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# ERROR MINIMIZATION IN APPROXIMATE SOLUTION OF INTEGRAL EQUATIONS

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In the present paper we shall study the approximate solution of Fredholm's integral equation

$$\varphi(x) - \lambda \int_0^1 K(x,t)\varphi(t)\,dt = f(x) \tag{1}$$

using the method of the degenerate kernel, i.e. by replacing the kernel $K(x,t)$ by the kernel

$$K_n(x,t) = \sum_{k=1}^{n} a_k(x)\, b_k(t) \,. \tag{2}$$

We assume that

$$\left\{ \begin{array}{l} K(x,t) \in L_2(0,1)\times(0,1), \quad f(x)\in L_2(0,1), \\ \lambda \text{ is not an eigenvalue of the kernel } K(x,t). \end{array} \right. \tag{3}$$

We suppose that the functions $a_k(x)$, $k = 1,\dots,n$ form an orthonormal system in $L_2(0,1)$. $L_n$ will stand for the subset formed by the functions $a_1(x),\dots,a_n(x)$.

The solution of the equation (1) can be approximated by the solution of the following equation

$$\psi_n(x) - \lambda \int_0^1 K_n(x,t)\,\psi_n(t)\,dt = f(x) \,. \tag{4}$$

We shall suppose that we can find the exact solution of (4), that is we do not take into consideration the error of the numerical solution of (4).

If the conditions (3) are satisfied, then

$$(E - \lambda K)^{-1} = E + \lambda \Gamma_K \,. \tag{5}$$

Here $E$ stands for the unit operator. In this paper we shall use the following theorem (see [1]), where $\| K - K_n \|$ denotes the norm of the operator $K - K_n$.

<u>Theorem 1</u>: If

$$(6) \qquad |\lambda| \, \| K - K_n \| \, (1 + |\lambda| \, \| \Gamma_k \|) < 1 ,$$

then $\lambda$ is not an eigenvalue of the kernel (2) and the inequality

$$(7) \qquad \| \varphi(x) - \psi_n(x) \|_{L_2} \leq \| K - K_n \| \, \frac{|\lambda| \, \| f(x) \| \, (1 + |\lambda| \, \| \Gamma_k \|)^2}{1 - |\lambda| \, \| K - K_n \| \, (1 + |\lambda| \, \| \Gamma_k \|)}$$

holds.

According to this theorem it is reasonable to find the minimum of $\| K - K_n \|$ when the $b_k(t)$ vary. We shall use the following notation

$$(8) \qquad \mathcal{E}(\mathfrak{M}, L_n) = \sup_{f \in \mathfrak{M}} \inf_{\alpha_1, \ldots, \alpha_n} \| f(x) - \sum_{k=1}^{n} \alpha_k \, a_k(x) \| .$$

$\mathfrak{M}$ is the image of the unit sphere in $L_2(0,1)$ under the operator $K$.

<u>Theorem 2</u>: Set

$$(9) \qquad \bar{b}_k(t) = \int_0^1 K(\xi, t) \, a_k(\xi) \, d\xi$$

and

$$(10) \qquad \bar{K}_n(x, t) = \sum_{k=1}^{n} a_k(x) \, \bar{b}_k(t) .$$

Then for arbitrary functions $b_1(t), \ldots, b_n(t)$ in (2), the inequality

$$(11) \qquad \| K - K_n \| \geq \| K - \bar{K}_n \| = \mathcal{E}(\mathfrak{M}, L_n)$$

holds.

<u>Definition 1</u>: The kernel (10) will be called the optimal degenerate kernel formed by the functions $a_1(x), \ldots, a_n(x)$.

<u>Remark</u>: To obtain the optimal degenerate kernel we can also use the method of moments for equation (1). This means the following: An approximate solution of (1) is assumed in the form

$$\psi(x) = f(x) + \sum_{k=1}^{n} \alpha_k a_k(x),$$

and the coefficients $\alpha_k$ are determined from the conditions

$$(\psi - \lambda K \psi - f,\ a_k) = 0, \quad k = 1, \dots, n;$$

(see [1]).

The convergence of the optimal degenerate kernels to the kernel $K(x, t)$ is stated in the following theorem:

Theorem 3: Let $\{a_n(x)\}_{n=1}^{+\infty}$ be a complete orthonormal sequence in $L_2(0, 1)$.

Then

$$\lim_{n \to +\infty} \varepsilon(\mathfrak{M}, L_n) = 0. \tag{12}$$

The following converse theorem holds.

Theorem 4: Let the solutionsof (4) converge to the solution of (1) for all $f(x)$ from a set dense in $L_2(0, 1)$. Let there exist a constant $A$ independent of $n$ so that, for the resolvent $\Gamma_n$ of the kernel (2), the inequality

$$\| \Gamma_n \| \leq A \tag{13}$$

holds for all $n$. Then the sequence $\{a_n(x)\}_{n=1}^{+\infty}$ is complete in $\mathfrak{M}$.

Remark: Theorem 4 can be strengthened at the cost of further assumptions on the kernel $K(x, t)$, but this is not important for our purpose.

We are able now to determine the minimum of $\varepsilon(\mathfrak{M}, L_n)$ by varying the subsets $L_n$ of dimension $n$.

Definition 2:

$$\inf_{L_n} \varepsilon(\mathfrak{M}, L_n) = d_n(\mathfrak{M}). \tag{14}$$

See [2].

Set

$$K^L(x,t) = \int_0^1 K(\xi, x) K(\xi, t)\, d\xi \tag{15}$$

$$K^R(x,t) = \int_0^1 K(x, \xi) K(t, \xi)\, d\xi \,. \tag{16}$$

The kernels $K^L(x,t)$, $K^R(x,t)$ have the same sequences of eigenvalues

$$0 < \lambda_1^2 \leqq \lambda_2^2 \leqq \dots \quad .$$

We shall denote by $\varphi_n^L(x)$ and $\varphi_n^R(x)$ the corresponding orthonormal eigen-functions of the kernels $K^L(x,t)$, $K^R(x,t)$ respectively.

<u>Definition 3</u>: The kernel

$$\hat{K}_n(x,t) = \sum_{k=1}^{n} \frac{\varphi_k^R(x)\varphi_k^L(t)}{|\lambda_k|} \tag{17}$$

will be called the $n$-th extremal degenerate kernel to the kernel $K(x,t)$. The subset $\hat{L}_n$ formed by the functions $\varphi_k^R(x)$, $k = 1, \dots, n$, will be called the extremal subset of dimension $n$ of the kernel $K(x,t)$.

The following theorem gives the reason for the preceding definition:

<u>Theorem 5:</u> For all $n$,

$$d_n(\mathfrak{M}) = \| K - \hat{K}_n \| = \frac{1}{|\lambda_{n+1}|} \quad . \tag{18}$$

The method of minimizing $\| K - K_n \|$ makes it possible to establish estimates from below of the error. This estimate is stated in the following theorem:

<u>Theorem 6</u>: Let $\varphi(x)$ be the solution of (1), and $\bar{\psi}_n(x)$ the solution of the equation

$$\bar{\psi}_n(x) - \lambda \int_0^1 \bar{K}_n(x,t)\, \bar{\psi}_n(t)\,dt = f(x) \quad (19)$$

Then

$$\sup_{\|\varphi\| \leq 1} \| \varphi(x) - \bar{\psi}_n(x) \| \geq |\lambda|\, d_n(\mathfrak{M}) \quad (20)$$

The right-hand side in (20) depends only on the kernel $K(x, t)$, and so (20) gives the estimate from below of the error made by replacing the solution of (1) by that of (19).

<u>Definition 4</u>: Denote by $\mathcal{K}(A)$ the set of kernels $K(x,t)$ satisfying

$$\| K(x,t) \|_{L_2 (0,1)\times(0,1)} \leq A \quad (21)$$

and put

$$d_n(A) = \sup_{K(x,t) \in \mathcal{K}(A)} d_n(\mathfrak{M}_K) \quad (22)$$

<u>Theorem 7</u>:

$$d_n(A) \sim \frac{A}{\sqrt{n}} \quad (23)$$

(Here $a_n \sim b_n$ denotes strong equivalence, i.e. $\lim_{n \to +\infty} \frac{a_n}{b_n} = 1$ .)

Assuming the kernel $K(x, t)$ to be sufficiently smooth, it is possible using the method described in [3], to describe the asymptotic behaviour of $d_n(\mathfrak{M})$. Thus we get the following theorems:

<u>Theorem 8</u>: Let either $\frac{\partial^s K}{\partial t^s}$ or $\frac{\partial^s K}{\partial x^s}$ be continuous and bounded in $(0,1)$ for $s = 0, \dots, m$. Then, for any positive $\varepsilon$, there holds

$$d_n(\mathfrak{M}) = O[n^{-(\frac{m}{2}+\frac{1}{4})+\varepsilon}] \quad (24)$$

For symmetric kernels we obtain this better result:

<u>Theorem 9</u>: Let $K(x, t)$ be symmetric and satisfy the assumptions of theorem 8. Then, for any positive $\varepsilon$, there holds

$$d_n(\mathfrak{M}) = O[n^{-(m+\frac{1}{2})+\varepsilon}] \quad . \tag{25}$$

If the kernel is analytic we obtain the following theorem:

<u>Theorem 10:</u> For any $\xi \in (0, 1)$ let there exist two functions $c(\xi)$, $r(\xi)$ satisfying

$$c(\xi) > 0, \quad 0 < r(\xi) < 1 ,$$

and such that

$$K(x, t) = \sum_{n=1}^{+\infty} \alpha_n(x, \xi)(t-\xi)^n \quad , \tag{26}$$

where

$$|\alpha_n(x, \xi)| < c(\xi)\, r^n(\xi) \tag{27}$$

for all $x \in (0, 1)$. Then there exists a positive constant $\beta$ such that

$$d_n(\mathfrak{M}) = O[e^{-\beta n}] \, . \tag{28}$$

The whole theory can be applied to the approximate solution of boundary problems for differential equations.

We denote by $G(x, t)$ Green's function of the problem

$$[p(x)\varphi'(x)]' - q(x)\varphi(x) = f(x), \tag{29}$$

$$\varphi(0) = \varphi(1) = 0 \, ,$$

where $q(x) \geq 0, p(x) \geq p_0 > 0$, and $q(x)$, $p'(x)$ are continuous in the interval $\langle 0, 1 \rangle$.

The set of Green's functions $G(x, t)$ of the problem (29) with $p(x), q(x)$ satfsfying the conditions

$$0 < p_0 \leq p(x) \leq p_1, \ 0 \leq q(x) \leq q_1$$

will be denoted by $\mathcal{G}(p_0, p_1, q_1)$.

Using the estimates of the eigenvalues of the Sturm-Liouville operator, we obtain

Theorem 11:

$$(30) \qquad \sup_{G \in \mathcal{G}(p_0, p_1, q_1)} d_n(\mathfrak{M}_G) \asymp \frac{1}{n^2}$$

for any $0 < p_0 \leq p_1$, $0 \leq q_1$.

(Here $a_n \asymp b_n$ denotes weak equivalence, i.e. $a_n = O(b_n)$, $b_n = O(a_n)$.)

The proofs of all these theorems and further results applying to boundary problems for differential equations will be published in the Czechoslovak Mathematical Journal.

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