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A NOTE ON THE SOUSLIN OPERATIONS

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The purpose of this paper is to give a simple and natural proof of Theorem 1 below, and to introduce two "Souslin"-like operations that seem to be important in the abstract theory of Souslin, "analytic", and "Borelian" sets: The details will appear in "Correspondence Technique in Abstract Descriptive Theory".

The operation $S_{\mathfrak{a}}$ in Section 2 is a good substitute for classical Souslin operation, the proofs of the main results are really clear, and the deep content of the classical Souslin operation is contained in $S_{\mathfrak{a}}$. The most of the troubles come when we want to prove that the last statement is true. The operation $S_{\mathfrak{a}}^*$ in Section 3 is something between the classical operation and operation $S_{\mathfrak{a}}$.

For a comment concerning Paul Meyer's approach see the remark following Theorem 1. The notation of [1] is used throughout. E.g. if M is a relation, then DM and EM stand, respectively, for the domain and the range of M.

1. Classical definition of Souslin sets.

Denote by S the set of all finite sequences in the set N of natural numbers, and let Σ be the set of all infinite sequences in N. If f and g are two relations we write $f \prec g$ to indicate that f is a restriction of g.

Given a collection of sets $\mathfrak{M}$, by a Souslin-$\mathfrak{M}$ set, or a Souslin set derived from $\mathfrak{M}$, is meant a set of the form

$$\mathcal{S}M = \bigcup\{\bigcap\{M s \mid s \prec \sigma\} \mid \sigma \in \Sigma\}$$

for some $M : S \to \mathfrak{M}$. The collection of all Souslin-$\mathfrak{M}$ sets is denoted by $\mathcal{S}(\mathfrak{M})$. In this classical definition we take mappings M from a certain ordered set into $\mathfrak{M}$. In this note we work with M's from collections of sets into $\mathfrak{M}$.

Definition 1. Let $\mathcal{B}$ and $\mathfrak{M}$ be collections of sets. A Souslin family over $\mathcal{B}$ in $\mathfrak{M}$ is a single valued relation M with $DM = \mathcal{B}$, and $EM \subset \mathfrak{M}$. The Souslin set of M is the set

(*) $$\mathcal{S}M = \bigcup\{\bigcap\{MB \mid x \in B \in \mathcal{B}\} \mid x \in \bigcup DM\}.$$

The collection of all $\mathcal{S}M$ with M in $\mathfrak{M}$ over $\mathcal{B}$, is denoted by $\mathcal{S}_{\mathcal{B}}(\mathfrak{M})$. The associated relation with a Souslin family M is the relation $\widetilde{M}$ consisting of all $\langle x, y\rangle$ such that $x \in \bigcup DM$, and $y \in MB$ for each $B \in \mathcal{B}$ with $x \in B$. The associated fragmentation is the family $\{x \to \bigcap\{MB \mid x \in B \in \mathcal{B}\} \mid x \in \bigcup DM\}$. The collection of all $\mathcal{S}M$ with the associated fragmentation disjoin (equivalently, when the associated relation is a fibration) is denoted by $\mathcal{S}^{d}_{\mathcal{B}}(\mathfrak{M})$. Sometimes the collections $\mathcal{S}^{s}_{\mathcal{B}}(\mathfrak{M})$ of all (*) with $\widetilde{M}$ single-valued, and $\mathcal{S}^{sd}_{\mathcal{B}}(\mathfrak{M})$ of all (*) with $\widetilde{M}$ single-valued an injective are important.

For each s in S put

$$\Sigma s = E\{\sigma \mid s < \sigma,\ \sigma \in \Sigma\}$$

Clearly $\{\Sigma s\}$ is an open base for the topology of coordinate convergence in Σ, if N is endowed with the discrete topology. The relation $\{s \to \Sigma s\}$ is one-to-one, and clearly if $M : S \to \mathcal{M}$ and $M' : E\{\Sigma s\} \to \mathcal{M}$ such that $Ms = M'\Sigma s$, then $\mathcal{S}M = \mathcal{S}M'$ where $\mathcal{S}M$ is defined by the classical definition, and $\mathcal{S}M'$ is define d by Definition 1. Thus $\mathcal{S}^d$, $\mathcal{S}^s$, and $\mathcal{S}^{sd}$ carries over to the classical case, and we can formulate the main property of the Souslin operation over $\{\Sigma s\}$.

Theorem 1. Let $\mathcal{M}$ be a collection of sets, and let $\Phi \in \mathcal{M}$. Then

(a) $\mathcal{S}(\mathcal{S}(\mathcal{M})) = \mathcal{S}(\mathcal{M})$,

(b) $\mathcal{S}^d(\mathcal{S}^d(\mathcal{M})) = \mathcal{S}^d(\mathcal{M})$,

(c) $\mathcal{S}^s(\mathcal{S}^s(\mathcal{M})) = \mathcal{S}^s(\mathcal{M})$,

(d) $\mathcal{S}^{sd}(\mathcal{S}^{sd}(\mathcal{M})) = \mathcal{S}^{sd}(\mathcal{M})$.

Remark. The classical proof is quite complicated, for (a) see [4],§ 36, for (b) see [5]. A nice proof of (a) was given in [3]; P. Meyer used ingeniously the projection technique that had been already developed for analytic sets. It should be remarked that the Meyer's method does not apply to the other sets.

Given a collection $\mathcal{M}$ of sets we denote by $\mu\mathcal{M}$ the set of all finite intersections of sets in $\mathcal{M}$. It is easy to see that

$$\mathcal{S}(\mathcal{M}) = \mathcal{S}(\mu\mathcal{M})$$

for each $\mathcal{M}$ containing Φ, and similarly for $\mathcal{S}^d$, $\mathcal{S}^s$, and $\mathcal{S}^{sd}$ which shows that if is enough to prove Theorem 1 for multiplicative $\mathcal{M}$. Indeed, if

$X = \mathcal{S}M$ with $Ms = \bigcap\{M(s,i) \mid i = 0, 1, \dots, n(s)\}$,

we define a mapping φ of Σ into Σ as follows: if $\sigma = \{\sigma(n) \mid n \in N\}$, then

$$\varphi(\sigma) = \sigma(0), \underbrace{0, 0, 0, \dots, 0}_{n(\{\sigma_0\})\text{-times}}, \quad \sigma(1)+1, \underbrace{0, \dots, 0}_{n(\sigma_1)\text{-times}}, \sigma(2)+1, \dots ,$$

and then define $M' : S \longrightarrow \mathcal{M}$ for sections of $\varphi(\sigma)$'s in the natural way, and $\emptyset$ otherwise. (This is the only reason for assuming $\emptyset \in \mathcal{M}$ in Theorem 1.) The conclusion follows from theorems 3,4 and 5.

Remark. It would be interesting to know for which collections of sets $\mathcal{B}$ Theorem 1 is true.

2. Operation S_Q .

Definition 1. Let Q be a topological space, and let $\mathcal{M}$ be a collection of sets. An S_Q -set derived from $\mathcal{M}$ is a set of the form $\mathcal{S}_{\mathcal{B}} M$ where $\mathcal{B}$ is a countable opem covering of Q , and M is in $\mathcal{M}$ over $\mathcal{B}$. The set of all S_Q -sets derived from $\mathcal{M}$ over Q is denoted by $S_Q(\mathcal{M})$. Thus $S_Q(\mathcal{M}) = \bigcup\{\mathcal{S}_{\mathcal{B}}(\mathcal{M}) \mid \mathcal{B}$ is a countable covering of Q $\}$.

Similarly we define $S_Q^d(\mathcal{M})$, $S_Q^s(\mathcal{M})$, and $S_Q^{sd}(\mathcal{M})$.

Proposition 1. Let $\mathcal{M}$ be a collection of sets with $\emptyset \in \mathcal{M}$ and let Q_1 and Q_2 be two topological spaces. Then:

(a) $S_{Q_2}(\mathcal{M}) \subset S_{Q_1}(\mathcal{M})$

if either there exists a continuous mapping of Q_1 onto Q_2, or if Q_2 is a closed subspace of Q_1 .

(b) $S_{Q_2}^{\alpha}(\mathcal{M}) \subset S_{Q_1}^{\alpha}(\mathcal{M})$

is true for $\alpha = d, s, sd$, if either there exists a one-to-one continuous mapping of Q_1 onto Q_2, or Q_2 is a closed subspace of Q_1 .

Remark. The assumption $\emptyset \in \mathcal{M}$ is needed for the case when Q_2 is a subspace of Q_1.

Definition 2. Given a space Q we denote by $Q^{\aleph_0}$ the space $\Pi\{Q \mid n \in N\}$, and we denote by $\aleph_0 Q$ the space $\Sigma\{Q \mid n \in N\}$.

Theorem 2. Let $\mathcal{M}$ be a collection of sets. If there exists a continuous mapping of Q onto $\aleph_0 Q$ then $S_Q(\mathcal{M})$ is closed under countable unions. If there exists a continuous mapping of Q onto $Q^{\aleph_0}$ then $S_Q(\mathcal{M})$ is closed under countable intersections, and

$$S_Q(S_Q(\mathcal{M})) = S_Q(\mathcal{M}) .$$

The same is true for S_Q^{s}.

Sketch of the proof of the last assertion. Assume that $M : \mathcal{L} \to S_Q(\mathcal{M})$ where $\mathcal{L}$ is a countable cover of Q. To prove that $\mathcal{S}_{\mathcal{L}} M \in S_Q(\mathcal{M})$ it is enough to show that $\mathcal{S}_{\mathcal{L}} M \in S_{Q_1}(\mathcal{M})$ where $Q_1 = Q \times Q^{\mathcal{L}}$. For each L in $\mathcal{L}$ we can choose $M_L : \mathcal{B}_L \to \mathcal{M}$ such that $\mathcal{B}_L$ is a countable covering of Q, and $\mathcal{S}_{\mathcal{B}_L} M = ML$. For each L in $\mathcal{L}$, and B in $\mathcal{B}_L$ denote by B_L the set

$$E\{\langle x, y\rangle \mid x \in L,\ pr_L y \in B\}$$

where pr_L is the projection onto the L-th coordinate space of $Q^{\mathcal{L}}$. Consider the open cover $\mathcal{B}'$ of Q' consisting of all B_L, $L \in \mathcal{L}$, $B \in \mathcal{B}_L$, and define $M' : \mathcal{B}' \to \mathcal{M}$ by setting

$$M'B_L = M_L B \text{ for } L \in \mathcal{L},\ B \in \mathcal{B}_L .$$

It is easy to see that $\mathcal{S}_{\mathcal{L}} M = \mathcal{S}_{\mathcal{B}'} M'$.

For the case of $\mathcal{S}^{d}$ and $\mathcal{S}^{sd}$ the same proof gives the following

<u>Theorem 3</u>. Let $\mathfrak{M}$ be a collection of sets, Q be a space. If there exists a one-to-one continuous mapping of Q onto $Q^{\aleph_0}$ then $S_Q^d(\mathfrak{M})$ is closed under countable intersections, and

$$S_Q^d(S_Q^d(\mathfrak{M})) = S_Q^\alpha(\mathfrak{M}) .$$

The same is true for S_Q^{sd} .

<u>Remark</u>. If $Q = (\aleph_0 R)^{\aleph_0}$ then the assumptions of Theorems 2 and 3 are fulfilled. The classical case of Q = $= \Sigma$ is obtained when taking a singleton for R .

The proof of Theorem 1 is now concluded by the following

<u>Theorem 4</u>. If $\mathfrak{M}$ is multiplicative then $\mathcal{S}(\mathfrak{M}) = S_\Sigma(\mathfrak{M})$, $\mathcal{S}^d(\mathfrak{M}) = S_\Sigma^d(\mathfrak{M})$, $S^s(\mathfrak{M}) = S_\Sigma^s(\mathfrak{M})$, and $\mathcal{S}^{sd}(\mathfrak{M}) = S^{sd}(\mathfrak{M})$.

<u>Proof</u>. Clearly the inclusions $\subset$ hold. To get the inclusions $\supset$,we must prove that any set

$$X = \mathcal{S}M$$

over an open case $\mathcal{B}$ can be expressed as

$$X = \mathcal{S}\, M'$$

over $\{\Sigma s\}$. This is obvious if $\{\Sigma s\}$ refines $\mathcal{B}$. Arrange $\mathcal{B}$ in a sequence $\{U_n\}$, and for any s of length k consider the set N′ of all n such that $U_n \supset \Sigma s$, and put $M'\Sigma s = \bigcap\{MU_n \mid n \in N''\}$ where N″ = N′ if the cardinal of N′ is at most n , and N″ is the n-th section of N′ otherwise. If $\{\Sigma s\}$ does not refine $\mathcal{B}$ then one finds a homeomorphism h of Σ onto Σ such that $h[\mathcal{B}]$ is refined by $\{\Sigma s\}$. For a more intuitive approach to this proof see Section 3.

3. Operation S_Q^*.

In the classical case all Souslin sets are defined over a distinguished fixed open base for the space Σ. In this section we study Souslin sets of certain Souslin families over open bases of the space; in the case of second countable spaces we get that these Souslin sets can be defined by bases contained in any given open base.

Definition 2. An S^*-family in $\mathcal{M}$ over a space Q is a Souslin family M in $\mathcal{M}$ over an open base $\mathcal{B}$ for Q such that

$$\bigcap\{MB \mid x \in B \in \mathcal{B}\} = \bigcap\{MB \mid B \in \mathcal{B}_x\}$$

for each x in Q, and each local base $\mathcal{B}_x \subset \mathcal{B}$ at x. The Souslin sets of S^*-families in $\mathcal{M}$ over Q form a set $S_Q^*(\mathcal{M})$. In the natural way the sets $S_Q^{*d}(\mathcal{M})$, $S_Q^{*s}(\mathcal{M})$, and $S_Q^{*sd}(\mathcal{M})$ are defined.

Evidently the restriction M' of an S^*-family M in $\mathcal{M}$ to a base $\mathcal{B} \subset DM$ is an S^*-family over Q and $\mathcal{S}M = \mathcal{S}M'$.

Theorem 5. Let Q be a second countable space and let $\mathcal{M}$ be a collection of sets. If $\mathcal{B}$ is any countable base for Q, then any element of $S_Q^*(\mathcal{M})$ is of the form $\mathcal{S}M$, where M is an S^*-family in $\mathcal{M}$ over Q such that $DM \subset \mathcal{B}$.

The proof follows immediately from the following

Lemma. Let $\mathcal{B}$ and $\mathcal{L}$ be two countable bases for a space Q. Let $\mathcal{B}_1$ be the set of all B which are contained in some element of $\mathcal{L}$. There exists a mapping $\varphi : \mathcal{B}_1 \longrightarrow \mathcal{L}$ such that, for each x and each local base $\mathcal{B}_x \subset \mathcal{B}_1$ at x, the set of all φB, $B \in \mathcal{B}_x$, is a local base at x.

Proof. Arrange $\mathcal{B}_1$ in one-to-one sequence $\{B_n\}$, and arrange $\mathcal{L}$ in a sequence $\{L_n\}$. For each n, let φB be a set $L \supset B_n$ in $\mathcal{L}$ with the following property: consider the set N' of all k with $L_k \supset B_n$; there exists the greatest $\ell \leqq n$ such that L is contained in the intersection of the first ℓ elements of $\{L_k \mid k \in N'\}$, we want this ℓ to be maximal for all possible L in $\mathcal{L}$ with $L \supset B_n$.

Theorem 6. If $\mathcal{M}$ is multiplicative then $\mathcal{S}(\mathcal{M}) = S^*_\Sigma(\mathcal{M})$, and similarly for $\mathcal{S}^d$, $\mathcal{S}^s$ and $\mathcal{S}^{sd}$.

Proof. The inclusion $\subset$ holds because every $\mathcal{S}M$ can be written as $\mathcal{S}M'$ with M' order-preserving, hence an S^*-family. At this point the multiplicativity is crucial. The inverse inclusion follows from the fact that if $X = \mathcal{S}M$ with M an S^*-family over a base $\mathcal{B} \subset \subset E\{\Sigma s\}$, then there exists an S^*-family M' over $\{\Sigma s\}$ with $EM = EM'$ and $\mathcal{S}M = \mathcal{S}M'$; the last statement follows from special order-properties of S.

Theorem 7. Let Q be a space, and let $\mathcal{M}$ be the collection of all closed sets in a space P. The following conditions (a) and (b) on a set $X \subset P$ are equivalent; and they are implied by condition (c). If Q is second countable then all the conditions are equivalent.

(a) $X \in S^*_Q(\mathcal{M})$;

(b) there exists a closed-graph-correspondence f of Q into P with $X = Ef$;

(c) $X \in S_Q(\mathcal{M})$.

Proof. I. Condition (c) implies condition (a) because if $X \in S^*_Q(\mathcal{M})$, then $X = \mathcal{S} M$ with M an S-family over a countable open covering $\mathcal{B}$ of Q , and given any open base $\mathcal{L}$ refining $\mathcal{B}$ we can define an S^*-family M' as follows:

$$M'L = \bigcap \{ MB \mid L \subset B \in \mathcal{B} \} .$$

Clearly

II. If Q is second-countable then any S^*_Q-set is an S_Q-set without any assumption on $\mathcal{M}$.

III. To prove that (a) and (b) are equivalent observe that the associated relation with an S^*-family is closed (thus (a) implies (b)), and if $f \subset Q \times P$ is closed and $\mathcal{B}$ is any open base for Q then $M : \mathcal{B} \to \mathcal{M}$ defined by

$$MB = \mathrm{cl}\, f[B]$$

is any S^*-family, and $f = \widetilde{M}$.

4. Remark . For the further development it is convenient to introduce the correspondence technique. Instead of collections $\mathcal{M}$ we consider paved spaces as introduced by P. Meyer [3] . A paved space is a pair $\langle P, \mathcal{M} \rangle$ where P is a set and $\mathcal{M}$ is a collection of subsets of P with $\emptyset \in \mathcal{M}$. An S-correspondence of a topological space Q into $\langle P, \mathcal{M} \rangle$ is a correspondence $f : Q \to$ $\to \langle P, \mathcal{M} \rangle$ such that the graph of f is associated with an S-family in $\mathcal{M}$ over Q . Similarlv S^*-correspondences are defined.

One can define "Souslin-product" of correspondences such that the proof of Theorem 2 is then a proof of the assertion that the Souslin product of S-correspondences is an S-correspondence. If we regard every topological space as a paved space with the pavement consisting of all closed sets, then we get that the Souslin product of upper semi-continuous compact-valued correspondences (shortly: usco-compact correspondences) is usco-compact, which gives the invariance of analytic spaces under the classical Souslin operation. One can define upper semi-continuity in general setting and get the concept of analytic set in abstract situation. The theory is developed in the paper referred to in the introduction.

R e f e r e n c e s

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