $P^{\wedge}$ has no further cycles nor critical points than those already present in P . (The assumption that T is a d system with unicity on P is preserved.)

Proposition 13. In $P^{\wedge}$, the set P generates $P^{\wedge}$ in the sense that $P^{\wedge} = P \, \hat{T} \, E^1$. Hence $P^{\wedge}$ has no further cycles nor critical points than those already present in P .

(Proof.) In any equivalence class x in $P^{\wedge}$ select some (x, λ); then

$$(x, \lambda) = (x, 0) \, \hat{T} \, \lambda \in (P \times (0)) \, \hat{T} \, \lambda$$

so that $x = x \, \hat{T} \, \lambda$ as asserted.

If $x \in P^{\wedge}$ is critical or on a cycle, then

$x = x \hat{\tau} R^1$ and as just shown, this latter set must intersect P; hence $x \in P$.

Proposition 14. $\hat{\tau}$ defines a gsd system on P^+, an ld system on P^-.

$$P = P^+ \cap P^-, \quad P^\wedge = P^+ \cup P^- .$$

$P = P^+$ iff τ is a gsd system, $P = P^\wedge$ iff τ is a gd system.

(Proof.) For the first statement use lemma 5. Second statement: if

$$x = x \hat{\tau} - \theta \in P^-, \quad x \in P, \quad \theta > 0,$$

then $x \hat{\tau} \lambda = (x \tau \lambda) \hat{\tau} - \theta \in P^-$ for $\lambda < \alpha_x$; thus $\hat{\tau}$ restricted to P^- is an ld system (with $\beta_x \equiv -\infty$).

Obviously $P \subset P^+ \cap P^-$. For the converse inclusion, take any $x \in P^+ \cap P^-$; then

$$x \to (x, \theta) \sim (x', -\theta'), \quad \theta, \theta' \geq 0 .$$

There are two cases. Either, for some ε' with $0 \leq \varepsilon' < \alpha_{x'}$,

$$x = x' \tau \varepsilon', \quad 0 = -\theta' - \varepsilon' .$$

The latter of these implies $\theta = \theta' = \varepsilon' = 0$, and thus $x \to (x, 0)$ is in P. Or

$$x' = x \tau \varepsilon, \quad -\theta' = \theta - \varepsilon, \quad 0 \leq \varepsilon < \alpha_x,$$

so that $0 \leq \theta = \varepsilon - \theta' \leq \varepsilon < \alpha_x$ and $x \tau \theta$ is defined, and thus $x \to (x, \theta) \sim (x \tau \theta, 0)$ is again in P. Thus $P^+ \cap P^- \subset P$.

The remaining statements have trivial proofs.

Proposition 15. $P^\wedge$, $\hat{\tau}$ are determined uniquely in the following sense. If $\dot{\tau}$ is a gd system on a set $\dot{P} \supset P$ with restriction τ, then there exists a map $h : P^\wedge \to \dot{P}$, identical on P, and with

(3) $\quad h(x \hat{\tau} \theta) = h(x) \dot{\tau} \theta \quad$ for $(x, \theta) \in P^\wedge \times E^1$.

If, furthermore, $\dot{P} = P \dot{\tau} E^1$, i.e. if P generates $\dot{P}$, then h is $1-1$ onto $\dot{P}$.

The proof is quite straightforward: h is obtained by showing that

$$h(x \hat{\tau} \theta) = x \dot{\tau} \theta, \qquad (x, \theta) \in P \times E^1,$$

defines a map as required; proposition 13 is used here. The inverse map may be defined similarly, if $\dot{P}$ has the indicated property.

For purposes of reference we collect these results.

Theorem 16. If τ is a d system with unicity on a set P, then there exists a gd system $\hat{\tau}$ on a set $P^\wedge \supset P$, such that τ is a restriction of $\hat{\tau}$ and that 13,14,15 hold.

Remark. It may be shown directly that the operation of forming $P^\wedge$ extends to a covariant functor on the obvious categories (morphisms are maps preserving the d system operators, as in (3)). Similarly for the operations of forming P^+ and P^-. Corresponding remarks apply to theorem 17 to follow.

(Construction 12 contd.) We proceed to show that theorem 14 may be significantly improved in case that the d system acts on a topological space.

Assume, then, that there is a topology $\mathcal{T}$ on P, compatible with the d system τ given initially. Then there is a natural cartesian topology for $P \times E^1$, and hence a quotient topology $\hat{\mathcal{T}}$ for $P^\wedge$ [1, p.74 ff.]. Since the mapping $\hat{\tau}$ defined by (1) (beginning of construction 12) is continuous, the topology $\hat{\mathcal{T}}$ is compatible with the previously constructed gd system $\hat{\tau}$ (in definition 6, (i) is trivial for global systems; continuity is obtained almost directly using the following commutative diagram

$$\begin{array}{ccc} (P \times E^1) \times E^1 & \longrightarrow & P \times E^1 \\ \downarrow & & \downarrow \\ P^\wedge \times E^1 & \xrightarrow{\hat{T}} & P^\wedge \end{array}$$

where the vertical maps are (induced by) the quotient mappings). The map p of construction 12 is easily shown to be interior, so that, after our identification of P with $p(P)$, $P \subset P^\wedge$ topologically.

<u>Theorem 17</u>. In the situation of theorem 16, let T be compatible with a topology τ on P. Then there is a topology $\hat{\tau}$ on $P^\wedge$ compatible with both $\hat{T}$ and τ. In proposition 14, P^- is open in $P^\wedge$; in proposition 15, h is continuous, and if $\dot{P}$, $\dot{T}$ have the last-indicated property, h is homeomorphic.

(Proof.) The only non-trivial proof concerns openness of P^-. It is easily established that, in $P \times E^1$, the least set saturated with respect to $\sim$ and containing $P \times E^1_-$ (i.e. mapped onto P^-) is

$$P^\alpha = \{(x, \theta) : x \in P,\ \theta < \alpha_x\}$$

By definition of the quotient topology (l. c.), P^- is open in $P^\wedge$ iff P^α is open in $P \times E^1$. Take any $(x, \theta) \in P^\alpha$, and any $\theta' \in E^1$ with

$$\max(0, \theta) < \theta' < \alpha_x$$

From lemma 7, there is a neighbourhood U of x in P such that $\alpha_y > \theta'$ for all $y \in U$. set $V = (-\infty, \theta')$ a neighbourhood of θ in E^1. Then $U \times V$ is a neighbourhood of (x, θ), and obviously $U \times V \subset P^\alpha$. This proves P^- is open.

Remarks. In the situation of the preceding theorem, from $P = P^+ \cap P^{-}$ (proposition 14) it follows that P is open in P^+. Thus we have the following diagram of inclusion maps

semi-syst:

local systems:

global systems:

$$\begin{tikzcd} P \arrow[r, "+inv"] \arrow[d, "open"] & P^- \arrow[d, "open"] \\ P^+ \arrow[r, "+inv"'] & P^\wedge \end{tikzcd}$$

In proposition 12, obviously we cannot assert that $P = P^-$ iff τ is an $\mathcal{ld}$ system. At least we have

<u>Lemma 18</u>. If τ is an $\mathcal{ld}$ system with unicity on a topological space P, then, in construction 12, P is open in P^-.

(Proof.) Since $P \subset P^-$, it suffices to show that P is open in $P^\wedge$. Now, the least subset of $P \times E^1$ containing $P \times \{0\}$ (i.e., mapped onto $P \subset P^\wedge$) and saturated with respect to $\sim$, is domain τ; this is open by definition 6.

3. <u>The openness condition; manifolds.</u>

Throughout this section we make the general assumption that there is given a $\mathcal{d}$ system τ with unicity on a topological space P; in particular, then, one has the objects $P^\wedge$, $\hat{\tau}$ of construction 12.

It will appear presently that the condition that P be open in $P^\wedge$ (the openness condition) is rather useful in that it yields significant results. In this connection we already have (17 and 18) that P is open in $P^\wedge$ if τ is an $\mathcal{ld}$ system. From the main result of this section (theorem 25) it follows that the openness condition does obtain if P is an

n-manifold.

From construction 12 we recall that

(4) $$P^\wedge = \bigcup_{\theta \in E^1} P \hat{\tau} \theta \quad .$$

Proposition 19. If P is open in $P^\wedge$, then $P^\wedge$ is locally homeomorphic to P.

(Proof.) For fixed $\theta \in E^1$, the map taking $x \in P^\wedge$ to $x \hat{\tau} \theta$ is a homeomorphism $P^\wedge \approx P^\wedge$ ($x \to x \hat{\tau} - \theta$ is the inverse mapping). Thus, first, P is homeomorphic to $P \hat{\tau} \theta$, and second, $P \hat{\tau} \theta$ is open if P is open; in particular, (4) is an open cover of $P^\wedge$.

Corollary 20. Let P be open in $P^\wedge$. If P is T_0 or T_1 or T_p or an n-manifold, then so is $P^\wedge$.

(Proof.) Each of the listed properties obtains iff it obtains locally; then apply proposition 19. In the case that P is an n-manifold, this reasoning yields that $P^\wedge$ is locally E^n. To show that then $P^\wedge$ is T_2 (and hence an n-manifold), proceed thus: P is an n-manifold, hence T_p; then $P^\wedge$ is T_p, hence T_2. This completes the proof.

I do not know whether $P^\wedge$ is T_2 if P is such.

Lemma 21. Let G be open in P and $A \subset E^1$ arbitrary. Then $G \tau A$ is open in P if either τ is an ld system or $A \subset E^1_+$ and P is open in $P^\wedge$.

(Proof.) Since $G \tau A = \bigcup_{\theta \in A} G \tau \theta$, it suffices to prove $G \tau \theta$ is open. In either case P is open in $P^\wedge$, so that G is open in $P^\wedge$, and hence $G \hat{\tau} \theta$ is also such. Therefore it suffices to prove the formula

$$G \tau \theta = P \cap (G \hat{\tau} \theta) .$$

Obviously the left side is a subset of the right one. Take $z \in P \cap (G \hat{\tau} \theta)$. Thus there exist $x \in G$, $y \in P$ with

$(x, \theta) \sim (y, 0)$ and we are to prove that $x \tau \theta$ is defined, since then $(x, \theta) \sim (x . \tau \theta, 0) \in G \tau \theta$. There are two cases, according as $\theta \geqslant 0$ or $\theta < 0$.

If $\theta \geqslant 0$, then from $(x, \theta) \sim (y, 0)$ there follows $y = x \tau \varepsilon$, $0 = \theta - \varepsilon$, i.e. $y = x \tau \theta$ is indeed defined. If $\theta < 0$ we have similarly $x = y \tau - \theta$ (this is the case when τ is an ld system); then, from lemma 3,

$$\beta_x = \beta_y + \theta < \theta < 0 < \alpha_x$$

and thus $x \tau \theta$ is again defined. This completes the proof.

Remark. This result was proved in [5] for the special case that τ is an ld system on an n-manifold. Thus this second assumption is unnecessary.

Proposition 22. If P is open in $P^\wedge$ and compact then $P = P^\wedge$.

(Proof.) By assumption, then, P is open-closed in $P^\wedge$. Any $z \in P^\wedge$ is of the form $z = x \hat{\tau} \theta$ for some $x \in P$, $\theta \in E^1$, so that it is in the same component of $P^\wedge$ as x ; hence z is in the same "composante" as x, i.e. in P. Thus $P^\wedge = P$.

Example 23. There exist gsd systems τ on metric P such that P is not open in any topological space $Q \supset P$ on which there is a gd system with restriction τ .

On $\langle 0, 1 \rangle \subset E^1$ take the gsd system generated by the motion of a point, initially at 1 , moving towards 0 with suitably decreasing velocity. Thus $x \tau \theta$ might be defined as $x / (1 + \theta x)$ for $0 \leq x \leq 1$, $\theta \geqslant 0$. Now assume there is a topological space $Q \supset P$ such that P is open in Q and that there is a gd system $\bar{\tau}$ on Q with restriction τ. Since P is open in Q, $\bar{\tau}$ defines an ld system on P which again is an extension of τ. Since 1 is not critical, there is an arc, say $1\,\tau\,\langle -\varepsilon, \varepsilon\rangle$ in P with 1 as interior point; but 1 is an end-point in P: contradiction.

Lemma 24. If P is open in $P^\wedge$ and $P^\wedge$ is compact, then $P = P^\wedge$.

(Proof.) By assumption, then, (4) is an open cover of a compact $P^\wedge$. Thus, for some $\theta_1, \dots, \theta_n$ in E^1,

$$P^\wedge = U_1^n\, P\,\hat{\tau}\,\theta_k \,. \tag{5}$$

Now we shall prove $\alpha_x = +\infty$ for all $x \in P$. Let $\theta = 1 + max\ \theta_k$; then

$$P^\wedge = P^\wedge\,\hat{\tau} - \theta = U_1^n\, P\,\hat{\tau} - \varepsilon_k \,, \quad \varepsilon_k > 0 \,.$$

Thus for every $x \in P \subset P^\wedge$ there is an $y \in P$ with $(x, 0) \sim (y, -\varepsilon_k)$; hence $y = x\,\tau\,\varepsilon_k$, so that $\varepsilon_k < \alpha_x$, and therefore

$$0 < min\ \varepsilon_k < \alpha_x \quad \text{for all } x \in P \,.$$

Now, if we had $\alpha_x < +\infty$ for some $x \in P$, then $0 < \alpha_{x\,\tau\,\theta} = \alpha_x - \theta \to 0$ with $\theta \to \alpha_x$ (lemma 3); this would contradict the last displayed relation. Thus $\alpha_x \equiv +\infty$.

Next, set $\theta' = min\ \theta_k$; from (5),

$$P^\wedge = P^\wedge\,\hat{\tau} - \theta = U_1^n\, P\,\hat{\tau} - \varepsilon_k \,, \quad \varepsilon_k > 0 \,.$$

However, from $\alpha_x \equiv +\infty$ there follows $P\,\hat{\tau}\,\varepsilon'_k = P\,\tau\,\varepsilon'_k \subset P$, and therefore $P^\wedge = U_1^n\, P\,\hat{\tau}\,\varepsilon_k \subset P$. This concludes the proof.

Remark. This result also holds for $P^\wedge$ quasi-compact [1, p.113].

Theorem 25. Let τ be a d system with unicity on an

n-manifold P. Then $P^\wedge$ is an n-manifold, and P is open in $P^\wedge$.

(Proof.) From corollary 20 it follows that it suffices to prove P is open in $P^\wedge$. It is easily shown that in $P \times E^1$, the least set containing $P \times \{0\}$ and saturated with respect to $\sim$ (cf. construction 12) is

$P^\vee = \{(x \tau \varepsilon, -\varepsilon) : x \in P,\ 0 \leq \varepsilon < \alpha_x\} \cup \{(x, \varepsilon) : x \in P, 0 \leq \varepsilon < \alpha_x\}$;

so that, as before, P is open in $P^\wedge$ iff $P^\vee$ is open in $P \times E^1$. To show this last, take any $(x, \varepsilon) \in P \times E^1$ with $0 \leq \varepsilon < \alpha_x$; then

(6) $\qquad (x, \varepsilon), \quad (x \tau \varepsilon, -\varepsilon)$

are general elements of $P^\vee$. Take λ with $\varepsilon < \lambda < \alpha_x$, and an open neighbourhood U of x in P with $U \times \langle 0, \lambda \rangle \subset$ $\subset$ domain τ (cf. lemma 7). Define a map $h : U \times (-\lambda, \lambda) \to P \times E^1$ (here $(-\lambda, \lambda)$ is the open segment) by

$$h(y, \theta) = \begin{cases} (y, \theta) & \text{if } \theta > 0 \\ (y \tau \theta, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

Easily, h is continuous and $1-1$ into $P^\vee$; also, $U \times (-\lambda, \lambda)$ is open in the $(n+1)$-manifold $P \times E^1$, so that, by the Preservation of Domain Theorem, h is a homeomorphism and image h is open. Obviously, both the points (6) are in image h; thus $P^\vee$ is indeed open. This concludes the proof.

<u>Corollary 26</u>. If τ is an *lsd* system with unicity on an n-manifold P, then there is a unique maximal *ld* system on P itself with restriction τ.

(Existence follows from 17,25 and 10; unicity from 17.)

<u>Corollary 27</u>. If τ is an *lsd* system with unicity on a

closed n-manifold P, then it is a gsd system and there is a unique gd system on P itself with restriction τ. (Use 9,17,25 and 22.)

Finally, theorem 25 makes it possible to apply known results on the structure of gd systems to lsd systems.

Theorem 28. Let τ be an lsd system with unicity on a 2-manifold P; let $x_o \in P$ be non-critical. Then there exists a simple arc S in P and a $\lambda > 0$ such that, for $0 \leq \theta \leq 2\lambda$, $S\tau\theta$ are disjoint simple arcs, $x_o \in S\tau\lambda$, $S\tau\langle 0, 2\lambda\rangle$ is a neighbourhood of x_o.

(Proof.) From 25, $P^\wedge$ is a 2-manifold and P is open in $P^\wedge$; from 13, x_o is non-critical in $P^\wedge$, $\hat{\tau}$. Now apply the Whitney-Bebutov theorem (e.g.,[4], theorem 2), taking initially a neighbourhood of $x_o \in P \subset P^\wedge$ small enough to be included in P. Then use [1.c., theorem 1] to show that S is a simple arc, concluding the proof.

For semi-systems τ on P we may define S-points as those points of P which are not of the form $x\tau\theta$ for any $x \in P$, $\theta > 0$. Thus from theorem 28 it follows that every lsd system with unicity on an n-manifold has no S-points. However, a stronger conclusion may be had.

An n-manifold with boundary (the term used, for $n = 2$, in [7] is merely 2-manifold) is a T_2 space such that every point is on some homeomorphic image of an euclidean n-simplex s^n whose interior maps onto an open set; a boundary point is then a point without neighbourhoods homeomorphic to E^n.

Proposition 29. Let τ be an lsd system with unicity on an n-manifold with boundary P. Then every S-point of P, τ is a boundary point of P; for every non-critical non-S-point $x_o \in P$ (even if it is a boundary point) the conclusion of 28 obtain.

(Proof.) If $x \in P$ is not a boundary point, it has an open neighbourhood U homeomorphic to E^n; from lemma 10, τ induces on U an *lsd* system with unicity, to which we may then apply theorem 28.

A another application of these results, let us determine all *lsd* systems with unicity on 1- manifolds. Obviously we need consider only connected 1- manifolds; topologically, there are only two of these: euclidean E^1 and the 1-sphere S^1.

<u>Example 30</u>. First consider only *gd* systems. On E^1, these are characterised easily: each $x \tau \theta$ (fixed x), if non-constant, is strictly monotonous, so that each non-empty limit set is a single critical point (in particular, there are no cycles). Thus one chooses an arbitrary closed $F \subset E^1$ as the critical points, and for each contiguous interval J, an arbitrary strictly monotone map of E^1 onto J, to determine motion within J (if φ is such a function, then $x \tau \theta = \varphi(\theta + \varphi^{-1}(x))$).

The (elementary) proofs of those statements closely parallel the complete discussion of solutions of the equation $dy/dx = f(y)$ in one scalar unknown with f continuous bounded.

Similarly for *gd* systems on S^1 :either S^1 is itself a complete cycle, or there is a critical point $x \in S^1$, whereupon on $S^1 - x \approx E^1$ there is induced a *gd* system which may then be treated as above.

This established, we may characterise all *lsd* systems with unicity on E^1 and S^1. To this end it suffices to determine the associated space $P^{\wedge}$ in both cases, and then apply theorem 17. From 26 and 22, $(S^1)^{\wedge} = S^1$, and 27 may be used. As concerns $(E^1)^{\wedge}$ it is an 1- manifold (theorem 26), but not S^1 (lemma 24). Thus $(E^1)^{\wedge} = E^1$ with the original space as an open subsegment.

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