Zdeněk Frolík

On the Souslin-graph theorem

Commentationes Mathematicae Universitatis Carolinae, Vol. 9 (1968), No. 2, 243--249

Persistent URL: http://dml.cz/dmlcz/105176

Commentationes Mathematicae Universitatis Carolinae
9, 2 (1968)

ON THE SOUSLIN-GRAPH THEOREM

Zdeněk FROLÍK, Praha

L. Schwartz in [13] turned the attention of analysts to a particular case of Theorem 1 below. The subsequent development in [10] is also contained in Theorem 1. Theorem 1 is a quite formal generalization of Theorem 2, which will be proved here. Everything what is needed for the proof is well-known, however the theorem is so useful, that a short survey might be in place. We use the term meager for "to be of the first category". All spaces are assumed to be separated.

Theorem 1. Assume that E is a T.L.S. which is inductively generated by homomorphisms from non-meager T.L.S., and that F is a locally convex T.L.S. which is analytic (see Definition below). Then:

if f is a homomorphism of E into F such that the graph of f is a Souslin set in $E \times F$ then f is continuous.

Theorem 1 follows immediately from the following

Theorem 2. Let $g: E_1 \rightarrow E$, and $h: F \rightarrow F_1$ be continuous homomorphisms, where E_1, E and F are T.L.S., and F_1 is a locally convex T.L.S. Assume that E_1 is non-meager, and that F is analytic. Then

if the graph of a homomorphism $f: E \rightarrow F$ is a Souslin set in $E \times F$ then $k = h \circ f \circ g$ is continuous.

Just the closely related terms Souslin and analytic should be explained. For a survey of the theory of analytic sets see [3], for abstract theory of Souslin sets [9],[6]; the latter paper also contains a try for an abstract theory of analytic sets. There we recall what is needed to understand Theorem 2.

A Souslin set derived from a collection of sets $\mathfrak{M}$, or simply a Souslin - $\mathfrak{M}$ set, is a set of the form

$$\mathcal{S}M = \bigcup\{\bigcap\{M_s \mid s \prec \sigma\} \mid \sigma \in \Sigma\} ,$$

where M is a single-valued relation with domain S - the set of all finite sequences in the set N of natural numbers - , which assigns to each $s \in S$ an element of $\mathfrak{M}$, Σ is the set of all (infinite!) sequences in N, and $s < \sigma$ means that s is a restriction of σ to an initial finite segment of N (thus $s \in S$). The collection of all Souslin -$\mathfrak{M}$ sets is denoted by $\mathcal{S}(\mathfrak{M})$. If $\mathfrak{M}$ is a collection of all closed sets in a topological space P then the Souslin-$\mathfrak{M}$ sets are called the Souslin sets in P.

Very important relation (for any $\mathfrak{M}$)

$$\mathcal{S}(\mathcal{S}(\mathfrak{M})) = \mathcal{S}(\mathfrak{M})$$

shows that $\mathcal{S}(\mathfrak{M})$ is closed under countable intersections and countable unions. Recently two simple proofs were given, the one in [10] uses abstract setting of contidion (2) below, that one in [6] uses the abstract setting of condition (1) below. The latter proof also works for some special cases of operation $\mathcal{S}$. There we only need to know that the intersection of two Souslin sets in a space is a Souslin set. It should be remarked that many important collections of sets in analysis are invariant under the operation $\mathcal{S}$, e.g. measurable sets, capacitable sets (in a proper setting), the sets enjoying the property of Baire (see Lemma 2 below).

The set Σ endowed with the topology of pointwise convergence (i.e. $\Sigma = N^N$ in the topological sense) is known to be homeomorphic to the space of irrational numbers. It is easy to observe that (see [4], Remark preceding Th.1)

$X \subset P$ is a Souslin set in a space P iff there exists a correspondence f (= a multivalued map) of Σ into P such that the graph of f is closed, and $X = Ef$ (= the abstract range of f, i.e. $f[\Sigma]$).

If M is given then f is defined by:

$$f\sigma = \bigcap \{ M_s \mid s \prec \sigma \} ,$$

and if f is given, then M is defined by:

$$M_s = cl\, f[\Sigma_s]$$

where $\Sigma_s = E\{\sigma \mid \sigma \in \Sigma, \ s \prec \sigma\}$. Now, a correspondence $f : Q \to P$ is called usco-compact if f is upper semi-continuous, and if the images of points are compact. For example, any continuous mapping and the inverse of any surjective proper mapping are usco-compact. The method of correspondences was introduced in [3] and [4].

<u>Definition</u>. A uniformizable (= completely regular) space P is said to be analytic if the following four equivalent conditions are fulfilled:

(1) P is the image of Σ (or any Polish space) under an usco-compact correspondence.

(2) P is the image of a $K_{\sigma\delta}$ under a continuous mapping ($K_{\sigma\delta}$ means a countable intersection of σ-compact spaces).

(3) P is absolute Souàlin, i.e., P is Souslin in any $Q \supset P$ (separated!).

(4) P is a Souslin set derived from the compact sets in some space.

Thus the class of all analytic spaces is closed under usco-compact correspondences, and it contains the Polish spaces and the compact spaces.

The proof of Theorem 2 follows immediately from the following three lemmas:

Lemma 1. If X is a Souslin set in the product $E \times F$ of two topological spaces, and if F is analytic, then the projection of X into E is Souslin. Thence, $X^{-1}[Y]$ is Souslin in E for each closed $Y \subset F$.

Lemma 2. Any Souslin set in any space has the property of Baire. More generally, the collection of all sets with the property of Baire is invariant under the Souslin operation $\mathcal{S}$.

Lemma 3. Let X be a non-meager set with the property of Baire in an inductively continuous group G (= a group endowed with a topology such that the translations are continuous). Then X - X is a neighborhood of the neutral element.

First we shall prove Theorem 2. Then a comment to the lemmas will be made.

Proof of Theorem 2. Choose a closed balanced neighborhood of zero in F_1. We shall prove that $\mathcal{k}^{-1}[K]$ is a neighborhood of zero in E_1.

The set $h^{-1}[K]$ is closed in F because of the continuity of h , hence $f^{-1}[h^{-1}[K]]$ is Souslin in E by Lemma 1. Since g is continuous, $k^{-1}[K]=g^{-1}[f^{-1}[h^{-1}[K]]]$ is obviously Souslin in E_1 . By Lemma 2 the set $Y=k^{-1}[K]$ has the property of Baire. Since E_1 is non-meager and $\bigcup\{n\cdot Y\}=E_1$, some n . Y must be non-meager, thence Y is non-meager because $\{x \to \frac{1}{n}\cdot x\}$ is a homeomorphism. Thus $Y-Y=2\cdot Y$ is a neighborhood of 0 , hence Y is a neighborhood of zero.

Remark. Notice that we have proved: if E_1 , E, F and F_1 are inductively continuous groups, g and h are continuous homomorphisms, f is a homomorphism whose graph is Souslin, E_1 is non-meager, and F is analytic, then for each symmetric closed set in F_1 such that $\bigcup_n n\cdot K=F_1$ (when $(n+1)K=K+n\cdot K$) there exists an n such that $k^{-1}[n\cdot K]$ is a neighborhood of the zero in E_1 . This shows how much of the linear structure is needed.

Proof of Lemma 1. Let K be any compactification of F and let π stand for the projection of $E\times K$ into E . By condition (3) above, the set F is Souslin in K , hence $E\times F$ is Souslin in $E\times K$. Since X is Souslin in $E\times F$, $X=X'\cap(E\times F)$ with X' Souslin in $E\times K$ (use the same representation). Thence X is Souslin in $E\times K$. The mapping π is closed (because K is compact!), and one sees that $\pi[K]$ is Souslin in X (if $X=\mathcal{S}M$, then $\pi[X]=\mathcal{S}\{s\to\pi[M s]\}$.

Remark. A proof of Lemma 1 based on condition (1) is given in [12].

The proof of Lemma 2 is given in [9], § 11, VII. Lemma 3 can be found in [1],[8],[9],[10],[14] under stronger assumptions, which, however, do not effect the proof.

R e f e r e n c e s

[1] E. ČECH: Topological spaces.Academia(Praha),1966.

[2] G. CHOQUET: Theory of Capacities.Ann.Inst.Fourier 5 (1953-4),131-294.

[3] G. CHOQUET: Ensembles K-analytiques et K-Sousliniens. Ann.Inst.Fourier 9(1959),75-81.

[4] Z. FROLÍK: A contribution to the descriptive theory of sets and spaces.General Topology and its relations to Modern Analysis and Algebra(Proc.Symp. Prague,Sept.1961),Academic Press.

[5] Z. FROLÍK: Baire sets that are Borelian subspaces.Proc. Roy.Soc.A 292(1967).

[6] Z. FROLÍK: Remarks to the abstract theory of Souslin sets.To appear elsewhere.

[7] Z. FROLÍK: A survey of Descriptive theory.Conferenze del Seminario di Matematica dell'Università di Bari.

[8] I. KELLEY: General Topology.

[9] K. KURATOWSKI: Topologie 1, Warszawa 1952.

[10] A. MARTINEAU: Sur le théorème du graphe fermé.Compt. Rend.Acad.Sci.263,870(1966).

[11] P. MEYER: Probability and Potentials.Blaisdell Publ. Co.1966.

[12] L.A. ROGERS - R. WILLMOTT: On the projection of Souslin sets.Matematika 13(1966),147-150.

[13] L. SCHWARTZ: Sur le théorème du graphe fermé.Compt. Rend.Acad.Sci.263,602(1966).

[14] F. TREVES: Topological vector-spaces,distributions and kernels.Academic Press 1967.

(Received March 25, 1968)