Vlastimil Dlab

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ON THE DEPENDENCE RELATION OVER MODULES

Vlastimil DLAB , Praha

(Preliminary communication)

In the present note, we shall give some necessary and sufficient conditions for a ring R to satisfy the following property: All modules over R admit a dependence theory analogous to that in abelian groups (in particular, every module over R admits an invariant rank). Our theorems generalize the results of Kertész [4] and Fuchs [3].

Without any substantial loss of generality, all rings to be considered here are (associative) rings R with identity and all R-modules - unitary left modules over R. For the terminology we refer to [2] or [1].

Let M be an R-module. An element $x \in M$ is said to depend on $X \subseteq M$ if there exist $\lambda \in R$ and $\lambda_i \in R$, $x_i \in X$ for $1 \leq i \leq n$ such that

$$0 \neq \lambda x = \sum_{i=1}^{n} \lambda_i x_i .$$

With respect to this dependence relation, any module over a ring is an A-dependence structure. For a large family of rings R including all rings with maximal or minimal condition for left ideals, the R-modules are GA-dependence structures (on the other hand, not all commutative rings belong to this family):

<u>Theorem 1</u>. Every R-module is a GA-dependence structure if and only if, for any proper left ideal L of R, there exists $\rho \notin L$ such that the quotient ideal $L:\rho$

is irreducible (i.e. it cannot be represented as the intersection of two left ideals different from $L : \rho$).

As to the similar question on LA -dependence structures, the answer is a consequence of the following more general theorem.

Theorem 2. Let $\mathcal{L}$ be a family of left ideals of R. Then the following statements are equivalent:

(i) For any R -modul M and any subset $S \subseteq M$ such that the order (annihilating ideal) of each element $a \in S$ belongs to $\mathcal{L}$, S is a LA -dependence structure (with respect to the dependence relation restricted to S).

(ii) Every ideal belonging to $\mathcal{L}$ is irreducible.

Corollary. Every R -module is a LA -dependence structure (and thus, any two its maximal independent subsets have the same cardinality) if and only if every left ideal L of R is irreducible, i.e. if and only if the family of all left ideals of R is a chain (linearly ordered by inclusion).

The last theorem reflects, in a general form, the situation in abelian groups.

Theorem 3. Let $\mathcal{L}$ be a family of left ideals of a ring R. Let P be a two-sided prime ideal of R such that the following two conditions are satisfied:

(i) If $L \in \mathcal{L}$ and $L : \rho = P$ for an element $\rho \in R$, then $L \subseteq P$.

(ii) If $L \in \mathcal{L}$ and $L \subseteq P$, then L is prime.

Then, any R -module M possesses the following property:

Let $S \subseteq M$ be the subset of all elements of order P. If M_1 and M_2 are two maximal independent subsets of M consisting of elements whose orders belong to $\mathcal{L}$, then

$S \cap M_1$ and $S \cap M_2$ are maximal independent subsets of S. If, moreover, P is irreducible, then S is a LA-dependence structure and thus $card\,(S \cap M_1) = card\,(S \cap M_2)$.

The above results were read by the author in the Makerere University College at Kampala, Uganda, on November 16, 1963 and in the 6th Austrian Mathematical Congress at Graz, September 14-18, 1964.

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