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Commentationes Mathematicae Universitatis Carolinae, Vol. 12 (1971), No. 1, 23--31

Persistent URL: http://dml.cz/dmlcz/105324

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12,1 (1971)

CONCERNING ALMOST DIVISIBLE TORSION FREE ABELIAN GROUPS

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A torsion free group G (all groups here are supposed to be abelian) will be called almost divisible if the set of all positive primes p with $pG \neq G$ is finite. In this note we shall give some conditions that are necessary and sufficient for an almost divisible group G to be completely decomposable. In the paper [2] of D.K. Harrison (see Proposition 5.2) such necessary and sufficient conditions are formulated for the groups of finite rank. But it was shown later (see [3]) that these conditions are not sufficient in general. However, the remark following Theorem 2 shows that the Harrison's conditions are sufficient whenever the corresponding type set is linearly ordered.

If G is a torsion free group, then $\mathcal{T}(G)$ will denote the type set of all non zero elements in G ; G is said to be homogeneous of the type κ if $\mathcal{T}(G)$ consists of one element κ only. For a type κ and a prime p the relation $\kappa(p) = \infty$ means that in any height belonging to κ the p-height is ∞ ; the symbols $G(\kappa)$, $G^*(\kappa)$ and $G^{**}(\kappa)$ represent the subgroups of G defined in

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AMS, Primary 20K20 Ref.Ž. 2.722.1

[1,§42]. The rank of a group G is denoted by $\kappa(G)$ and $\kappa_p(G)$ stands for its p-rank (see [5]). But in this note we shall use the relation $\kappa_p(G) = 0$ only; this last relation says that for any finite set $M \subseteq G$ the p-primary component of the torsion group $\{M\}_*/\{M\}$ is reduced ($\{M\}_*$ denotes here the least pure subgroup of G containing M).

First of all we shall prove the following helpful assertion.

<u>Lemma.</u> Let G be an almost divisible torsion free group and let, for a type $u \in \mathcal{T}(G)$, the following conditions be fulfilled:

(a) The group $G(u) / G^*(u)$ is torsion free and belongs to some Baer's class Γ_α ;

(b) for any prime p the inequality $u(p) \neq \infty$ implies

$$\kappa_p(G(u)/G^*(u)) = 0 \quad .$$

Then the group $G^*(u)$ is a direct summand of $G(u)$, $G(u) = G_u \dotplus G^*(u)$, where G_u is completely decomposable and homogeneous of the type u , or $G_u = 0$.

<u>Proof</u>. If $G^*(u) = G(u)$, then $G_u = 0$, therefore we may suppose that $G^*(u) \neq G(u)$. The group $G(u)$ as a pure subgroup of G is likewise almost divisible and so is the factor group $\overline{G} = G(u) / G^*(u)$ as well. In view of (a), the group $\overline{G}$ is torsion free and the type of any of its non zero elements is $\geqq u$. Thus, if p is a prime with $u(p) = \infty$, then $p\overline{G} = \overline{G}$. But if $u(p) \neq \infty$, then by (b) $\kappa_p(\overline{G}) = 0$.
In the last case, each pure rank one subgroup of $\overline{G}$ is

of zero p-rank (see [4,Corollary 2]), therefore each non zero element of $\overline{G}$ has a finite p-height in $\overline{G}$ (see [6,Lemma 6.1]). Now we deduce from the finiteness of the set of all primes p with $\tau(p) \neq \infty$ that $\overline{G}$ is homogeneous of the type τ. Thus the inequality $p\overline{G} \neq \overline{G}$ implies $\tau(p) \neq \infty$ and therefore $\kappa_p(\overline{G}) = 0$. By (a), $\overline{G}$ belongs to some Baer's class Γ_α and in view of [4,Corollary 4] $\overline{G}$ is completely decomposable. Evidently, τ is the type of any element $g \in G(\tau) \dot{-} G^*(\tau)$, hence, according to the Baer's lemma [1,the note following Theorem 46.5] $G^*(\tau)$ is a direct summand in $G(\tau)$. Thus we have $G(\tau) = G_\tau \dot{+} G^*(\tau)$, $G_\tau \cong G(\tau)/G^*(\tau) = \overline{G}$, therefore G_τ is completely decomposable and homogeneous of the type τ.

Now we are in a position to prove a theorem concerning almost divisible groups with the linearly ordered type set (in natural order of the types).

Theorem 1. Let G be an almost divisible torsion free group with the linearly ordered type set $\mathcal{T}(G)$. Then G is completely decomposable if and only if for any $\tau \in \mathcal{T}(G)$ the condition (a) together with the condition

(b*) $\kappa_p(G/G^*(\tau)) = 0$ whenever $\tau(p) \neq \infty$

are fulfilled.

Proof. If G is completely decomposable and $G = \sum_{i \in I} J_i$ is a complete decomposition of G, then $\mathcal{T}(G)$ coincides also with the set of the types of all rank one groups J_i $(i \in I)$. Thus for any $\tau \in \mathcal{T}(G)$ the torsion free group $G(\tau)/G^*(\tau)$ is completely decomposable and homogeneous of the type τ; evidently, $G(\tau)/G^*(\tau) \in$

$\in \Gamma_\alpha$ $(1 \leq \alpha \leq 2)$. The group $G / G^*(\tau)$ is completely decomposable as well and the types of its direct summands are $\leq \tau$. Hence, if $\tau(p) \neq \infty$, then $G/G^*(\tau)$ is p-reduced and in view of [4, Corollary 1] we have $\kappa_p(G/G^*(\tau)) = 0$. Thus in this case the conditions (a), (b^*) are fulfilled.

Now, let us suppose that G satisfies (a) and (b^*); we shall show that G is completely decomposable. From the hypothesis it follows immediately that $\mathcal{T}(G)$ is finite. Let us put $\mathcal{T}(G) = \{\tau_1 < \dots < \tau_m\}$. Then we shall prove the complete decomposability of G by induction on $m = \operatorname{card} \mathcal{T}(G)$.

For $m = 1$ the group G is homogeneous of the type τ_1 and $G^*(\tau_1) = 0$. Then the inequality $pG \neq G$ for a prime p implies $\tau_1(p) \neq \infty$ and in view of (b^*) we have $0 = \kappa_p(G/G^*(\tau_1)) = \kappa_p(G)$. Hence, by [4, Corollary 4], G is completely decomposable.

Thus, suppose $m \geq 2$ and let our assertion hold whenever the cardinality of the corresponding type set is $m-1$. Since $G(\tau_1) = G$, we can apply our Lemma to G for $\tau = \tau_1$ and we get

(1) $$G = H \dotplus G^*(\tau_1),$$

where the group H is completely decomposable. If we put $G^*(\tau_1) = G_1 = G(\tau_2)$, then by (1) G_1 is also almost divisible and $\mathcal{T}(G_1) = \{\tau_2 < \dots < \tau_m\}$. We shall now verify that G_1 fulfils (a) and (b^*) for all types of $\mathcal{T}(G_1)$. In fact, if $\tau \in \mathcal{T}(G_1)$, then $\tau_1 < \tau_2 \leq \tau$ and hence $G(\tau) \subseteq G(\tau_2) = G_1$, which implies

$G_1(\varkappa) = G(\varkappa)$; analogously, we obtain $G^*(\varkappa) \subseteq \subseteq G^*(\varkappa_1) = G_1$, therefore $G^*(\varkappa) = G_1^*(\varkappa)$. Thus we have $G_1(\varkappa) / G_1^*(\varkappa) = G(\varkappa) / G^*(\varkappa)$, which means that G_1 fulfils (a) for each $\varkappa \in \mathcal{T}(G_1)$. By (1), we can write for any $\varkappa \in \mathcal{T}(G_1)$

$$(2) \quad G / G^*(\varkappa) = (H \dotplus G_1) / G^*(\varkappa) \cong H \dotplus G_1 / G^*(\varkappa) = \\ = H \dotplus G_1 / G_1^*(\varkappa) ;$$

thus for $\varkappa(p) \neq \infty$ it is $\kappa_p(G / G^*(\varkappa)) = 0$ and hence by (2)

$$\kappa_p(H \dotplus G_1 / G_1^*(\varkappa)) = 0 .$$

Following [4,Corollary 2], we get $\kappa_p(G_1 / G_1^*(\varkappa)) = 0$, therefore the condition (b*) is satisfied by G_1 . Under the inductive hypothesis G_1 and in view of (1) G is completely decomposable as well. Thus the proof of our theorem is finished.

If the group G is torsion free of finite rank and H any of its pure subgroups, then $\kappa_p(G) = \kappa_p(H) + \kappa_p(G/H)$ for every prime p (see [6,Theorem 6]). In particular, we obtain that $\kappa_p(G) = 0$ implies $\kappa_p(G/H) = 0$ for each pure subgroup H of G . We shall use this last fact in the proof of the following theorem. Let us recall that if G is torsion free and p any prime, then $G[p^\infty]$ will denote the greatest p-divisible subgroup of G . Evidently, $\kappa_p(G[p^\infty]) = \kappa(G[p^\infty])$ (see [5,Theorem 1]).

<u>Theorem 2</u>. Let G be an almost divisible torsion free group of finite rank with the linearly ordered type set $\mathcal{T}(G)$. Then G is completely decomposable if and only if $\kappa_p(G) = \kappa(G[p^\infty])$ for every prime p .

Proof. If G is completely decomposable, then for any prime p, $G = G[p^\infty] \dotplus G_1$ where G_1 is also completely decomposable and p-reduced. Then, by [4, Corollary 1] $\kappa_p(G_1) = 0$. Since $\kappa_p(G) = \kappa_p(G[p^\infty]) + \kappa_p(G_1)$ (see [6, Theorem 6]), we get $\kappa_p(G) = \kappa_p(G[p^\infty]) = \kappa(G[p^\infty])$.

To prove the converse consider $\kappa_p(G) = \kappa(G[p^\infty])$ for all primes p. For the proof of complete decomposability of G it suffices to show that G fulfils (b*) only, (a) being trivial. Let $\mathcal{T}(G) = \{t_1 < \ldots < t_n\}$, take $t \in \mathcal{T}(G)$ and suppose $t(p) \neq \infty$ for some prime p. In order to prove the relation $\kappa_p(G/G^*(t)) = 0$ we shall distinguish two cases: $G[p^\infty] = 0$ and $G[p^\infty] \neq 0$. If $G[p^\infty] = 0$, then $\kappa_p(G) = \kappa_p(G[p^\infty]) = 0$ and in view of the preceding remark we have $\kappa_p(G/G^*(t)) = 0$. If $G[p^\infty] \neq 0$, then there exists an integer $j \leqq n$ with $t_j(p) = \infty$; since $t_1 \leqq t$ and $t(p) \neq \infty$, it is certainly $1 < j$. Let i denote the smallest integer with $t_i(p) = \infty$; we shall show that $G[p^\infty] = G(t_i)$. The relation $t_i(p) = \infty$ implies the inclusion $G(t_i) \subseteq$ $\subseteq G[p^\infty]$. But if $0 \neq g \in G[p^\infty]$ and $t_k = \mathrm{type}(g)$, then $t_k(p) = \infty$, therefore $i \leqq k$. Hence we conclude $t_i \leqq t_k$ and $g \in G(t_i)$.

Thus we have shown that $G[p^\infty] = G(t_i)$ and also $G[p^\infty] =$ $= G^*(t_{i-1})$ $(2 \leqq i)$. By [6, Theorem 6] we have

$$\kappa_p(G) = \kappa_p(G[p^\infty]) + \kappa_p(G/G[p^\infty]) \ ;$$

since $\kappa_p(G[p^\infty]) = \kappa(G[p^\infty]) = \kappa_p(G)$, we get

$$(3) \qquad 0 = \kappa_p(G/G[p^\infty]) = \kappa_p(G/G^*(t_{i-1})) \ .$$

From $\mathfrak{u}(p) \neq \infty$ it follows $\mathfrak{u} \not\leq \mathfrak{u}_{i-1}$ and hence $G^*(\mathfrak{u}_{i-1}) \subseteq G^*(\mathfrak{u})$. Thus we have

$$G/G^*(\mathfrak{u}) \cong (G/G^*(\mathfrak{u}_{i-1}))/(G^*(\mathfrak{u})/G^*(\mathfrak{u}_{i-1}))$$

and by (3) $r_p(G/G^*(\mathfrak{u})) = 0$. This means that G fulfils (b*) and Theorem 2 is proved.

Remark. The preceding theorem may be likewise formulated in the following way (see [2,Proposition 5.2]; for the definition of the regularity of a group see also [2, § 5]): Let G be an almost divisible torsion free group of finite rank with the linearly ordered type set $\mathfrak{T}(G)$. Then the group G is completely decomposable if and only if it is regular.

Till now we have considered groups with the linearly ordered type set $\mathfrak{T}(G)$ only. In order to investigate the general case we shall use [1,Theorem 48.6]. Thus we get the following assertion:

Theorem 3. An almost divisible torsion free group G is completely decomposable if and only if the conditions (a),(b) and

(c) $\quad G^*(\mathfrak{u}) = G(\mathfrak{u}) \cap G^{**}(\mathfrak{u})$

are fulfilled for each type $\mathfrak{u}$ $\mathfrak{T}(G)$.

Proof. Firstly, assume that G is completely decomposable and that $G = \sum_{\lambda\in\Lambda} J_\lambda$ is one of its complete decompositions. Denote by $T(G)$ the set of all types of the groups J_λ $(\lambda \in \Lambda)$; evidently, $T(G) \subseteq \mathfrak{T}(G)$. For $\mathfrak{u} \in T(G)$ let $A_{\mathfrak{u}}$ denote the direct sum of all groups J_λ of the type $\mathfrak{u}$; certainly, it is $G(\mathfrak{u})/G^*(\mathfrak{u}) \cong A_{\mathfrak{u}}$. If $\mathfrak{u}(p) \neq \infty$, then $A_{\mathfrak{u}}$ is a p-reduced completely decomposable group and in view of [4,Corollary 1] $0 = \varkappa_p(A_{\varkappa}) = \varkappa_p(G(\varkappa)/G^*(\varkappa))$. Thus for $\varkappa \in T(G)$ the conditions (a) and (b) are fulfilled. But if $\varkappa \in \mathcal{T}(G) \dot{-} T(G)$, then $G(\varkappa) = G^*(\varkappa)$ and the conditions (a), (b) are trivial. The condition (c) follows from [1,Theorem 48.6].

Further, suppose that G fulfils the conditions (a), (b),(c), and prove that G is completely decomposable. If $\varkappa \in \mathcal{T}(G)$, then by Lemma there exists a direct decomposition of the form $G(\varkappa) = G_{\varkappa} \dotplus G^*(\varkappa)$ where the group $G_{\varkappa}$ is completely decomposable and homogeneous of the type $\varkappa$. Now, the proof proceeds in the same way as that of sufficiency in [1,Theorem 48.6]. Thus, firstly, it may be shown that the subgroups $G_{\varkappa}$ ($\varkappa \in \mathcal{T}(G)$) generate their direct sum $\sum_{\varkappa} G_{\varkappa}$, and then we should get $G = \sum_{\varkappa} G_{\varkappa}$. The last relation is proved in [1,Theorem 48.6] under the assumption that $\mathcal{T}(G)$ satisfies the maximum condition, but in our case $\mathcal{T}(G)$ is finite, G being almost divisible. Since each $G_{\varkappa}$ is completely decomposable, so is the group $G = \sum_{\varkappa} G_{\varkappa}$ as well, which finishes the proof of the theorem.

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(Oblatum 7.10.1970)