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Václav Chvátal

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ON FINITE AND COUNTABLE RIGID GRAPHS AND TOURNAMENTS

V. CHVÁTAL, Praha

Let V be a non-void set and E a binary relation on V, $E \subset V \times V$. Let f be a transformation of V. If $(x,y) \in E$ implies $(f(x), f(y)) \in E$, then f is called compatible with the relation E.

Let $C(E)$ denote the set of all transformations compatible with a relation E. Then $C(E)$ with the binary operation $\circ$ ($\circ$ is defined, as usual, by the compositions of transformations) is a semigroup, and its unity element is the identity transformation.

The pair $[V,E]$ will be considered as a graph, where V is the set of vertices, E the set of edges. The transformations in $C(E)$ will be called endomorphisms of $[V,E]$. If, for every $x,y \in V$, precisely one of the cases $(x,y) \in E$, $(y,x) \in E$ holds, then the graph $[V,E]$ is called a tournament. We emphasize that a tournament contains all loops; thus every constant transformation is an endomorphism.

An $f \in C(E)$ is called an automorphism of the graph $[V,E]$ if f is 1-1 mapping; an $f \in C(E)$ is called a proper endomorphism of the graph $[V,E]$ if f is not 1-1.

Let $C(E)$ contain $|V| + 1$ elements (here $|V|$ denotes the cardinal of V), namely the identity and all the constant transformations of V. Then the graph $[V,E]$ is called rigid.[x)]

x) We remark that the expression "rigid graph" is often used in a different sense.

The purpose of this paper is to prove some theorems concerning rigid graphs, and to show how rigid tournaments can be constructed for $|V| > 5$.

Theorem 1. There exists no rigid graph for $|V| = 3$ nor for $|V| = 4$; there exists just one rigid graph for $|V| = 2$.

Theorem 2. There exist two [x)] rigid tournaments for $|V| = 5$.

Theorem 3. There exist at least three rigid tournaments for $|V| \geqq 6$.

Theorem 4. There exists a countable rigid tournament.

First, we shall prove some lemmas.

Lemma 1. Let $[V,E]$ be a rigid graph, $|V| > 1$; then $(x,x) \in E$ for all $x \in V$.

Proof. If $E = \emptyset$, then $C(E)$ contains all transformations of V and $[V,E]$ is not a rigid graph. Hence E contains some couple (u,v) , and all the constants are endomorphisms; thus $(x,x) \in E$ for all $x \in V$.

In the sequel we shall confine ourselves to graphs with all the loops.

Lemma 2. Let $[V,E]$ be a rigid graph, $x,y \in V$, $x \neq y$, $(x,y) \in E$. Then $(y,x) \notin E$.

Proof. Let $(x,y) \in E$ and $(y,x) \in E$. Define a transformation f by $f(x) = y$, $f(u) = x$ for all $u \neq x$. Then $f \in C(E)$, and we obtain a contradiction.

Lemma 3. Let $|V| \geqq 3$, $[V,E]$ be a rigid graph.

If we define $G(x) = \{u: (x,u) \in E , u \neq x\}$

$G^{-1}(x) = \{u: (u,x) \in E , u \neq x\}$,

then $|G(x)| \geqq 1$, $|G^{-1}(x)| \geqq 1$ for all $x \in V$.

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x) Two rigid tournaments are explicitly given in the proof; it may be easily shown that there are no other ones.

Proof. Let $|G(x)| = |G^{-1}(x)| = 0$. Define $f(x) = x)$ and $f(u) = y$, $y \neq x$, for all $u \neq x$. Then $f \in C(E)$ and this is a contradiction.

Let $|G(x)| = 0$, $|G^{-1}(x)| > 0$. Define $f(x) = x$ and $f(u) = y$, $y \in G \ (x)$, for all $u \neq x$. Then $f \in C(E)$ and we have a contradiction.

Similarly for $|G^{-1}(x)| = 0$, $|G(x)| > 0$.

Lemma 4. Let $[V,E]$ be a rigid graph. Then there exists an $x \in V$, for which $|G(x)| = |G^{-1}(x)| = 1$ does not hold.

Proof. Indeed, assume the relation for all $x \in V$. Put $f(x) = G(x)$ for all $x \in V$. Then $f \in C(E)$ and we obtain a contradiction.

Lemma 5. Let $[V, E]$ be a tournament, $|V| \geqq 3$, $(x,z) \in E$, $(z,y) \in E$, $f \in C(E)$, $f(x) = f(y)$. Then $f(z) = f(x) = f(y)$.

Proof. $(f(x), f(z)) \in E$, $(f(z), f(x)) \in E$ and $[V,E]$ is a tournament; hence $f(x) = f(z)$.

Lemma 6. Let $[V,E]$ be a tournament such that $C(E)$ contains a non-identical automorphism. Then there exist at least three different points $x,y,z \in V$, for which $|G(x)| = |G(y)| = |G(z)|$ holds.

Proof. Evidently $|G(x)| = |G(f(x))|$ for all $x \in V$, and there exists a $u \in V$ for which $f(u) \neq u$. If $f \circ f(u) = u$, then $(u,f(u))$, $(f(u),u) \in E$, and this is a contradiction.
One cannot have $f \circ f(u) = f(u)$, because f is a 1-1 transformation. Hence $|G(u)| = |G(f(u))| = |G(f \circ f(u))|$.

Now, we shall prove our theorems.

Proof of theorem 1. Using lemmas 1,2,3,4 it is easy to show that no other graphs except G_1, G_2, G_3, G_4 on fig. 1 are rigid for $V = 2,3,4$. We find easily that the graph G_1

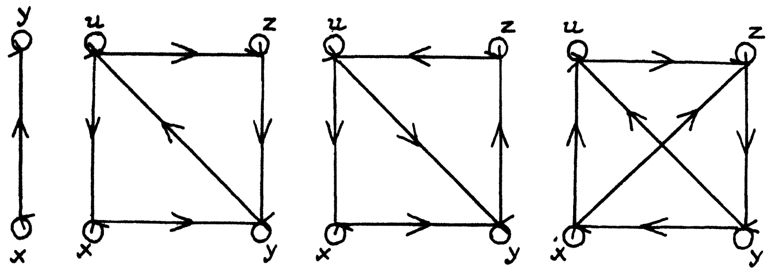

Fig. 1

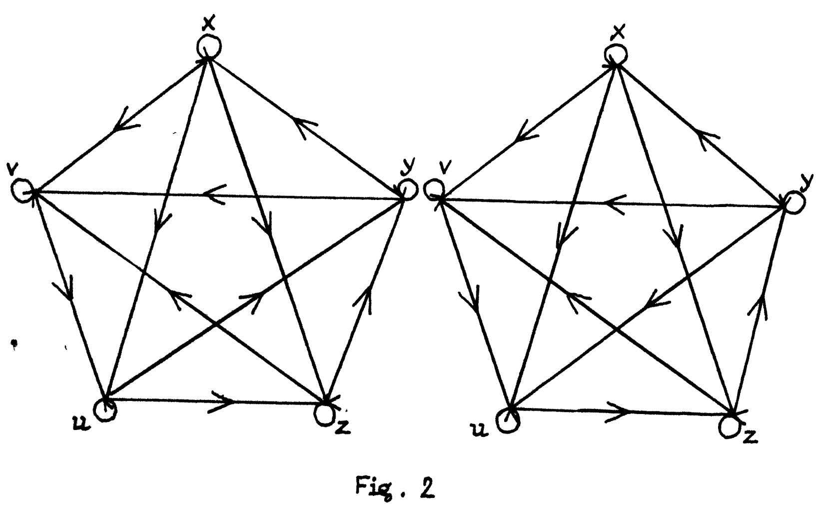

Fig. 2

is rigid, and the others have the following endomorphisms:

$G_2\begin{pmatrix}xyzu\\zyzu\end{pmatrix}$, $G_3\begin{pmatrix}xyzu\\yyzu\end{pmatrix}$, $G_4\begin{pmatrix}xyzu\\uyzu\end{pmatrix}$.

Proof of theorem 2. Both the tournaments T_1 , T_2 on fig.2 are rigid. We shall denote by p_n the number of those $x \in V$ for which $|G(x)| = n$ (n a positive integer). For T_1 and T_2 we then obtain

T_1 : $p_1 = 1$, $p_2 = 3$, $p_3 = 1$,

T_2 : $p_1 = 2$, $p_2 = 1$, $p_3 = 2$.

By lemma 6 , the tournament T_2 has no non-identical automorphism.

Let the tournament T_1 have an automorphism f . It follows that $f(x) = x$, $f(y) = y$. But $(z,u), (u,v), (z,v) \in E$, and thus f must be the identity.

It remains to investigate the proper endomorphisms.

If $(x,y), (y,z), (z,x) \in E$, put $\Delta xyz = \{(x,y), (y,z), (z,x)\}$, and $\Delta xyz \sim \Delta uvw$ if $\Delta xyz \cap \Delta uvw \neq \emptyset$. If $\Delta xyz \sim \Delta uvw$, $f \in C(E)$ and $f(x) = f(y)$, then it follows from lemma 5 that $f(x) = f(y) = f(z) = f(u) = f(v) = f(w)$.

Now, it is easy to show that every proper endomorphism of T_1 , T_2 is constant.

For T_1 there is $\Delta xzy \sim \Delta xuy$, $\Delta xuy \sim \Delta vuy$, $\Delta vuy \sim \Delta vuz$; and it follows from lemma 5 that $f(x) = f(v) \Longrightarrow f(x) = f(z)$, if $f \in C(E)$.

For T_2 there is $\Delta xzy \sim \Delta yuz$, $\Delta yuz \sim \Delta vuz$; and it follows from lemma 5 that $f(x) = f(v) \Longrightarrow f(x) = f(z)$, $f(v) = f(y) \Longrightarrow f(x) = f(y)$, $f(x) = f(u) \Longrightarrow f(v) = f(u)$, if $f \in C(E)$.

Hence T_1 and T_2 have no proper endomorphism except the constants.

Proof of theorem 3. We shall construct the rigid tournaments for $|V| \geqq 6$.

Let $[V_o , E_o]$ be a rigid tournament, $|V_o| = n$, $n \geqq 5$, $p_{n-2} \in \langle 1,2 \rangle$. Denote by x_o , y_o the points for which $|G(x_o)| = n - 2$, $(y_o , x_o) \in E$, and if $p_{n-2} = 2$ then $|G(y_o)| = n - 2$. Now set $V = V_o \cup \{x\}$, $E = E_o \cup E_x$, $E_x = \{(x,u): u \in V_o , u \neq x_o\} \cup \{(x_o , x), (x,x)\}$. Then the tournament $[V,E]$ is rigid.

Indeed, assume that $[V,E]$ has a non-identical automorphism f . If $f(x) = x$, then $[V_o , E_o]$ has the non-identical automorphism f_o , defined by $f_o(u) = f(u)$ for all $u \in V_o$; but this is a contradiction.

If $f(x) \neq x$, then there must be $f(x_o) = x$, $f(x) = x_o$, because $|G(x)| = |G(x_o)| = n - 1$ and $u \neq x$, $u \neq x_o \Rightarrow$ $\Rightarrow |G(u)| < n - 1$.

Hence $(x,x_o) \in E$, and this is a contradiction.

Now assume that $[V,E]$ has a proper non-constant endomorphism f , and write $f^{-1}(u) = \{v: f(v) = u\}$. If $f^{-1}(u) \cap$ $\cap V_o \neq \emptyset$, we may choose an element of $f^{-1}(u) \cap V_o$ and denote it $g(u)$. Then $g \circ f$ is a transformation of V_o .

Let $(u,v) \in E_o$. If $g \circ f(u) = g \circ f(v)$, then evidently $(g \circ f(u), g \circ f(v)) \in E_o$. If $g \circ f(u) \neq g \circ f(v)$, then $(f(u), f(v)) \in E$ implies $(g \circ f(u), g \circ f(v)) \in E_o$. Hence $g \circ f \in C(E_o)$.

Assume that $g \circ f$ is the identity. Then $u, v \in V_o$, $u \neq v$ imply $f(u) \neq f(v)$. One must have $f(x) = f(u)$ for some $u \in V_o$, because f is not 1-1 . But there exists a

$v \in V_o$ for which $(v,u) \in E$, $v \neq x_o$ and $(f(u), f(v))$, $(f(v), f(u)) \in E$; this is a contradiction.

Assume that $g \circ f$ is a constant. Then $f(u) = v$ for all $u \in V_o$ and $(f(x),v)$, $(v,f(x)) \in E$. It follows that $f(x) = f(v)$, so that f is a constant transformation; but this contradicts our assumption.

It results that $[V_o, E_o]$ is not rigid, and this is a contradiction. Thus we have proved that $[V,E]$ is rigid.

Setting $|V| = n$, one has $p_{n-2} = 2$. It follows that one can construct two sequences of rigid tournaments. Then

$p_1 = 2$, $p_2 = p_3 = \dots p_{n-3} = 1$, $p_{n-2} = 2$

for the sequence derived from T_2 , and

$p_1 = 1$, $p_2 = 3$, $p_3 = 0$, $p_4 = p_5 = \dots p_{n-3} = 1$,

$p_{n-2} = 2$

for the sequence derived from T_1 .

If we take complements of graphs from the second sequence preserving loops, we obtain a sequence of rigid tournaments distinct from both;for this sequence there is

$p_1 = 2$, $p_2 = p_3 \dots p_{n-5} = 1$, $p_{n-4} = 0$, $p_{n-3} = 3$,

$p_{n-2} = 1$.

Proof of theorem 4.

In this part we shall denote vertices by positive integers.

If we construct the second sequence of rigid tournaments and proceed to infinity, we obtain a countable tournament $[N,E]$, where N is the set of all positive integers and $E = B \cup S$,

$B = \{(1,2),(3,1),(4,1),(5,1),(2,3),(2,4),(5,2),(3,4),(5,3),$
$(4,5),(1,1),(2,2),(3,3),(4,4),(5,5)\}$

$S = \{(x,y): x,y \in N,\ x > 5\ ,\ y < x - 1\ \text{VEL}\ y = x + 1\ \text{VEL}\ y = x\} \cup \{(5,6)\}$

There is $\Delta 123 \sim \Delta 124 \sim \Delta 245 \sim \Delta 345 \sim \Delta 456 \sim \Delta 567 \ldots \ldots \sim \Delta\, n\, n{+}1\ \ n+2 \sim \Delta\, n\, n{+}1\ \ n+2\ \ n+3 \sim \ldots$

and for no other set Δ except these. Moreover, using lemma 5, there is for $f \in C(E)$

$f(1) = f(5) \Rightarrow f(1) = f(3)$,

$f(u) = f(v) \Rightarrow f(u) = f(u+1)$ if $u > 5$, $u > v + 1$,

It follows that if f is an endomorphism of $[N,E]$ and there exist $x,y \in N$, $x \neq y$, $f(x) = f(y)$, then f is a constant.

Let us assume that $[N,E]$ has a non-constant endomorphism f ; then $x \neq y \Rightarrow f(x) \neq f(y)$.

The edge $(4,5)$ is an element of three distinct sets $\Delta 245$, $\Delta 345$, $\Delta 456$, and no other edge is an element of three or more sets Δ . It follows that $f(4) = 4$, $f(5) = 5$, because the edge $(f(4), f(5))$ is an element of three sets Δ . The edge $(f(5),f(6))$ is an element of two sets Δ , hence $f(6) = 6$. Similarly, $f(u) = u$ for all $u > 6$.

If $f(u) \neq u$ for some $u \in \{1,2,3\}$, then T_1 has the automorphism f_o , defined by $f_o(u) = f(u)$, which is not the identity transformation; this is a contradiction.

Thus f is the identity, and we have proved that $[N,E]$ is rigid.

Remark to theorem 4. If we derive a countable tournament from T_1 , we obtain the tournament $[N,E']$, where

$E' = \{(x,y): (x,y) \in E\ \text{ET}\ (x,y) \neq (2,4)\} \cup \{(4,2)\}$;

however this tournament is not rigid since it has the endomorphism f, defined by $f(n) = n + h$, where h is an arbitrary positive integer.

Applications of the results.

1. Algebra. A set M with a binary operation $\circ$, which assigns to any ordered pair of elements M some element of M, is called a grupoid. If $u \circ v = v \circ u$ for all $u, v \in M$, then M is called a commutative grupoid. The elements $u \in M$ with $u \circ u = u$ are called idempotents. If f is a transformation of M and for every $u, v \in M$ there is $f(u) \circ f(v) = f(u \circ v)$, then f is called a homomorphism of the grupoid.

Let $[V,E]$ be a rigid tournament. We may define a binary operation $\circ$ on V by

$$u \circ v = u \text{ for } (v,u) \in E$$
$$u \circ v = v \text{ for } (u,v) \in E .$$

Evidently, the set V with the binary operation $\circ$ is a commutative grupoid such that all elements are idempotents and that each homomorphism is either constant or the identity transformation. Thus

There exists a commutative grupoid G such that all elements are idempotents and that each homomorphism is either constant or the identity transformation for $5 \leqq |G| \leqq \aleph_0$.

2. Rigid closure spaces. If P is a set with a rule which assigns to any set $M \subset P$ its closure $\bar{M}$ in such a manner that the axioms

$$\emptyset = \bar{\emptyset} \qquad (I)$$

$$M \subset \bar{M} \qquad (II)$$

$$\overline{M_1 \cup M_2} = \bar{M}_1 \cup \bar{M}_2 \qquad (III) \text{ (see [1])}$$

are fulfilled, then P is called a closure space. A transforformation f of P is called continuous if $f(\overline{M}) \subset \overline{f(M)}$, where $f(M) = \{x: x = f(u), u \in M\}$.

Let $[V,E]$ be a rigid tournament, and set $\overline{Y} = \{x: (u,x) \in E, u \in Y\}$ for any set $Y \subset P$. The set V with the so defined closure is a closure space, all continuous transformations of which are either constant or identical. Thus:

There exists a closure space P such that all continuous transformations of P are either constant or the identity transformations provided that $5 \leq |P| \leq \aleph_0$.

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References:

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