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# ON INTERPRETABILITY IN SET THEORIES

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Denote by ZF the Zermelo-Fraenkel set theory (with regularity but without choice) and by GB the Gödel-Bernays set theory (the same restriction). Both theories are supposed to be formulated as formal systems with one sort of variables and one binary predicate $\in$ . Every ZF-formula can be considered as a particular GB-formula by means of an obvious relative interpretation.

In a discussion with Professor G. Kreisel in summer 1969 I formulated the following

<u>Problem</u>: Does for every ZF-formula $\varphi$ relative interpretability of $\langle ZF, \varphi \rangle$ in ZF imply relative interpretability of $\langle GB, \varphi \rangle$ in GB ?

Denoting, for every theory $T$ which is either an extension of ZF or an extension of GB, by $\mathcal{J}_T$ the set of all ZF-formulas such that $\langle T, \varphi \rangle$ is relatively interpretable in $T$ , our problem reads: Is $\mathcal{J}_{ZF} \subseteq \mathcal{J}_{GB}$ ?

We shall prove a theorem which implies the negative answer of our problem. The theorem also implies that $\mathcal{J}_{ZF}$ is not recursively enumerable (whereas $\mathcal{J}_{GB}$ is, which is easy to show). I discussed the problem with Professors G. Kreisel, J.R. Shoenfield and R. Solovay; I thank them for

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their interest and for the encouraging advice to look for a counterexample. Discussions with my wife on her work [4] were not only an exciting pleasure for me but also helped me to find a solution.

First, let us recall some known facts on finitary relative consistency proofs useful in the sequel and yielding a background of our problem. Presupposed is the knowledge of the notion of a relative interpretation in the sense of Tarski [7] and some familiarity with Feferman's fundamental work [2].

<u>Lemma 1</u>. For every ZF-formula $\varphi$, $ZF \vdash \varphi$ iff $GB \vdash \varphi$; equivalently, for every ZF-formula $\varphi$, $\mathit{Con}\,(ZF, \varphi)$ iff $\mathit{Con}\,(GB, \varphi)$.

See [6] for a finitary proof; in fact, Shoenfield constructs a primitive recursive function associating with every ZF-formula $\varphi$ and every GB-proof of $\varphi$ a ZF-proof of $\varphi$.

Although we shall be dealing with set theories, we shall explicitly use only variables ranging over the set of natural numbers; the letters $x, y, \dots$ will be used for this purpose. $\varsigma(x)$ is an arbitrary but fixed bi-numeration of the set of axioms of ZF in ZF. If $\varphi$ is a ZF-formula then $\varsigma \cup \{\overline{\varphi}\}$ means the formula $\varsigma(x) \vee x \approx \overline{\varphi}$ which bi-numerates the axioms of $(ZF, \varphi)$ in ZF.

<u>Lemma 2</u>. For each ZF-formula $\varphi$, $\varphi \in \mathcal{J}_{ZF}$ iff $ZF \vdash \mathit{Con}_{\varsigma \cup \{\overline{\varphi}\} \restriction n}$ for every $n$.

See [2] Theorem 8.10 (and also 6.3, 6.9 and 5.9) for the proof of the implication $\Longrightarrow$ (cf. also [5], footnote 22). The converse implication is easy to prove using reflexivity of $(ZF, \varphi)$ and observing that

$$ZF \vdash [(\underset{\cdot}{Con}_{\S \cup \{\bar{\varphi}\} \upharpoonright \bar{n}})^{*} \rightarrow \underset{\cdot}{Con}_{\S \cup \{\bar{\varphi}\} \upharpoonright \bar{n}}]$$

( $*$ denotes the image of the respective formula in the interpretation in question).

Hence, having proved $ZF \vdash \underset{\cdot}{Con}_{\S \cup \{\bar{\varphi}\} \upharpoonright \bar{n}}$ for every $n$, we have the following: (i) $(ZF, \varphi)$ is relatively interpretable in ZF, (ii) consequently, $\varphi$ is relatively consistent w.r.t. ZF and (iii) is relatively consistent w.r.t. GB. But the question remains whether $(GB, \varphi)$ is relatively interpretable in GB and we are led to our problem whether $\mathcal{J}_{ZF} \subseteq \mathcal{J}_{GB}$ .

A counterexample is a ZF-formula $\varphi$ such that $(ZF, \varphi)$ is relatively interpretable in ZF, but $(GB, \varphi)$ is not relatively interpretable in GB. Such a $\varphi$ is consistent with GB, and also $\neg\varphi$ is consistent with GB, for otherwise the identical interpretation of GB would be an interpretation of $(GB, \varphi)$ in GB.

Theorem. Suppose that ZF is $\omega$ -consistent. Let $W$ be a recursively enumerable set of ZF-formulas such that, for every $\varphi$, $\varphi \in W$ implies $Con(ZF, \varphi)$ . Then there is a $\varphi$ such that $\varphi \in \mathcal{J}_{ZF} - W$ . In fact, there is a primitive recursive function associating with every RE-formula $\vartheta(x)$ a formula $\varphi$ such that, if $W$ is the set numerated by $\vartheta(x)$ in ZF and if every element of $W$ is a ZF-formula consistent with ZF, then $\varphi \in \mathcal{J}_{ZF} - W$ .

Proof. Let $W = \{n ; (\exists m) A(m, n)\}$ where $A$ is primitive recursive. Let $\alpha(x, y)$ be a PR-formula such that $\alpha(x, y)$ bi-numerates $A$ in ZF and $\bigvee_x \alpha(x, y)$ numerates $W$ in ZF. (Cf.[2] 3.11.) Using the diagonal lemma 5.1 [2] we can construct a ZF-formula $\varphi$ such that

$$ZF \vdash \varphi \leftrightarrow \bigwedge_x (\alpha(x, \bar{\varphi}) \rightarrow \neg Con_{\S \cup \{\bar{\varphi}\} \restriction x}).$$

(a) $Con(ZF, \varphi)$. Otherwise we have $ZF \vdash \bigvee_x \alpha(x, \bar{\varphi})$ and therefore $\varphi \in W$, which implies $Con(ZF, \varphi)$.

(b) $\varphi \notin W$. Otherwise we have $A(m, \varphi)$ for some $m$; then $ZF \vdash \alpha(\bar{m}, \bar{\varphi})$ and $(ZF, \varphi) \vdash \neg Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$. But since $(ZF, \varphi)$ is consistent and reflexive (see [2], p.89) we have $(ZF, \varphi) \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$ which contradicts the consistency of $(ZF, \varphi)$.

(c) $\varphi \in \mathcal{J}_{ZF}$. We show $ZF \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$ for every $m$; then $\varphi \in \mathcal{J}_{ZF}$ by Lemma 2. Since $(ZF, \varphi) \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$ by the reflexivity, it suffices to show $(ZF, \neg\varphi) \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$. But $\neg\varphi$ is equivalent in ZF to $\bigvee_x (\alpha(x, \bar{\varphi}) \,\&\, Con_{\S \cup \{\bar{\varphi}\} \restriction x})$.
Now for each $m$ we have $ZF \vdash \neg\alpha(\bar{m}, \bar{\varphi})$ since $\varphi \notin W$ by (b) and since $\alpha$ bi-numerates $A$ in ZF. Hence we have

$$(ZF, \neg\varphi) \vdash \bigvee_x (x > \bar{m} \,\&\, Con_{\S \cup \{\bar{\varphi}\} \restriction x})$$

for each $m$, which implies $(ZF, \neg\varphi) \vdash Con_{\S \cup \{\bar{\varphi}\} \restriction \bar{m}}$. This completes the proof.

<u>Corollary 1</u>. If ZF is $\omega$-consistent then $\mathcal{J}_{ZF} - \mathcal{J}_{GB} \neq 0$. For, evidently, $\varphi \in \mathcal{J}_{GB}$ implies $Con(ZF, \varphi)$ and $\mathcal{J}_{GB}$ is recursively enumerable. (A formula $\varphi$ belongs to $\mathcal{J}_{GB}$ iff there are two GB-formulas defining classes and membership in the sense of the interpretations and,

in addition, GB-proofs of the interpretations of all the finitely many - 15, say - axioms of $(GB, \varphi)$.

Corollary 2. Let $GB_1$ be a consistent finitely axiomatized extension of GB (for example, by adding the axiom of existence of measurable cardinals, assuming that this extension is consistent). If ZF is $\omega$-consistent then $\mathcal{J}_{ZF} - \mathcal{J}_{GB_1} \neq 0$.

Corollary 3. If ZF is $\omega$-consistent then $\mathcal{J}_{ZF}$ is not recursively enumerable. (By the theorem, every recursively enumerable subset of $\mathcal{J}_{ZF}$ is a proper subset.)

Discussion. (1) A historical remark. The Cohen's pioneering proof of the independence of the continuum hypothesis (CH) can be understood as a proof that, for every $n$, $ZF \vdash Con_{\S \cup \{\neg \overline{CH}\} \restriction \overline{n}}$ (see [1]) and therefore yields a relative interpretation of $(ZF, \neg CH)$ in ZF. But it follows from our theorem that a relative interpretation of $(ZF, \neg CH)$ in ZF does not automatically yield an interpretation of $(GB, \neg CH)$ in GB. Such an interpretation was constructed in [8] by exploring the Cohen's proof (see also various relative interpretations of GB + additional axiom in GB constructed in [9] using the notion of Boolean valued models). It can be said that construction of a relative interpretation is the most natural kind of a relative consistency proof; but perhaps it is the matter of one's taste. (In fact, Vopěnka constructed a parametrical relative interpretation called a parametric syntactic model in [3]; but if $(GB, \varphi)$ has a parametric relative interpretation in GB such that the range of parameters is described by a ZF-formula, then $(GB, \varphi)$ has a (non-parametric) relative interpretation in GB, see [3], Theorem 4.)

(2) Is $\mathcal{J}_{GB} \subseteq \mathcal{J}_{ZF}$ ? It is true that if $(GB, \varphi)$ has a "nice" relative interpretation in GB then $\varphi \in \mathcal{J}_{ZF}$ . E.g. it suffices that $M^*$ is absolute from below (i.e. $GB \vdash M^*(X) \longrightarrow M(X)$ ) and, in addition, both $M^*(a)$ and $M^*(a) \,\&\, M^*(b) \,\&\, a \in^* b$ are equivalent in GB to some ZF-formulas. (Here $X$ is a class variable and $a, b$ are set variables.) One can formulate more general conditions, but the problem in full generality seems to be open.

(3) By Lemma 2, $\mathcal{J}_{ZF}$ is a $\Pi^0_2$ set and by Corollary 3, it is not a $\Sigma^0_1$ set. I do not know whether $\mathcal{J}_{ZF}$ is a $\Pi^0_1$ set and/or a $\Delta^0_2$ set.

R e f e r e n c e s

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