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ON THE EXISTENCE OF TRACES OF DISTRIBUTIONS BELONGING TO $B_{p,\theta}^{(\kappa)}$.

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(Preliminary communication)

1. Introduction. Recently S.L. Sobolev's and O.V. Běsov's spaces $W_2^{(k)}$ and $B_{p,\theta}^{(k)}$ were studied in considerable detail, and generalized in several directions. This paper deals with one generalization of these spaces, with the help of Fourier transforms. In general, the spaces considered cannot be identified with spaces of distributions; however, traces of some elements may be defined. We study the question as to when is the trace of an element in the space $L_2 = W_2^{(0)}$ or $B_{p,\theta}^{(0)}$. In the case that every element of the considered space has a trace on some fixed hyperplane then the space is a space of distributions, and the traces have the usual sense.

2. Notation. In this paper, points and subsets of Euclidean n-space E_n are denoted by capital letters, $[X, Y]$ is the scalar product of X and Y, $\mathcal{K}(M)$ the convex hull of M, $J_i = (\delta^{1i}, \dots, \delta^{ni})$ where $\delta^{ik} = 0$ for $k \neq i$ and $\delta^{ii} = 1$. Furthermore, put $n = r + s$ where r, s are positive integers, $X = (x_1, \dots, x_n) = (x, y)$ where $x = (x_1, \dots, x_r)$, $y = (y_1, \dots, y_s) = (x_{r+1}, \dots, x_n)$; $P_i = (\alpha^{(i)}, \beta^{(i)}) (i = 1, \dots, m)$ where $\alpha^{(i)} = (\alpha_1^{(i)}, \dots, \alpha_r^{(i)})$, $\beta^{(i)} = (\beta_1^{(i)}, \dots, \beta_s^{(i)})$.

Also set
$$\mathcal{P}_K(X) = \sum_{j=1}^{m} |x|^{\alpha^{(j)}} |y|^{\beta^{(j)}}$$
where $|x|^{\alpha} = |x_1|^{\alpha_1} |x_2|^{\alpha_2} \dots |x_r|^{\alpha_r}$, $|y|^{\beta} = |y_1|^{\beta_1} \dots |y_s|^{\beta_s}$.

Let $K = \mathcal{K}(\{P_1, \dots, P_m\})$, and let K° be the interior of K.

The space of all measurable functions for which

$$|u|_{W_2^{(K)}} = \left(\int_{E_n} |u(X)|^2 |\mathcal{P}_K(X)|^2 \, dX\right)^{\frac{1}{2}} < +\infty$$

is denoted by $W_2^{(K)}(E_n)$. This space is a Hilbert space.

Let $p > 1$, $\Theta > 1$. Let χ_ℓ be a characteristic function of the interval

$$\begin{aligned} &\langle 2^{\ell-1}, 2^{\ell} \rangle && \text{if } \ell > 0, \\ &\langle -1, 1 \rangle && \text{if } \ell = 0, \\ &\langle -2^{|\ell|}, -2^{|\ell|-1} \rangle && \text{if } \ell < 0, \end{aligned}$$

and set $\chi_{k_1, \dots, k_n}(X) = \chi_{k_1}(x_1) \dots \chi_{k_n}(x_n)$.
Let $\mathcal{F}$ stand for the Fourier transform and $\mathcal{F}^{-1}$ for the inverse of $\mathcal{F}$. The space of all functions u for which

$$|u|_{\mathcal{B}^{(0)}_{p,\Theta}} = \left(\sum_{k_1, \dots, k_n = -\infty}^{\infty} |\mathcal{F}^{-1} \chi_{k_1, \dots, k_n} u|^{\Theta}_{L_p}\right)^{\frac{1}{\Theta}} < +\infty$$

will be denoted by $\mathcal{B}^{(0)}_{p,\Theta}(E_n)$. The space of all functions u for which $u \mathcal{P}_K \in \mathcal{B}^{(0)}_{p,\Theta}(E_n)$ will be denoted by $\mathcal{B}^{(K)}_{p,\Theta}(E_n)$.

Put

$$|u|_{\mathcal{B}^{(K)}_{p,\Theta}} = |u \mathcal{P}_K|_{\mathcal{B}^{(0)}_{p,\Theta}}.$$

$\mathcal{B}^{(K)}_{p,\Theta}(E_n)$ is a Banach space.

Let π be the set of all X such that $[X, A] = 0$ where $A = (0, \dots, 0, a_{r+1}, \dots, a_n)$, $a_i \neq 0$ for $i = r+1, \dots, n$. Then π is a hyperplane in E_n. Let $B_1, \dots, B_{n-1}$ be a basis of π, and $B_1, \dots, B_{n-1}, B_n$ a basis of E_n.

Let T be the transformation defined by $Tu = v$, where

$$v(x_1, \dots, x_n) = \frac{1}{|\det B|} \int_{-\infty}^{\infty} u(B^{-1}X) \, dx_n$$

provided that the integral on the right hand side converges a.e.

Here $B = \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix}$ is an $m \times m$ matrix. The transformation T is said to be the dual imbedding on the hyperplane π.

The aim of this paper is to establish a necessary and sufficient condition on K in order that T be a continuous mapping of $W_2^{(K)}(E_n)$ into $L_2(E_{n-1})$ and a sufficient condition for T to be a continuous transformation of $B_{p,\theta}^{(K)}(E_n)$ into $B_{p,\theta}^{(0)}(E_{n-1})$. The main result is contained in the Theorems 10,11. The connection of T with the imbedding operator is established in Remark 12.

A finite set $\{P_1, \ldots, P_m\}$ is said to be well distributed with respect to the point P if

$$\bigcap_{i=1,\, P_i \neq P}^{m} \pi_P(P_i) = \emptyset ,$$

where $\pi_P(P_i)$ is the set of all $X \in E_n$ for which $[X - P, P_i - P] < 0$.

3. Remark. $\{P_1, \ldots, P_m\}$ is well distributed with respect to P if and only if $\{P_1, \ldots, P_m\} \div \{P\}$ is well distributed with respect to P. If $\{P_1, \ldots, P_m\}$ is well distributed with respect to P, then $P \in \mathcal{H}(\{P_1, \ldots, P_m\})$.

4. Notation. Put $I_1 = J_{r+1}, \ldots, I_s = J_n$, $S = \mathcal{H}(\{I_1, \ldots, I_s\})$, $I_{ik} = 2I_i - I_k$, $i, k = 1, \ldots, s$; let S^0 be the set of all points of the form $\alpha_1 I_1 + \ldots + \alpha_s I_s$ where $\sum_{i=1}^{s} \alpha_i = 1$, $\alpha_i > 0$ $(i = 1, \ldots, s)$.

Let $k \neq l$; $k, l = 1, \ldots, s$. Put

$${}^{(k,l)}P_i = (\alpha_1^{(i)}, \ldots, \alpha_r^{(i)}, \beta_1^{(i)}, \ldots, \beta_{k-1}^{(i)}, \beta_{k+1}^{(i)}, \ldots, \beta_{l-1}^{(i)}, \beta_{l+1}^{(i)}, \ldots, \beta_n^{(i)}, \beta_k^{(i)} + \beta_l^{(i)})$$

and ${}^{(k,l)}K = \mathcal{H}(\{{}^{(k,l)}P_1, \ldots, {}^{(k,l)}P_m\})$.

It is obvious that ${}^{(k,l)}K \subset E_{n-1}$, $n - 1 = r + (s - 1)$.

5. **Definition.** Assume $s = 1$. The set K is said to be $(r, 1)$-admissible if the set $\{P_1, \dots, P_m\}$ is well distributed with respect to the point I_1 .

Let $s > 1$. Then the set K is said to be (r, s)-admissible if the following conditions are satisfied:

1) $K \cap S \neq \emptyset$.

2) If $K \cap S = \{I_k\}$ then the set $\{P_1, \dots, P_m\} \cup \{I_{k1}, \dots, I_{ks}\}$ is well distributed with respect to the point I_k .

3) $\max\limits_{i=1,\dots,m} \sum\limits_{j=1}^{s} \beta_j^{(i)} > 1$.

4) If $N \subset \{1, \dots, s\}$, $N \neq \emptyset$ then $\max\limits_{i=1,\dots,m} \sum\limits_{j \in N} \beta_j^{(i)} < 1$.

5) The set ${}^{(k,l)}K$ is $(r, s-1)$-admissible for every k, l, $k \neq l$, $k, l = 1, \dots, s$.

6. **Remark.** The (r, s)-admissible sets are defined by finite induction. If K satisfies conditions 1)-4), then ${}^{(k,l)}K$ fulfils conditions 1),3),4) automatically and the latter ones need not be verified.

7. **Theorem.** If $K^\circ \cap S \neq \emptyset$ then K is (r, s)-admissible.

8. **Theorem.** Let $K \supset \{R', R''\}$ and $R' = (0, \dots, 0, r_1', \dots, r_s')$, $R'' = (0, \dots, 0, r_1'', \dots, r_s'')$. Let $r_i' > 0$, $r_i'' \geq 0$, $\sum\limits_{i=1}^{s} r_i'' < 1 < \sum\limits_{i=1}^{s} r_i''$.
Then K is (r, s)-admissible.

9. **Notation.** For $p > 1$ let $p \cdot K$ denote the set of all $X \in E_n$ for which $\frac{1}{p} X \in K$.

10. **Theorem.** The transformation T is a continuous linear mapping of $W_2^{(K)}(E_n)$ into $L_2(E_{n-1})$ if and only if

the set $2K$ is $(\varkappa, \lambda)$-admissible.

11. Theorem. Under the hypotheses of Theorem 8 on the set $p \cdot K$, T is a continuous transformation of $\mathcal{B}^{(K)}_{p,\Theta}(E_n)$ into $\mathcal{B}^{(0)}_{p,\Theta}(E_{n-1})$.

12. Remark. Under the hypotheses of Theorem 10, the space $\mathcal{W}^{(K)}_2(E_n)$ is isometrically isomorphic with the space $W^{(K)}_2(E_n) = \varphi^{-1}\mathcal{W}^{(K)}_2(E_n)$ which is a space of distributions. Under the hypotheses of Theorem 11 the space $\mathcal{B}^{(K)}_{p,\Theta}(E_n)$ is isometrically isomorphic with the space $B^{(K)}_{p,\Theta}(E_n) = \varphi^{-1}\mathcal{B}^{(K)}_{p,\Theta}(E_n)$ which is also a space of distributions. In latter cases the transformation $f \to \varphi^{-1} T \varphi f$ is continuous and linear, and maps $W^{(K)}_2(E_n)$ into $L_2(E_{n-1})$ and $B^{(K)}_{p,\Theta}(E_n)$ into $B^{(0)}_{p,\Theta}(E_{n-1})$. The distribution $\varphi^{-1} T \varphi f$ is the trace of the distribution $f . 2\pi$ on the hyperplane π. For particular cases of the set K the spaces considered coincide with S.L. Sobolev's and O.V. Běsov's spaces.

13. Remark. If some differential properties of traces are required, the following reasoning may be applied: $u \in W^{(K)}_2(E_n)$ if and only if $D^{\alpha}u = \dfrac{\partial^{\alpha_1 + \dots + \alpha_n} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \in W^{(K_1)}_2(E_n)$ where $K_1 = K - \alpha$. The case of the space $B^{(K)}_{p,\Theta}(E_n)$ may be treated in a similar manner.

14. Illustrative example. Let $P_1 = (0, 0)$, $P_2 = (0, \frac{3}{8})$, $P_3 = (\frac{3}{8}, 0)$, $P_4 = (\frac{3}{8}, \frac{3}{8})$.

Then

$$|u|_{W_2^{(K)}(E_n)} = |\varphi u(x_1, x_2)(1 + |x_1|^{\frac{3}{8}} + |x_2|^{\frac{3}{8}} + |x_1 x_2|^{\frac{3}{8}})|_{L_2(E_n)} .$$

The inequality $|u|_{L_2(\pi)} \leq C |u|_{W_2^{(K)}(E_n)}$ holds for all

$u \in W_2^{(\kappa)}(E_2)$. if and only if π is not parallel to any coordinate axis. In this case $u \in W_2^{(k)}(\pi)$ for all $k < \frac{1}{4}$. If $p > \frac{3}{4}$ and π is not parallel to any coordinate axis, then the inequality $|u|_{B^{(0)}_{p,\theta}(\pi)} \leq C|u|_{B^{(\kappa)}_{p,\theta}(E_2)}$ holds. If $p > \frac{8}{3}$ then the latter inequality holds for every π.

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