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ON SOME CLASSES OF POINT ALGEBRAS

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1. Introduction. In this note we give a characterization of the following classes of point algebras. (See [1], [2] for a general definition of point algebra even though it will not be needed here.) Let S be a nonempty set, n an integer ≥ 2 and k a positive integer such that $2k \leq n$. Define on $S^n (= S \times S \times \ldots \times S,\ n\ S's)$ the following binary operations:

(i) $$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$$
$$= (b_{n-k+1}, \ldots, b_n, b_{k+1}, \ldots, b_{n-k}, b_1, \ldots, b_k),$$

(ii) $$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$$
$$= (a_{n-k+1}, \ldots, a_n, a_{k+1}, \ldots, a_{n-k}, a_1, \ldots, a_k),$$

where, if $n = 2k$, the right side of (i) is $(b_{k+1}, \ldots, b_n, b_1, \ldots, b_k)$. Similarly for the right side of (ii). In the remainder of this note we will denote the groupoid on S^n, where $|S| = u$, obtained by the binary operation in (i), by the symbol $G(n, k, u)$ and the groupoid on S^n obtained by the binary operation in (ii), by the symbol $H(n, k, u)$. It is the point algebras $G(n, k, u)$ and

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$H(m, k, u)$ that we characterize.

2. The characterizations. We first prove the following lemma which is also of independent interest.

Lemma. Let G and H be groupoids such that I_G and I_H denote the (possibly empty) sets of idempotents of G and H, respectively. If

(i) each of G and H satisfies the identity $x \cdot yx = x$ (or each of G and H satisfies $xy \cdot x = x$),

(ii) $|G| = |H|$,

(iii) $|I_G| = |I_H|$ and

(iv) $|G - I_G| = |H - I_H|$, then $G \approx H$.

Proof. (In this proof we assume that each of G and H satisfies $x \cdot yx = x$. If each of G and H satisfies $xy \cdot x = x$ the proof is analogous.) Define a mapping Θ from G into H as follows. If I_G, and hence I_H, is nonempty where $I_G = \{a_\lambda\}_{\lambda \in \Lambda}$ and $I_H = \{b_\lambda\}_{\lambda \in \Lambda}$ then for each $a_\lambda \in I_G$ let $(a_\lambda)\Theta = b_\lambda$. Clearly $\Theta | I_G$ is $1-1$ and onto I_H. Of course, if I_G, and thus I_H, is empty this step is omitted. Now suppose $G - I_G$, and thus $H - I_H$, is nonempty. We note that if $x \in G - I_G$ then $x^2 \in G - I_G$ and $x^2 \cdot x^2 = x$. Similarly for $y \in H - I_H$. Thus let Γ be an indexing set such that $A = \{\{x_\gamma, x_\gamma^2\} \mid \gamma \in \Gamma\}$ and $B = \{\{y_\gamma, y_\gamma^2\} \mid \gamma \in \Gamma\}$ are partitions of $G - I_G$ and $H - I_H$, respectively. If $\gamma \in \Gamma$ then let $x_\gamma \Theta = y_\gamma$ and $x_\gamma^2 \Theta = y_\gamma^2$. Then clearly $\Theta | G - I_G$ is $1-1$ and onto $H - I_H$. Of course if $G - I_G$, and thus $H - I_H$, is empty we omit this step.

Thus Θ is a $1-1$ mapping onto H. Here we note that for any $x \in G$, $x^2\Theta = (x\Theta)^2$. Therefore if $x, y \in G$ then $(xy)\Theta = (x(y \cdot y^2))\Theta = y^2\Theta = (y\Theta)^2 = x\Theta(y\Theta(y\Theta)^2) = x\Theta y\Theta$. Hence Θ is an isomorphism.

One easily shows that $G(n, k, \kappa)$ satisfies the identity $x \cdot yz = z$ and that $H(n, k, \kappa)$ satisfies the identity $xy \cdot z = x$. Furthermore the idempotents of $G(n, k, \kappa)$ as well as of $H(n, k, \kappa)$ are precisely the elements in S^n of the norm, $(a_1, \ldots, a_k, a_{k+1}, \ldots, a_{n-k}, a_1, \ldots, a_k)$, of which there are κ^{n-k}. If κ is finite then of course there are $\kappa^{n-k}(\kappa^k - 1)$ non-idempotents. If κ is infinite then clearly there are κ idempotents and κ non-idempotents. Thus we have the following corollary.

<u>Corollary</u>. Let G be a groupoid of order κ^n, where κ is a cardinal and n is an integer ≥ 2. Assume G satisfies the identity $x \cdot yz = z$ $(xy \cdot z = x)$. Also assume G has κ^{n-k} idempotents where k is a positive integer and $2k \leq n$. If κ is infinite assume G has κ non-idempotents, too. Then $G \approx G(n, k, \kappa)$ $(H(n, k, \kappa))$.

R e f e r e n c e s

[1] M. SAADE: Point algebras. Notices Amer.Math.Soc.16 (1969),94.

[2] ______ : Generating operations of point algebras. J.Combinatorial Theory (to appear).

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