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ERROR - ESTIMATES FOR THE METHOD OF LEAST SQUARES OF FINDING EIGENVALUES AND EIGENFUNCTIONS

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In [1],[2], we considered the approximation of eigenvalues and eigenfunctions of a DS-operator. In this paper, we shall present a priori and a posteriori error-estimates for the method of least squares of finding eigenvalues and eigenfunctions. Upper and lower error bounds are found.

We assume throughout that A be a DS-operator with its domain in a real separable Hilbert space H, i.e., A is a symmetric operator in H such that the set of its eigenvalues is of the first category on the real axis and the spectrum $\sigma(A)$ is the closure of this set. Let $\{\Psi_i\}_{i=1}^{\infty}$ be a totally complete system. Suppose A is such that the eigenvalues $\{\lambda_i\}_i$ of A satisfy the relations

(I) $$0 < |\lambda_1| < |\lambda_2| \leq |\lambda_3| \leq \ldots$$

and λ_1 is simple.

Let R_m and $\mathcal{R}_m$ be subspaces of H determined by functions $\{\Psi_i\}_{i=1}^{m}$ and $\{A\Psi_i\}_{i=1}^{m}$, respectively. Let φ_1 be a normalized eigenfunction of A corresponding to the eigenvalue λ_1. We shall denote the orthogonal projection of φ_1 on R_n and $\mathfrak{R}_n$ by $\varphi_1^{(n)}$ and ${}^{(n)}\varphi_1$, respectively. By T we shall mean the restriction of A to R_n. Since $0 \notin \sigma(A)$, it follows that T and T^{-1} are continuous linear operators on R_n and $\mathfrak{R}_n$ respectively.

It has been shown in [1] that q_n is an approximation to $|\lambda_1|$, where

$$q_n = \min_{\substack{u \in R_n \\ \|u\|=1}} \|Au\| .$$

From Theorem 3 of [2] it follows that there exist $\{u_n\}_{n=1}^{\infty}$ such that the following conditions are satisfied:

$$\text{(II)} \quad \begin{aligned} &1)\ u_n \in R_n,\ \|u_n\| = 1, \\ &2)\ \|Au_n\| = q_n, \\ &3)\ \lim_{n\to\infty} u_n = \varphi_1, \\ &4)\ (u_n, \varphi_1) > 0 \quad \text{for } n = 1, 2, 3, \ldots . \end{aligned}$$

1. In this section, we shall derive upper and lower bounds for $q_n - |\lambda_1|$. Before going further we note this useful fact:

Since $\|\varphi_1\| = 1$, it follows from the definition of orthogonal projection that

$$\text{(1)} \quad \begin{aligned} \|\varphi_1 - \varphi_1^{(n)}\|^2 &= 1 - \|\varphi_1^{(n)}\|^2 , \\ \|\varphi_1 - {}^{(n)}\varphi_1\|^2 &= 1 - \|{}^{(n)}\varphi_1\|^2 . \end{aligned}$$

Now, we present a group of two results, which is useful to have on record for later use.

<u>Lemma 1.</u> With the assumption of (I), the following inequalities are valid for each positive integer m :

a) $\lambda_1^2 \cdot \| T^{-1\,(m)}\varphi_1 \|^2 \geq 2 \cdot \|^{(m)}\varphi_1 \|^2 - \|\varphi_1^{(m)}\|^2$,

b) $|\lambda_1| \, \| T^{-1\,(m)}\varphi_1 \| \geq 1 - \|\varphi_1 - {}^{(m)}\varphi_1 \|$.

<u>Proof</u>. a) It follows from the definition of $\varphi_1^{(m)}$ that

$$\|\varphi_1 - \varphi_1^{(m)}\|^2 \leq \|\varphi_1 - \lambda_1 T^{-1\,(m)}\varphi_1 \|^2 \quad .$$

We have therefore

(2) $1 - \|\varphi_1^{(m)}\|^2 \leq 1 + \lambda_1^2 \cdot \|T^{-1\,(m)}\varphi_1\|^2 - 2\lambda_1 \cdot (\varphi_1, T^{-1\,(m)}\varphi_1)$.

The proof of a) follows at once from (2), because

$$\lambda_1(\varphi_1, T^{-1\,(m)}\varphi_1) = (A\varphi_1, T^{-1\,(m)}\varphi_1) = (\varphi_1, {}^{(m)}\varphi_1) = \|^{(m)}\varphi_1\|^2 .$$

b) By Theorem 2 of [1] we have

$$\|Au\| > |\lambda_1| \cdot \|u\| \quad \text{for any } u \in \mathfrak{D}(A) .$$

Letting $u = \varphi_1 - \lambda_1 T^{-1\,(m)}\varphi_1$ we see that

$$|\lambda_1| \cdot \|\varphi_1 - {}^{(m)}\varphi_1\| = \|Au\| \quad ,$$

whence follows

(3) $$\|\varphi_1 - {}^{(m)}\varphi_1\| \geq \|\varphi_1 - \lambda_1 T^{-1\,(m)}\varphi_1\| \quad .$$

It follows from $\|\varphi_1\| \leq \|u\| + |\lambda_1| \cdot \|T^{-1\,(m)}\varphi_1\|$ that

(4) $$|\lambda_1| \cdot \|T^{-1\,(m)}\varphi_1\| \geq 1 - \|\varphi_1 - \lambda_1 T^{-1\,(m)}\varphi_1\| \quad .$$

Now, if we insert (3) in (4), we obtain the stament b).

Corollary 1. For any n, we have $\| {}^{(n)}\varphi_1 \|^2 \leq \| \varphi_1^{(n)} \|^2$.
Hence $\| \varphi_1 - {}^{(n)}\varphi_1 \|^2 \geq \| \varphi_1 - \varphi_1^{(n)} \|^2$.

Proof: By the definition ϱ_n, we have $\varrho_n \geq |\lambda_1| > 0$ and

$$(5) \qquad \| T^{-1}\, {}^{(n)}\varphi_1 \| \leq \frac{1}{\varrho_n} \cdot \| {}^{(n)}\varphi_1 \| \quad .$$

The corollary follows easily from (5) and Lemma 1.

Remark 1. From the totally completeness of $\{\Psi_i\}_{i=1}^{\infty}$ and the assumption $0 \in \sigma(A)$ it follows that

$$\lim_{n \to \infty} \| \varphi_1^{(n)} \| = \lim_{n \to \infty} \| {}^{(n)}\varphi_1 \| = 1$$

and therefore

$$\lim_{n \to \infty} {}^{(n)}\varphi_1 = \lim_{n \to \infty} \varphi_1^{(n)} = \varphi_1 \quad .$$

Consequently, from Lemma 1 it follows $\lim_{n \to \infty} |\lambda_1| \cdot \| T^{-1}\, {}^{(n)}\varphi_1 \| = 1$.

Remark 2. There exists some $n_0 \geq 0$ such that

$$2 \cdot \| {}^{(n)}\varphi_1 \|^2 - \| \varphi_1^{(n)} \|^2 \geq (1 - \| \varphi_1 - {}^{(n)}\varphi_1 \|)^2 \quad \text{for } n \geq n_0 \, .$$

Proof. From Remark 1 it follows that there exists n_0 such that $\| \varphi_1 - {}^{(n)}\varphi_1 \|^2 \leq \frac{2}{3} \cdot \| \varphi_1 - {}^{(n)}\varphi_1 \|$ for $n \geq n_0$. It follows that

$$\| {}^{(n)}\varphi_1 \|^2 \geq 1 - \frac{2}{3} \cdot \| \varphi_1 - {}^{(n)}\varphi_1 \| \qquad \text{for } n \geq n_0 \, .$$

When this is substituted in

$$2 \cdot \| {}^{(n)}\varphi_1 \|^2 - \| \varphi_1^{(n)} \|^2 \geq 2 \cdot \| {}^{(n)}\varphi_1 \|^2 - 1 = 3 \cdot \| {}^{(n)}\varphi_1 \|^2 + \| \varphi_1 - {}^{(n)}\varphi_1 \|^2 - 2 \, ,$$

we obtain the statement.

An important tool in the proof of the next theorem is furnished by the following lemma.

Lemma 2. If we denote the product (u_n, φ_1) by $\alpha_1^{(n)}$, then under the assumption (I) we have

$$(\alpha_1^{(n)})^2 \geq 1 - \frac{q_n^2 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} \qquad \text{for any } n .$$

Proof. By Lemma 1 of [1], we have

$$q_n^2 - \lambda_1^2 = \sum_{i=1}^{\infty} (\lambda_i^2 - \lambda_1^2) \| u_i^{(n)} \|^2 , \tag{6}$$

where $u_i^{(n)}$ is the orthogonal projection of u_n on H_i and H_i is the closure of linear manifold generated by the eigenfunctions of A associated with the eigenvalue λ_i. Since $|\lambda_2| > |\lambda_1|$ and $\|u_n\| = 1$, it follows from (6) that

$$q_n^2 - \lambda_1^2 \geq (\lambda_2^2 - \lambda_1^2)(\|u_n\|^2 - \|u_1^{(n)}\|^2) ,$$

so that
$$\| u_1^{(n)} \|^2 \geq 1 - \frac{q_n^2 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2} .$$

Now $u_1^{(n)} = (u_n, \varphi_1) \cdot \varphi_1$ and thus the proof is complete.

The following theorem is of fundamental importance.

Theorem 1. Let A be a DS-operator and $\{\psi_i\}_{i=1}^{\infty}$ a totally complete system. Suppose the eigenvalues $\{\lambda_i\}_i$ of A satisfy the relations $0 < |\lambda_1| < |\lambda_2| \leq |\lambda_3| \leq \dots$ and λ_1 is simple. Construct the sequence of numbers $\{q_n\}_{n=1}^{\infty}$ such that

$$q_n = \min_{\substack{u \in R_n \\ \|u\| = 1}} \| A u \|$$

where $R_n = \mathcal{L}\{\psi_i\}_{i=1}^{n}$.

Let ${}^{(n)}\varphi_1$ be the orthogonal projection of a normalized eigenfunction φ_1 corresponding to λ_1 on $\mathcal{R}_n =$

$= \mathcal{L}\{A\psi_i\}_{i=1}^{n}$ and n_o be a positive integer such that ${}^{(n_o)}\varphi_1 \neq 0$ and ${}^{(n_o-1)}\varphi_1 = 0$. Then there exist constants C_1 and $C_2 \neq 0$ which do not depend on n such that

(7) $$C_2 \cdot \|\varphi_1 - {}^{(n)}\varphi_1\|^2 \leq q_m - |\lambda_1| \leq C_1 \|\varphi_1 - {}^{(n)}\varphi_1\|^2$$

for $n \geq n_o$.

<u>Proof</u>. Suppose $n \geq n_o$. Then $\|{}^{(n)}\varphi_1\| \neq 0$. By the definition of q_n

(8) $$q_n - |\lambda_1| \leq C \cdot (\|{}^{(n)}\varphi_1\|^2 - \lambda_1^2 \|T^{-1}\, {}^{(n)}\varphi_1\|^2) ,$$

where

$$C = \|T^{-1}\, {}^{(n)}\varphi_1\|^{-1} \cdot (\|{}^{(n)}\varphi_1\| + |\lambda_1| \cdot \|T^{-1}\, {}^{(n)}\varphi_1\|)^{-1} .$$

From Lemma 1 and (8) it follows that

(9) $$q_n - |\lambda_1| \leq C \cdot (-\|{}^{(n)}\varphi_1\|^2 + \|\varphi_1^{(n)}\|^2) \qquad \text{for } n \geq n_o .$$

Since

$$\frac{\|{}^{(n)}\varphi_1\|}{\|T^{-1}\, {}^{(n)}\varphi_1\|} \geq |\lambda_1| ,$$

we have

$$C \leq \frac{1}{2\,|\lambda_1| \cdot \|T^{-1}\, {}^{(n)}\varphi_1\|^2}$$

From this and Lemma 1 we obtain

$$C \leq \frac{|\lambda_1|}{2(1 - \|\varphi_1 - {}^{(n)}\varphi_1\|)^2} \qquad \text{for } n \geq n_o .$$

Letting

$$C_1 = \frac{1}{2}\,|\lambda_1| \cdot (1 - \|\varphi_1 - {}^{(n)}\varphi_1\|)^{-2} ,$$

from(9) and (1) it follows

$$q_n - |\lambda_1| \le C_1 \cdot \|\varphi_1 - {}^{(n)}\varphi_1\|^2 \quad \text{for } n \ge n_0 .$$

To prove the second part of (7) we construct u_n such that the conditions (I) are satisfied. Then

$$q_n^2 - \lambda_1^2 = \|Au_n - A\varphi_1\|^2 + 2\lambda_1^2 (u_n - \varphi_1, \varphi_1) \ge$$
$$\ge \lambda_1^2 \|\varphi_1 - {}^{(n)}\varphi_1\|^2 + 2\lambda_1^2 (\alpha_1^{(n)2} - 1) ,$$

where $\alpha_1^{(n)} = (u_n, \varphi_1)$.

Using Lemma 2, we have

$$q_n^2 - \lambda_1^2 \ge \lambda_1^2 \cdot \|\varphi_1 - {}^{(n)}\varphi_1\|^2 - 2\lambda_1^2 \cdot (q_n^2 - \lambda_1^2) \cdot (\lambda_2^2 - \lambda_1^2)^{-1} ,$$

whence with the notation

$$x = q_n - |\lambda_1|, \; a = (\lambda_2^2 - \lambda_1^2)(\lambda_2^2 + \lambda_1^2)^{-1} \cdot \lambda_1^2 \cdot \|\varphi_1 - {}^{(n)}\varphi_1\|^2, \; b = 2|\lambda_1|$$

one finds

(10) $$x(x + b) \ge a .$$

After some computation we find that the solution of (10) satisfies the inequality $x = q_n - |\lambda_1| \ge$ $\ge C_2 \cdot \|\varphi_1 - {}^{(n)}\varphi_1\|^2$, where

$$C_2 = 2|\lambda_1| \cdot (\lambda_2^2 - \lambda_1^2) \cdot (5\lambda_2^2 + 3\lambda_1^2)^{-1} .$$

Thus the proof is complete.

Remark 3. Theorem 1 is valid in the case when λ_1 is a multiple eigenvalue of A .

Remark 4. From the proof of Theorem 1 it follows that the right hand side of the inequality (7) is valid for any DS-operator such that $0 \notin \sigma(A)$.

2. Bearing in mind the considerations of the previous section, we now find a priori bounds for the approximations u_n to an eigenfunction φ_1. To establish these bounds we require the following Lemma 3.

Lemma 3. Under the hypotheses as in Theorem 1, we have for $n \geq n_0$

(a) $\|Au_n - A\varphi_1\|^2 \leq (\varrho_n^2 - \lambda_1^2)\cdot(\lambda_2^2 + \lambda_1^2)\cdot(\lambda_2^2 - \lambda_1^2)^{-1}$,

(b) $\|u_n - \varphi_1\|^2 \leq 2(\varrho_n^2 - \lambda_1^2)(\lambda_2^2 - \lambda_1^2)^{-1}$,

(c) $\varrho_n^2 - \lambda_1^2 \leq D\cdot\|\varphi_1 - {}^{(n)}\varphi_1\|^2$,

where $D = \lambda_1^2\cdot(1 - \|\varphi_1 - {}^{(n)}\varphi_1\|)^{-2}$.

Proof. In a similar way, by methods analogous to those employed in the proof of Theorem 1, we can obtain

(11) $\varrho_n^2 - \lambda_1^2 \leq \|T^{-1}\,{}^{(n)}\varphi_1\|^{-2}\cdot(\|\varphi_1^{(n)}\|^2 - \|{}^{(n)}\varphi_1\|^2)$.

From Lemma 1 and (1) it follows the inequality (c).

To prove (a) we write

$$\|Au_n - A\varphi_1\|^2 = \|Au_n\|^2 + \lambda_1^2 - 2(Au_n, A\varphi_1) = \\ = \varrho_n^2 - \lambda_1^2 + 2\lambda_1^2(1 - \alpha_1^{(n)}).,$$

where $\alpha_1^{(n)} = (u_n, \varphi_1)$.

Since $\alpha_1^{(n)} \in \langle 0, 1\rangle$, we see that

$$\|Au_n - A\varphi_1\|^2 \leq \varrho_n^2 - \lambda_1^2 + 2\lambda_1^2\cdot(1 - (\alpha_1^{(n)})^2) ,$$

and the inequality (a) follows from Lemma 2.

The proof of (b) follows at once from Lemma 2, because $(\alpha_1^{(n)})^2 \leq \alpha_1^{(n)}$, $\|u_n\| = 1$ and $\|\varphi_1\| = 1$.

The following theorem is a consequence of Lemma 3.

Theorem 2. Under the hypotheses as in Theorem 1 there exist the constants C_2 and C_3 which do not depend on n such that for $n \geq n_0$

$$|\lambda_1| \cdot \|\varphi_1 - {}^{(n)}\varphi_1\| \leq \|Au_n - A\varphi_1\| \leq C_2 \cdot \|\varphi_1 - {}^{(n)}\varphi_1\|$$

$$\|\varphi_1 - \varphi_1^{(n)}\| \leq \|u_n - \varphi_1\| \leq C_3 \cdot \|\varphi_1 - {}^{(n)}\varphi_1\| ,$$

where $\varphi_1^{(n)}$ is the orthogonal projection of φ_1 on $R_n = \mathcal{L}\{\Psi_i\}_{i=1}^{n}$.

Proof. The right sides of these inequalities follow at once from Lemma 3. Since $\lambda_1 \neq 0$ from the definition of orthogonal projection it follows

$$\|Au_n - A\varphi_1\|^2 = \lambda_1^2 \left\| \frac{Au_n}{\lambda_1} - \varphi_1 \right\|^2 \geq \lambda_1^2 \cdot \|\varphi_1 - {}^{(n)}\varphi_1\|^2$$

$$\|u_n - \varphi_1\| \geq \|\varphi_1 - \varphi_1^{(n)}\| .$$

Thus all is proved.

Remark 5. Theorem 2 is valid in the case when λ_1 is a multiple eigenvalue of A .

3. In this section, we find a posteriori bounds for the errors in the approximations q_n and u_n to the eigenvalue λ_1 and the eigenfunction φ_1 , respectively.

Under the hypotheses as in Theorem 1, we construct the sequence $\{u_n\}_{n=1}^{\infty}$ such that the condition (I) is satisfied. To simplify our notation in this section let $\delta_n = \|Au_n - e q_n u_n\|$, where $e = \operatorname{sign} \lambda_1$. Our next principal result is Theorem 3. An important tool in the proof of this theorem is furnished by the following lemma.

Lemma 4. Suppose n is such that $\alpha_1^{(n)} > 0$ and $|\lambda_2| > q_n$. Then

(a) $q_n - |\lambda_1| \le D_1 \sigma_n^2$,

where $D_1 = \dfrac{4q_n^2 + \lambda_2^2 - \lambda_1^2}{q_n \cdot (\lambda_2^2 - q_n^2)}$,

(b) $q_n - |\lambda_1| \ge D_2 \sigma_n^2$,

where $D_2 = \frac{1}{2}(\lambda_2^2 - \lambda_1^2) \cdot q_n^{-3} \cdot (\sqrt{2} + \sqrt{(\frac{\lambda_2}{\lambda_1})^2 + 1})^2$,

(c) $\| Au_n - A\varphi_1 \| \le D_3 \sigma_n$,

where $D_3 = 5\lambda_2^2 \cdot [(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - q_n^2)]^{-\frac{1}{2}}$,

(d) $\| Au_n - A\varphi_1 \| \ge D_4 \sigma_n$,

where $D_4 = (\lambda_2^2 - \lambda_1^2)^{\frac{1}{2}} \cdot [\sqrt{2}|\lambda_1| + \sqrt{\lambda_2^2 + \lambda_1^2}]^{-1} \cdot (\frac{|\lambda_1|}{q_n})^{\frac{1}{2}}$,

(e) $\| u_n - \varphi_1 \| \le D_5 \sigma_n$,

where $D_5 = 5 \cdot |\lambda_2| \cdot [(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - q_n^2)]^{-\frac{1}{2}}$.

Proof. Since $\lambda_1 \cdot e = |\lambda_1|$ and $\| Au_n \| = q_n$, we have

(12) $\| Au_n - e q_n \varphi_1 \|^2 = 2q_n (q_n - |\lambda_1| \cdot \alpha_1^{(n)})$,

where $\alpha_1^{(n)} = (u_n, \varphi_1)$.

If we subtract the following identity

$$\|\varphi_1 - u_n\|^2 \cdot q_n^2 = 2 q_n^2 \, (1 - \alpha_1^{(n)})$$

from (12), we obtain

$$(13) \quad 2\alpha_1^{(n)} q_n (q_n - |\lambda_1|) = y(y + 2q_n \cdot \|\varphi_1 - u_n\|) ,$$

where $y = \|Au_n - e q_n \varphi_1\| - q_n \cdot \|\varphi_1 - u_n\|$.

Since $q_n \geq |\lambda_1|$, it follows that $y \geq 0$ and

$$\delta_n = \|Au_n - e q_n \varphi_1 + e q_n \varphi_1 - e q_n u_n\| \geq y .$$

Hence we have from (13)

$$(14) \quad 2\alpha_1^{(n)}(q_n - |\lambda_1|) \leq \delta_n \cdot (\delta_n + 2q_n \cdot \|\varphi_1 - u_n\|) .$$

It follows immediately from Lemma 3

$$\|u_n - \varphi_1\| \leq 2 \cdot \sqrt{\frac{q_n}{\lambda_2^2 - \lambda_1^2}} \cdot \sqrt{q_n - |\lambda_1|} .$$

Using this in (14), we obtain

$$ax^2 \leq c + bx ,$$

where

$$x = \sqrt{q_n - |\lambda_1|}, \; a = 2q_n\alpha_1^{(n)}, \; b = 4q_n\delta_n \cdot \sqrt{\frac{q_n}{\lambda_2^2 - \lambda_1^2}} .$$

After some computation we may find that $x \geq D_1 \cdot \delta_n^2$. This proves (a).

To prove (b) observe that

$$(15) \quad \delta_n \leq \|Au_n - e \cdot q_n \cdot \varphi_1\| + q_n \cdot \|\varphi_1 - u_n\| .$$

By the definition of n_0 in Theorem 2, we have that $n \geq n_0$. Since $q_n \geq |\lambda_1|$, it now follows from Lemma 3 that

$$\|\varphi_1 - u_n\| \leq \frac{2 \cdot \sqrt{q_n}}{\sqrt{\lambda_2^2 - \lambda_1^2}} \cdot \sqrt{q_n - |\lambda_1|} .$$

Assume that $q_n > |\lambda_1|$. Then, by (15) and (12)

(16) $$\delta_n \leq C \cdot \sqrt{q_n - |\lambda_1|} \quad ,$$

where

$$C = 2 \cdot \sqrt{\frac{q_n^3}{\lambda_2^2 - \lambda_1^2}} + \sqrt{2q_n} \cdot \sqrt{1 + \frac{|\lambda_1|(1-\alpha_1^{(n)})}{q_n - |\lambda_1|}} \ .$$

Since $\alpha_1^{(n)} > 0$, we see from Lemma 2 that

(17) $$1 + \frac{|\lambda_1|(1-\alpha_1^{(n)})}{q_n - |\lambda_1|} \leq 1 + \frac{|\lambda_1|(q_n + |\lambda_1|)}{\lambda_2^2 - \lambda_1^2} = \frac{\lambda_2^2 + |\lambda_1| \cdot q_n}{\lambda_2^2 - \lambda_1^2} .$$

The inequality (b) now follows from (16) and (17) in the case of $q_n > |\lambda_1|$. It is readily verified that (b) is also valid in the case of $q_n = |\lambda_1|$.

The proof of (c) and (e) follows at once from (a) and Lemma 3 because $q_n + |\lambda_1| \leq 2q_n$

It is readily verified that

$$\| A u_n - A\varphi_1 \|^2 \geq q_n^2 - \lambda_1^2 \geq 2 \cdot |\lambda_1| \cdot (q_n - |\lambda_1|)$$

and from (b) it follows the inequality (d). This completes the proof.

From Lemma 4 (c) and from $\delta_n \xrightarrow[n \to \infty]{} 0$ it follows $\lim\limits_{n \to \infty} A u_n = A\varphi_1$. Consequently, there exists n_1 such that for $n \geq n_1$

(18) $$\operatorname{sign} (A u_n , u_n) = \operatorname{sign} \lambda_1 = e \ .$$

Therefore

(19) $$\| A u_n - e q_n u_n \|^2 = 2q_n \cdot (q_n - |(A u_n, u_n)|) \text{ for } n \geq n_1 \ .$$

From Lemma 4 and (19) we deduce the following

Theorem 3. Under the hypotheses as in Theorem 1 there exist the constants K_1, K_2, K_3, K_4, K_5 which do not depend on n and an integer n_1 such that for $n \geq n_1$

$$K_2 \cdot \varepsilon_n^2 \leq q_n - |\lambda_1| \leq K_1 \cdot \varepsilon_n^2 \quad ,$$

$$K_4 \cdot \varepsilon_n \leq \| A u_n - A\varphi_1 \| \leq K_3 \cdot \varepsilon_n \quad ,$$

$$\| u_n - \varphi_1 \| \leq K_5 \cdot \varepsilon_n \quad ,$$

where $\varepsilon_n = q_n - |(Au_n, u_n)|$.

Remark 5. From (18) it follows that

$$\lim_{n\to\infty} [q_n \cdot \operatorname{sign}(Au_n, u_n)] = \lambda_1 .$$

4. In all previous sections we have been concerned with setting up error bounds of approximations for λ_1 and φ_1 .

In order to obtain error bounds for λ_i , $i > 1$, we shall assume that

(III) λ_i is not an accumulation point of the spectrum $\sigma(A)$.

For the sake of simplicity, we shall suppose that

(IV) λ_i is simple and $0 \notin \sigma(A)$.

Select μ in such a way that

(V)
1) $\mu \notin \sigma(A)$,

2) $|\mu - \lambda_i| < |\mu - t|$ for any $t \in \sigma(A)$, $t \neq \lambda_i$.

From Theorem 3 of [1] it follows that $\lim_{n\to\infty} q_n =$

$= |\mu - \lambda_i|$, where $q_m = \min_{\substack{u \in R_m \\ \|u\|=1}} \|Au - \mu u\|$.

Then $\mu + q_m$ or $\mu - q_m$ is the approximation to λ_i. Denote this approximation by $\lambda_i^{(m)}$. Let φ_i be a normalized eigenfunction corresponding to λ_i, and $\varphi_i^{(m)}$ and ${}^{(m)}\varphi_i$ orthogonal projections of φ_i on $R_m = \mathcal{L}\{\psi_j\}_{j=1}^{m}$ and $\mathcal{R}_m = \mathcal{L}\{A\psi_j\}_{j=1}^{m}$, respectively.

If we apply the above results with $(A - \mu I)$ in place of A, then we obtain error bounds of approximations for λ_i and φ_i. As an immediate consequence of Theorems 1,2,3 and the following Lemma 5, we have

Theorem 4. Under the assumptions (III) - (V) we construct $\{u_m\}_{m=1}^{\infty}$ such that the following conditions are satisfied:

1) $u_m \in R_m$, $\|u_m\| = 1$,

2) $q_m = \|Au_m - \mu u_m\|$,

3) $(u_m, u_{m+1}) \geq 0$.

Then there exist an integer m_1 and the constants C_1, C_2, C_3, C_4, C_5, K_1, K_2, K_3, K_4 which do not depend on m such that for $m \geq m_1$

(a) $C_2 \delta_m^2 \leq |\lambda_i - \lambda_i^{(m)}| \leq C_1 \delta_m^2$,

$\kappa_m \leq \|u_m - \varphi_i\| \leq C_3 \delta_m$,

$C_5 \kappa_m \leq \|Au_m - A\varphi_i\| \leq C_4 \delta_m$,

where $\delta_m = \|\varphi_i - {}^{(m)}\varphi_i\|$ and $\kappa_m = \|\varphi_i - \varphi_i^{(m)}\|$.

(b) $K_2 \varepsilon_m^2 \le |\lambda_i - \lambda_i^{(m)}| \le K_1 \varepsilon_m^2$,

$$\|u_m - \varphi_i\| \le K_3 \varepsilon_m ,$$

$$\|Au_m - A\varphi_i\| \le K_4 \varepsilon_m ,$$

where $\varepsilon_m = q_m - |(Au_m, u_m) - \mu|$

and $\lambda_i^{(m)} = \mu + q_m \cdot \operatorname{sign}[(Au_m, u_m) - \mu]$.

Lemma 5. Let ${}^{(m)}\varphi_i^{(\mu)}$ be the orthogonal projection of φ_i on $\mathcal{R}_m^{(\mu)} = \mathcal{L}\{(A-\mu I)\Psi_j\}_{j=1}^m$. Under the assumptions (III) - (V) we have

$$D_1 \cdot \|\varphi_i - {}^{(m)}\varphi_i\| \le \|\varphi_i - {}^{(m)}\varphi_i^{(\mu)}\| \le D_2 \cdot \|\varphi_i - {}^{(m)}\varphi_i\| ,$$

where

$$D_1 = \left|\frac{\lambda_i}{\lambda_i - \mu}\right| \cdot \inf_{t \in \sigma(A)} \left|1 - \frac{\mu}{t}\right| ,$$

$$D_2 = \left|\frac{\lambda_i}{\lambda_i - \mu}\right| \cdot \sup_{t \in \sigma(A)} \left|1 - \frac{\mu}{t}\right| .$$

Proof. It follows by the definition of ${}^{(m)}\varphi_i^{(\mu)}$ that

(20) $\|\varphi_i - {}^{(m)}\varphi_i^{(\mu)}\| = \min_{u \in R_m} \|\varphi_i - (A - \mu I)u\|$.

Since $0 \notin \sigma(A)$ and $\mu \notin \sigma(A)$, there exist A^{-1} and $(A - \mu I)^{-1}$. Then

(21) $\|\varphi_i - (A-\mu I)u\| = \|B[(A-\mu I)^{-1}\varphi_i - Au]\|$, $u \in R_m$,

where $B = (A - \mu I)A^{-1}$ and I is the identity operator. Letting $u = A^{-1}\, {}^{(m)}\varphi_i \cdot \dfrac{\lambda_i}{\lambda_i - \mu}$, it follows from (20) and (21) that

(22) $\| \varphi_i - {}^{(n)}\varphi_i^{(\mu)} \| \le \left| \frac{\lambda_i}{\lambda_i - \mu} \right| \cdot \| B(\varphi_i - {}^{(n)}\varphi_i) \|$.

Then, since A is a DS-operator, we have

(23) $$\| B \| \le \sup_{t \in \sigma(A)} \left| 1 - \frac{\mu}{t} \right| .$$

(24) $\| B v \| \ge \| v \| \cdot \inf_{t \in \sigma(A)} \left| 1 - \frac{\mu}{t} \right|$ for any $v \in \mathcal{R}(A)$.

Thus, by (23) and (22)

$$\| \varphi_i - {}^{(n)}\varphi_i^{(\mu)} \| \le D_2 \cdot \| \varphi_i - {}^{(n)}\varphi_i \| .$$

It is readily verified that

(25) $\| \varphi_i - {}^{(n)}\varphi_i^{(\mu)} \| = \min_{u \in R_n} \| B [\varphi_i - Au] \| \cdot \left| \frac{\lambda_i}{\lambda_i - \mu} \right|$.

It follows now from (24) and (25) that

$$\| \varphi_i - {}^{(n)}\varphi_i^{(\mu)} \| \ge D_1 \cdot \| \varphi_i - {}^{(n)}\varphi_i \| .$$

<u>Remark 6.</u> In the case of multiple eigenvalue Theorem 4 is valid , if u_n satisfies 3*) $(u_n, u_{n+1}) \ge \ge \varepsilon > 0$ in place of 3).

R e f e r e n c e s

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