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A METATHEOREM FOR FIXED POINT THEORIES

Richard B. THOMPSON, Tucson [1]

A topological space X is said to have the fixed point property if each map $f: X \longrightarrow X$ leaves some point invariant. We will denote by $\mathcal{F}$ the class of all compact Hausdorff spaces which have the fixed point property. A space is described as satisfying the Lefschetz fixed point theorem if every fixed point free self-map has a Lefschetz number of zero.

Definition. A fixed point theory is a class $\mathcal{C}$ of compact Hausdorff spaces, each of which satisfies the Lefschetz fixed point theorem. $\mathcal{C}$ is called complete if $\mathcal{F} \subseteq \mathcal{C}$.

It is easily seen that $\mathcal{F}$ itself is an example of a complete fixed point theory.

We refer to a theory $\mathcal{C}$ as closed under products if the Cartesian product of any pair of spaces in $\mathcal{C}$ is also in $\mathcal{C}$ and as closed under cones if the cone on each member of $\mathcal{C}$ is again in $\mathcal{C}$. Of the fixed point theories that have been studied in [1],[2],[3],[5],[6] and [7], all but the retracts of convexoids in [2] are known to be closed under products, while the weak semicomplexes of [6] and [7] are also

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closed under cones.

Theorem. A fixed point theory cannot be both complete and closed under products or cones.

Proof. Knill gives an example in [4,sec.3] of a contractible subspace B of $\mathbb{R}^3$ such that B is in $\mathcal{F}$ but neither the cone $C(B)$ nor $B \times [0,1]$ is in $\mathcal{F}$. If $\mathcal{C}$ is a complete theory, then B and $[0,1]$ are in $\mathcal{C}$. However, $C(B)$ and $B \times [0,1]$ are both acyclic and hence must be in $\mathcal{F}$ if they are in $\mathcal{C}$. Since these spaces are not in $\mathcal{F}$, $\mathcal{C}$ is not closed under either products or cones.

The net effect of this theorem is that there can exist no one ideal setting for the application of Lefschetz-type algebraic methods to fixed point questions. This situation is somewhat analogous to the well-known use of the existence of a non-measurable set to show that there can be no countable, additive, translation invariant measure defined on all subsets of the real numbers which agrees with length on intervals.

The fixed point theories described in [1],[2], and [3] are all incomplete, since they are limited to locally connected situations and hence exclude such spaces as the pseudo-arc. Additionally, applying the remark just prior to the theorem, we obtain the following result about the theories of Lefschetz [5] and the author [6 and 7].

Corollary. The class of weak semicomplexes and their subclass, the quasi-complexes, both constitute incomplete fixed point theories.

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