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A REPRESENTATION THEOREM FOR TIME-SCALE FUNCTIONALS

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In this note we establish a representation for a class of nonlinear maps on a function space into a space of set functions. Our work is closely related to the investigations of MARTIN and MIZEL [7] and MIZEL and SUNDARESAN [8]. As a corollary, this representation provides a characterization of certain maps, called time-scale functionals, which are of interest in the study of hereditary thermo-mechanical effects. For the role of time-scale functionals in the study of thermorheologically simple materials refer to [2,3,4,9, 10]. We restrict ourselves here to a mathematical formulation.

We write $\mathcal{R}$ for the real line and $\mathcal{R}^+$ for the non-negative real line. Let P denote the collection of all intervals of the form $[a,b) = \{s \in \mathcal{R}^+ : a \leq s < b\}$, where $0 \leq a \leq b < \infty$, and let R denote the collection of all finite unions of intervals in P. The collection R is precisely the ring of sets in $\mathcal{R}^+$ generated by the semi-ring P. For any set $\mathcal{A} \subset \mathcal{R}$ and $\sigma \in \mathcal{R}$ we write $\mathcal{A} - \sigma$ for the set $\{(s - \sigma) \in \mathcal{R} : s \in \mathcal{A}\}$.

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The LEBESGUE measure (length) of a set $\mathcal{A} \in \mathrm{R}$ will be denoted by $|\mathcal{A}|$. By a <u>history</u> we mean an essentially bounded LEBESGUE measurable function $f: \mathcal{R}^+ \rightarrow \mathcal{R}$. The vector space of all histories is denoted by $\mathcal{H}$. The characteristic function of a set $\mathcal{A} \subset \mathcal{R}^+$ is a history which we denote by $\chi_{\mathcal{A}}$. For a history f and a non-negative real number σ, define a new history f_σ by

$$f_\sigma(s) = f(s+\sigma), \quad s \in \mathcal{R}^+ .$$

For any $\theta \in \mathcal{R}$, we write θ^+ to denote the <u>constant history with value</u> θ

$$\theta(s) = \theta, \quad s \in \mathcal{R}^+ .$$

Here the term <u>almost everywhere</u> means <u>except for a set of LEBESGUE measure zero</u>. See [5] for set- and measure-theoretical terminology.

Throughout this note we are concerned with a map $f \mapsto \mu_f$ which assigns to each history $f \in \mathcal{H}$ a non-negative set function μ_f defined and additive on the ring R. We call such a map:

<u>local</u> if, and only if, for each $\mathcal{A} \in \mathrm{R}$

$$\mu_f(\mathcal{A}) = \mu_k(\mathcal{A}),$$

whenever $f\chi_{\mathcal{A}} = k\chi_{\mathcal{A}}$ almost everywhere;

<u>translation invariant</u> if, and only if, for each $\mathcal{A} \in \mathrm{R}$ and $\sigma \in \mathcal{R}^+$

$$\mu_f(\mathcal{A}) = \mu_{f_\sigma}(\mathcal{A}-\sigma)$$

provided σ is such that $\mathcal{A}-\sigma \in \mathrm{R}$ (Strictly speaking, we define here only <u>left</u> translation invariance, since this is sufficient for our purpose. The rôle of translation invariance in the investigation of hereditary phenomenae has been considered in some detail by COLEMAN and MIZEL [1] and LEITMAN and Mizel [6] ;

<u>continuous</u> if, and only if, for each $a \in R$

$$\mu_{f_n}(a) \to \mu_f(a)$$

as $n \to \infty$, whenever $f_n \to f$ as $n \to \infty$ boundedly almost everywhere; and

<u>non-vanishing</u> if, and only if, there is no history $f \in \mathcal{H}$ such that μ_f vanishes on R.

For such maps we have the following

<u>Representation theorem</u>. <u>A map $f \mapsto \mu_f$ which is local, translation invariant, and continuous is completely characterized by a uniquely determined non-negative continuous function φ on $\mathcal{R}$. This characterization is realized through the formula</u>

$$(*) \qquad \mu_f(a) = \int_a \varphi(f(s))\,ds \ ,$$

<u>where $f \in \mathcal{H}$ and $a \in R$. Furthermore, the given map is non-vanishing if, and only if, φ is positive on $\mathcal{R}$.</u>

<u>Proof</u>. Let φ be a non-negative continuous function on $\mathcal{R}$. Then for each history $f \in \mathcal{H}$, the formula $(*)$ certainly defines a non-negative additive set function μ_f on R. The map $f \mapsto \mu_f$ thus defined is easily seen to be local and translation invariant. That it is also continuous follows from LEBESGUE's dominated convergence theorem.

Conversely, let the map $f \mapsto \mu_f$ be local, translation invariant, and continuous. Then there is a non-negative continuous function φ on $\mathcal{R}$ such that

$$\mu_{\theta^+}(\mathfrak{a}) = \varphi(\theta)\,|\mathfrak{a}|$$

for every $\theta \in \mathcal{R}$ and $\mathfrak{a} \in \mathrm{R}$. Since it suffices to verify this assertion for intervals $\mathfrak{a} \in \mathrm{P}$, we let $\mathfrak{a} = [a, b)$, where $0 \leq a \leq b < \infty$. This translation invariance of the map $\theta^+ \mapsto \mu_{\theta^+}$ and the fact that $\theta^+_a \equiv \theta^+$ together imply

$$\mu_{\theta^+}\{[a, b)\} = \mu_{\theta^+}\{[0, b-a)\}$$

or, equivalently,

$$\mu_{\theta^+}(\mathfrak{a}) = \mu_{\theta^+}([0, |\mathfrak{a}|)) \quad .$$

Suppose that $|\mathfrak{a}|$ is rational. Then by repeated use of the previous argument and the additivity of μ_{θ^+} it follows that

$$\mu_{\theta^+}([0, |\mathfrak{a}|)) = \mu_{\theta^+}([0, 1))\,|\mathfrak{a}| \quad .$$

But this formula holds for all $\mathfrak{a} \in \mathrm{P}$. Indeed, if $|\mathfrak{a}|$ is not rational, choose an increasing (positive) rational sequence $\{c_n\}$ and a decreasing rational sequence $\{d_n\}$ such that each has limit $|\mathfrak{a}|$. From the monotonicity of the set function μ_{θ^+} we conclude

$$\begin{aligned}\mu_{\theta^+}([0, 1))\,c_n = \mu_{\theta^+}([0, c_n)) &\leq \mu_{\theta^+}([0, |\mathfrak{a}|)) \\ &\leq \mu_{\theta^+}([0, d_n)) = \mu_{\theta^+}([0, 1))\,d_n \quad ,\end{aligned}$$

for each n. Hence, letting $n \to \infty$, we obtain

$$\mu_{\theta^+}([0, 1))\,|\mathfrak{a}| \leq \mu_{\theta^+}([0, |\mathfrak{a}|)) \leq \mu_{\theta^+}([0, 1))\,|\mathfrak{a}| \,.$$

Thus, for any set $\mathcal{a} \in \mathrm{P}$, we have shown that

$$\mu_{\theta^+}(\mathcal{a}) = \mu_{\theta^+}([0,1))\,|\mathcal{a}| \quad .$$

Now define φ on $\mathcal{R}$ by

$$\varphi(\theta) = \mu_{\theta^+}([0,1)), \quad \theta \in \mathcal{R} \; .$$

The function φ is clearly non-negative. To see that it is also continuous, observe that $\theta_n^+ \to \theta^+$ as $n \to \infty$ implies $\theta_n^+ \to \theta^+$ as $n \to \infty$ boundedly almost everywhere. Hence, the continuity of the map $\theta^+ \mapsto \mu_{\theta^+}$ implies that $\varphi(\theta_n) \to \varphi(\theta)$ as $n \to \infty$. The assertion is verified.

Let h be a history which is simple with respect to R; that is, h may be written as the finite sum

$$h = \sum_{i=1}^{N} \partial_i \chi_{a_i}$$

where $\{\theta_i\}$ is a collection of N real numbers and $\{a_i\}$ is a collection of N mutually disjoint sets in R. The local property of the map $f \mapsto \mu_f$ together with the additivity of μ_f implies

$$\mu_h(\mathcal{a}) = \sum_{i=1}^{N} \varphi(\theta_i)\,|\mathcal{a} \cap a_i| + \varphi(0)\,|\mathcal{a} - \bigcup_{i=1}^{N} a_i| \quad ,$$

for each $\mathcal{a} \in \mathrm{R}$. For any history f there can be found a sequence of histories $\{f_n\}$, each simple with respect to R, such that $f_n \to f$ as $n \to \infty$ boundedly almost everywhere. Moreover, for each simple f_n we already have

$$\mu_{f_n}(\mathcal{a}) = \int_{\mathcal{a}} \varphi(f_n(s))\,ds \; ,$$

for every $\mathcal{a} \in \mathrm{R}$. The continuity of the function φ and the continuity of the map $f \mapsto \mu_f$ together with LEBESGUE's dominated convergence theorem then imply

$$\mu_f(\mathcal{a}) = \lim_{n\to\infty} \mu_{f_n}(\mathcal{a}) = \lim_{n\to\infty} \int_{\mathcal{a}} \varphi(f_n(s))\,ds =$$

$$= \int_{\mathcal{a}} \lim_{n\to\infty} \varphi(f_n(s))\,ds = \int_{\mathcal{a}} \varphi(f(s))\,ds \ ,$$

for every $\mathcal{a} \in \mathcal{R}$. The function φ is clearly uniquely determined and the characterization is established.

Finally, it follows from the definition of φ that the map $f \mapsto \mu_f$ is non-vanishing if and only if φ is positive. The Representation Theorem is proved.

<u>Remark 1</u>. For each history f, the set function μ_f on $\mathcal{R}$ is seen to be absolutely continuous with respect to LEBESGUE measure. Hence, μ_f is countably additive on $\mathcal{R}$ and, by classical arguments, possesses a unique extension to a BOREL measure on $\mathfrak{R}^+$.

<u>Remark 2</u>. It may be shown directly that a local map $f \mapsto \mu_f$ has a <u>separate additivity property</u> closely related to that considered by MARTIN and MIZEL [7]: for each set $\mathcal{a} \in \mathcal{R}$,

$$\mu_f(\mathcal{a}) + \mu_h(\mathcal{a}) - \mu_{f+h}(\mathcal{a}) = \mu_{0^+}(\mathcal{a}) \ ,$$

whenever f and h are histories with separate support in $\mathcal{a}$; that is, $fh\chi_{\mathcal{a}} = 0$ almost everywhere.

Remark 3. If, at the outset, the set functions μ_f were assumed to be real-valued and countably additive on $\mathcal{R}$, then our results carry over to signed measures, with the obvious modifications. We thank Professor V.J. MIZEL for this observation.

In order to explore one of the consequences of the Representation Theorem, let $f \mapsto \mu_f$ be local, translation invariant, continuous, and non-vanishing. For each history f, define a function $\mathcal{S}_f$ on $\mathcal{R}^+$ by

$$(**) \qquad \mathcal{S}_f(\xi) = \mu_f([0,\xi)) \equiv \int_0^{\xi} \varphi(f(s))\,ds, \quad \xi \in \mathcal{R}^+ .$$

The function $\mathcal{S}_f$ is a monotone increasing map of $\mathcal{R}^+$ onto $\mathcal{R}^+$. To see this, observe that, since φ is positive, $\mathcal{S}_f$ is continuous and monotone increasing on $\mathcal{R}^+$. Let c_f be the essential supremum of $|f|$ on $\mathcal{R}^+$. Since φ is positive and continuous on $[-c_f, c_f] \subset \mathcal{R}$ it attains a positive minimum and maximum, say, m_f and M_f on this interval. But then

$$m_f \xi \leq \mathcal{S}_f(\xi) \leq M_f \xi$$

for all $\xi \in \mathcal{R}^+$, so that $\mathcal{S}_f$ maps $\mathcal{R}^+$ onto $\mathcal{R}^+$.

By direct computation, the map $f \mapsto \mathcal{S}_f$ can be shown to reflect the properties of the map $f \mapsto \mu_f$. Thus, the map $f \mapsto \mathcal{S}_f$ satisfies:

(i) (local) for each $\xi \in \mathcal{R}^+$,

$$\mathcal{S}_f(\xi) = \mathcal{S}_h(\xi)$$

whenever $f\chi_{[0,\xi)} = k\chi_{[0,\xi)}$ almost everywhere;

(ii) (<u>continuous</u>) for each $\xi \in \mathcal{R}^+$,

$$\mathcal{S}_{f_n}(\xi) \to \mathcal{S}_f(\xi)$$

as $n \to \infty$ whenever $f_n \to f$ as $n \to \infty$ boundedly almost everywhere; and

(iii) (<u>additive</u>) for each $\xi \in \mathcal{R}^+$ and $\sigma \in \mathcal{R}^+$,

$$\mathcal{S}_f(\xi + \sigma) = \mathcal{S}_f(\xi) + \mathcal{S}_{f_\sigma}(\xi) \quad .$$

A <u>time-scale functional</u> (See introductory remarks.) is a map $f \mapsto \mathcal{S}_f$ which assigns to each history $f \in \mathcal{H}$ a monotone increasing map $\mathcal{S}_f$ of $\mathcal{R}^+$ onto $\mathcal{R}^+$.

The Representation Theorem has the following

<u>Corollary</u>. <u>A time-scale functional may be characterized by a uniquely determined positive continuous function φ on $\mathcal{R}$ through formula (**) if and only if, it is local, continuous, and additive (properties (i) - (iii))</u>.

This conclusion follows from the Representation Theorem upon observing that a time-scale functional $f \mapsto \mathcal{S}_f$ induces a (non-vanishing) map $f \mapsto \mu_f$ according to

$$\mu_f([a, b)) = \mathcal{S}_f(b) - \mathcal{S}_f(a) \quad ,$$

where $0 \leq a \leq b < \infty$. Furthermore, the properties (i), (ii), and (iii), defined for $f \mapsto \mathcal{S}_f$, are equivalent to locality, continuity, and translation invariance defined for $f \mapsto \mu_f$. That (i) is equivalent to locality and (ii) is equivalent to continuity is obvious. Finally, to see that (iii) implies translation invariance, let $0 \leq \sigma \leq a \leq b < \infty$.

Then

$$\begin{aligned}\mu_f([a,b)) &= \mathcal{S}_f(b) - \mathcal{S}_f(a) \\ &= \mathcal{S}_f((b-\sigma)+\sigma) - \mathcal{S}_f((a-\sigma)+\sigma) \\ &= \mathcal{S}_f(\sigma) + \mathcal{S}_{f_\sigma}(b-\sigma) - \mathcal{S}_f(\sigma) - \mathcal{S}_{f_\sigma}(a-\sigma) \\ &= \mathcal{S}_{f_\sigma}(b-\sigma) - \mathcal{S}_{f_\sigma}(a-\sigma) \\ &= \mu_f([a-\sigma, b-\sigma)) \ .\end{aligned}$$

Conversely, property (iii) can be deduced from translation invariance by setting $\sigma = a$ and reversing the previous argument.

R e f e r e n c e s

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