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THE FIRST MEASURABLE CARDINAL AND THE GENERALIZED CONTINUUM HYPOTHESIS

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Let $\aleph_\vartheta$ be the first cardinal number such that on a set of this cardinality there is a non-trivial ultrafilter j closed with respect to countable intersections.

In the present paper, there are found upper estimates of the cardinality $2^{\aleph_\vartheta}$ on the basis of estimates of $2^{\aleph_\alpha}$ for all $\aleph_\alpha < \aleph_\vartheta$.

The paper is written in the Gödel-Bernays set theory. The proofs are carried out by the method of models. Nevertheless, it is possible to obtain all the results by classical means within the classical set theory.

Definition 1. Let V^* be the class of all functions with domain ω . The letters $f, g, \ldots$ denote variables for elements of the class V^* . Put

$$f \stackrel{*}{=} g \equiv \{\alpha;\ f(\alpha) = g(\alpha)\} \in j;\ f \stackrel{*}{\in} g \equiv \{\alpha;\ f(\alpha) \in g(\alpha)\} \in j .$$

By a set formula is meant, the p.p.f from [1] such that there are no symbols for the special classes contained. If φ is a set formula, then φ^* is the formula obtained from φ by replacing the symbol $\in$ by the symbol $\stackrel{*}{\in}$ and by restricting the variables to the class V^* .

Metatheorem 1. Let $\varphi(x_1, \ldots, x_n)$ be a set formula which does not contain free variables other than $x_1, \ldots, x_n$. Then the following statement is provable in set theory:

$\varphi^*(f_1,\dots,f_n) \equiv \{\alpha;\ \varphi(f_1(\alpha),\dots f_n(\alpha)\} \in j$
for every $f,\dots f_n \in V^*$.
For proof see [3].

It is known that there is a model Γ of set theory such that its universal class is V^* and such that the membership relation for sets of Γ is the relation $\overset{*}{\in}$ from definition 1 (see [3]).

It is also known that there is a function G, defined on the class On, such that $G(\alpha)$ is an ordinal number of Γ for every $\alpha \in$ On, that $G(\beta) \overset{*}{\in} G(\alpha)$ for $\beta \in \alpha$, and that for every ordinal f of the model Γ there is an $\alpha \in$ On with $f \overset{*}{=} G(\alpha)$ (see [2], [3]).

If $x \in V$, we denote by k_x the element of V^* defined by

$$(\alpha)[\alpha \in \omega_\vartheta \rightarrow k_x(\alpha) = x].$$

Since $\aleph_\vartheta$ is the first measurable cardinal, $G(\alpha) \overset{*}{=} k_\alpha$ holds for every $\alpha \in \omega_\vartheta$. Put $d = G(\omega_\vartheta)$. Obviously $d \overset{*}{\in} k_{\omega_\vartheta}$. It is easy to show that the ultrafilter j can be chosen in such a manner that $d(\alpha) = \alpha$ for every $\alpha \in \omega_\vartheta$. In what follows, the ultrafilter j is assumed to possess this property.

Since Γ is a standard model (see [3]), the function $G(\omega_\alpha)$ is a cardinal of Γ for every cardinal ω_α. If ω_α is a strongly inaccessible cardinal, then $G(\omega_\alpha)$ is a strongly inaccessible cardinal in the model Γ.

Let M be the class of all strongly inaccessible cardinals. The symbol $(\alpha)^*\varphi(\alpha)$ is an abbreviation for $\{\alpha ; \varphi(\alpha) \& \alpha \in \omega_{\vartheta}\} \in j$.

Lemma 1. $(\alpha)^* (\alpha \in M)$.

Proof. As $d = G(\omega_{\vartheta})$ and ω_{ϑ} is strongly inaccessible, d is strongly inaccessible in Γ . By the metadefinition 1, $\{\alpha ; d(\alpha) \in M\} \in j$.

Definition 2. Put $d_0(\alpha) = 2^{\aleph_\alpha}$ for every $\alpha \in \omega_{\vartheta}$. Let $d_0 = G(\vartheta_0)$.

Lemma 2. card $\vartheta_0 = 2^{\aleph_\vartheta}$.

Proof. By the metatheorem, d_0 is the cardinality of the power set of d in the model Γ . Hence card $\vartheta_0 \leq$ $\leq 2^{\aleph_\vartheta}$. Hence, it suffices to prove that there exists a 1 - 1 mapping of $\mathcal{P}(\omega_{\vartheta})$ into the power set of d in Γ . For $m \subseteq \omega_{\vartheta}$ put $r_m(\alpha) = d(\alpha) \cap m$ (for every $\alpha \in \omega_{\vartheta}$). Evidently $r_m \overset{*}{\subseteq} d$. If $m \neq m'$, then $r_m \overset{*}{\neq} r_{m'}$.

Theorem 1. Let $\gamma \in \omega_{\vartheta}$. Then
$(\alpha)^* [2^{\aleph_\alpha} \leq \aleph_{\alpha+\gamma}] \rightarrow 2^{\aleph_\vartheta} \leq \aleph_{\vartheta+\gamma}$.

Proof. Put $d_\delta(\alpha) = \aleph_{\alpha+\delta}$. Obviously, the elements d_δ for $\delta \leq \gamma$ form the set of all cardinals between d and d_0 . Hence, there are at most as many cardinals between ω_ϑ and ϑ_0 .

Theorem 2. $(\alpha)^* [2^{\aleph_\alpha} \leq \aleph_{\alpha+\alpha}] \rightarrow 2^{\aleph_\vartheta} \leq \aleph_{\vartheta+\vartheta}$.

Proof. Similarly as in the previous proof, all the cardinals between d and d_0 (in Γ) are the d_δ's for $d \leq \omega_\vartheta$.

Analogous theorems may be obtained for other estimates of the cardinalities $2^{\aleph_\alpha}$.

References:

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