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ON PROJECTIVE LIMITS OF PROBABILITY SPACES

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The aim of this paper is to correct some results in the interesting paper of M.M. Rao [3].

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1. Pure Probabilities

1.1. Definition (see [3], 4.1). Let $P : \mathcal{A} \longrightarrow [0,1]$ be a finitely additive set function on an algebra $\mathcal{A} \subset \exp X$. A ring $\mathcal{R} \subset \mathcal{A}$ is called P-pure if

(i) $A_n \in \mathcal{R}$ for $n \in N$ (N is the set of all non-negative integers), $A_n \searrow \emptyset$ imply $P[A_{n_0}] = 0$ for some n_0,

(ii) $P[A] = \inf \{ \sum_{n \in N} P[A_n] \mid A_n \in \mathcal{R}$ and $\bigcup_{n \in N} A_n \supset A \}$ for each $A \in \mathcal{A}$.

If there exists a P-pure ring then P is said to be pure.

Remark. Any pure P is σ-additive ([3], 4.2) but the converse is not true as it will be shown below (beforehand, David Preiss constructed another counter-example).

1.2. Lemma (cf.[2], 7(ii)). Let $P : \mathcal{A} \longrightarrow [0,1]$ be a non-atomic probability, let $\mathcal{R} \subset \mathcal{A}$ be a P-pure ring, $E \in \mathcal{R}$,

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$P[E] > 0$. Then there exist $E_1, E_2 \in \mathcal{R}$ such that $E_1 \cup E_2 \subset E$, $E_1 \cap E_2 = \emptyset$ and $\frac{1}{4} P[E] > P[E_i] > 0$ for $i = 1, 2$.

Proof. As P is non-atomic there are $A_1, A_2 \in \mathcal{A}$ such that $A_1 \cup A_2 \subset E$, $A_1 \cap A_2 = \emptyset$, $P[A_1] = P[A_2] = \frac{1}{8} P[E]$. There exist $B_i^j \in \mathcal{R}$ $(i = 1, 2;\ j \in N)$ such that $\bigcup_{j \in N} B_i^j \supset A_i$ and $P[\bigcup_{j \in N} B_i^j] < \frac{1}{4} P[E]$ for $i = 1, 2$. Obviously $P[B_2^k \cap E] > 0$ for some $k \in N$. As $\bigcup_{j \in N} (B_1^j \cap E) \setminus B_2^k \supset$ $\supset A_1 \setminus \bigcup_{j \in N} B_2^j = A_1 \setminus [\bigcup_{j \in N} B_2^j \setminus A_2]$ and $P[A_1] = \frac{1}{8} P[E]$, $P[\bigcup B_2^j \setminus A_2] < \frac{1}{8} P[E]$ one has $P[\bigcup (B_1^j \cap E) \setminus B_2^k] > 0$. Hence $P[(B_1^l \cap E) \setminus B_2^k] > 0$ for some $l \in N$. The sets $E_1 = (B_1^l \cap E) \setminus B_2^k$ and $E_2 = B_2^k \cap E$ have the required properties.

1.3. Proposition (cf. [2], 7(iii)). Let $P : \mathcal{A} \to [0,1]$ be a non-atomic probability (on a σ-algebra $\mathcal{A}$) and let $\mathcal{R} \subset \mathcal{A}$ be a P-pure ring, $E \in \mathcal{R}$, $P[E] > 0$. Then there exists $A \in \mathcal{A}$ such that $A \subset E$, $\operatorname{card} A \geq \exp \aleph_0$ and $P[A] = 0$.

Proof will be only sketched here (it is essentially the same as the proof of 7(iii) in [2]): by means of Lemma 1.2 one can (inductively) construct the sets $E(a_0, a_1, \ldots, a_m)$, $m \in N$, $a_i = 0,1$ for $i = 0, 1, \ldots, m$, such that $P[E(a_0, a_1, \ldots, a_m)] > 0$,

$E(a_0, a_1, \ldots, a_m, 0) \cap E(a_0, a_1, \ldots, a_m, 1) = \emptyset$,

$E \supset E(a_0, a_1, \ldots, a_m) \supset E(a_0, a_1, \ldots, a_{m+1})$, and put $A = \bigcap_{n \in N} E_n$ where $E_n = \bigcup \{E(a_0, a_1, \ldots, a_n) \mid a_i = 0, 1$ for $0 \leq i \leq n\}$.

Remarks. Sierpiński proved (supposing continuum-hypothesis) that there exists a non-atomic probability space all null-sets of which are at most countable (see e.g. [4]); such a probability is not pure due to 1.3 (cf. [2], 7(iv)). The properties of pure probabilities are very similar to those of compact ones (for definition of compact measure see [2]), e.g. indirect product of pure probabilities is pure. It is even pretty possible that these two notions (compact, pure) are not really distinct; this is the case for countably-generated (in the sense of Carathéodory) probabilities; the proofs will soon be published.

2. Projective Limits

M.M. Rao gave conditions for σ-additivity of projective limits in terms of extensions of given probabilities ([3], 4.5 - 4.7). However, some of them are not correctly formulated (see 2.3).

2.0. Notations. Below, D is a set directed by the relation $\leq$ (i.e. $R \circ R = R$, $R \cap R^{-1}$ = diagonal, $R \circ R^{-1} = D \times D$ where $R \subset D \times D$ realizes $\leq$), $\{\mathcal{F}_\alpha\}_{\alpha \in D}$ is a family of σ-algebras $\subset \exp X$ such that $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ for $\alpha \leq \beta$; $\mathcal{F} = \bigcup_{\alpha \in D} \mathcal{F}_\alpha$, $\sigma\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}$. Given probabilities $P_\alpha : \mathcal{F}_\alpha \longrightarrow [0,1]$ for $\alpha \in D$ such that $P_\alpha[E] = P_\beta[E]$ for $E \in \mathcal{F}_\alpha \cap \mathcal{F}_\beta$, $P : \mathcal{F} \longrightarrow [0,1]$ is the finitely additive set function such that $P[E] = P_\alpha[E]$ for $E \in \mathcal{F}_\alpha$.

2.1. Proposition (see 2.0). The following conditions are equivalent:

(i) P is σ-additive;

(ii) for any $\alpha \in D$ there exists a probability $\overline{P_\alpha} : \sigma\mathcal{F} \to [0,1]$ that extends P_α and for every such extensions the following statement holds:

for every $A_m \in \mathcal{F}\ (m \in N),\ A_m \searrow \emptyset$ and $\varepsilon > 0$ there are $\alpha_0 \in D,\ m_0 \in N$ such that $\overline{P_\alpha}[A_m] < \varepsilon$ for $\alpha \geq \alpha_0,\ m \geq m_0$ (= mapping $\langle \alpha, m \rangle \longmapsto \overline{P_\alpha}[A_m]$ is continuous on $D \times N$);

(iii) for any $\alpha \in D$ there exists a probability $\overline{P_\alpha} : \sigma\mathcal{F} \to [0,1]$ that extends P_α and $\lim\limits_{m \to \infty} (\sup\limits_{\alpha \in D} \overline{P_\alpha}[A_m]) = 0$ for every $A_m \in \sigma\mathcal{F}\ (m \in N)$ with $A_m \searrow \emptyset$ (= mapping $m \longmapsto \overline{P_\alpha}[A_m]$ is continuous on N uniformly for all $\alpha \in D$).

Proof. Implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i) are immediate. (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii): to show the existence of the required extensions one can use for $\overline{P_\alpha}$ the (unique) extension of P on $\sigma\mathcal{F}$. If $\overline{P_\alpha}$'s are arbitrary extensions of P_α's and $A_m \in \mathcal{F},\ A_m \searrow \emptyset$ then $P[A_{m_0}] < \varepsilon$ for some m_0 and $A_{m_0} \in \mathcal{F}_{\alpha_0}$ for some α_0 . Hence $\overline{P_\alpha}[A_{m_0}] = P_\alpha[A_{m_0}] = P[A_{m_0}] < \varepsilon$ for $\alpha \geq \alpha_0$ and $\overline{P_\alpha}[A_m] \leq \overline{P_\alpha}[A_{m_0}]$ for $m \geq m_0$.

Remark. The condition in 2.1(iii) can be reformulated like this:

$\{\overline{P}_\alpha \mid \alpha \in D\} \subset ca(X, \sigma\mathcal{F})$ is weakly sequentially compact (see [1], IV.9.1) or like this:

$\overline{P}_\alpha$'s are uniformly λ-continuous for some $\lambda \in ca(X, \sigma\mathcal{F})$ (see [1], IV.9.2). But these conditions need not hold <u>for every</u> family $\{\overline{P}_\alpha\}$ of extensions (see 2.3).

2.2. <u>Proposition</u> (see 2.0). Let $D = N$ (N naturally ordered). The following conditions are equivalent:

(i) P is σ-additive;

(iv) for any $k \in N$ there exists a probability $\overline{P}_k : \sigma\mathcal{F} \longrightarrow [0,1]$ that extends P_k and for <u>every such extensions</u> and for every $A_n \in \mathcal{F}$ $(n \in N)$, $A_n \searrow \emptyset$ it holds $\lim\limits_{n\to\infty} (\sup\limits_{k \in N} \overline{P}_k[A_n]) = 0$ (= mapping $n \mapsto \overline{P}_k[A_n]$ is continuous on N uniformly for all $k \in N$);

(v) for any $k \in N$ there exists a probability $\overline{P}_k : \sigma\mathcal{F} \longrightarrow [0,1]$ that extends P_k and such that $\lim\limits_{k\to\infty} \overline{P}_k[A]$ exists for any $A \in \sigma\mathcal{F}$ (= mapping $k \mapsto \overline{P}_k[A]$ is continuous on N for any A).

<u>Proof</u>. Implication (iv) $\Longrightarrow$ (i) is clear, implication (v) $\Longrightarrow$ (i) is the theorem of Nikodým (see [1], III.7.4). (i) $\Longrightarrow$ (iv) and (i) $\Longrightarrow$ (v): the existence of extensions $\overline{P}_k$ is obvious as in the proof of 2.1.

Let $\overline{P}_k$'s be arbitrary extensions of P_k's, $A_n \in \mathcal{F}$, $A_n \searrow \emptyset$, $\varepsilon > 0$. For some n_1 it holds $P[A_{n_1}] < \varepsilon$, for some k_1 it holds $A_{n_1} \in \mathcal{F}_{k_1}$, hence $\overline{P}_k[A_{n_1}] = P_k[A_{n_1}] = P[A_{n_1}] < \varepsilon$ for $k \geq k_1$. For $i = 0, 1, \ldots, k_1 - 1$

there are ℓ_i such that $\overline{P_i}[A_{\ell_i}] < \varepsilon$; put $n_0 = \max\{n_1, \ell_0, \ell_1, \dots, \ell_{n_1-1}\}$; then $\overline{P_k}[A_{n_0}] < \varepsilon$ for any $k \in N$.

2.3. Examples. (a) The condition in 4.5 of [3] does not necessarily hold for arbitrary extensions $\overline{P_\alpha}$: Lebesgue probability on $[0,1]$ is the projective limit of all its restrictions to finite subalgebras and any such restriction can be extended as convex combination of Dirac measures. The family $\{\overline{P_\alpha}\}$ containing all these extensions works very wildly and does not satisfy any expected condition.

(b) This example shows (for $D = N$) that a family $\{\overline{P_k}\}$ of extensions need not be terminally uniformly λ-continuous for any finite measure λ on $\mathcal{F}$:
For $k \in N$, $\mathcal{F}_k \subset \exp[0,1]$ is the algebra of all the finite unions of intervals with end-points $\frac{r}{2^k}$, $r = 0, 1, \dots, 2^k$, P_k is the restriction of the Lebesgue probability on $[0,1]$ to $\mathcal{F}_k$, $\overline{P_k} = \frac{1}{2^k} \sum_{s=1}^{2^k} \delta_{x(s,k)}$ where $x(s,k) = \frac{2s-1}{2^{k+1}}$ and δ_x is the Dirac measure supported by x .

R e f e r e n c e s

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