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GENERALIZED-PURE-HEREDITARY RADICAL CLASSES OF ABELIAN GROUPS

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Abstract: In this paper the investigation of subgroup closure properties of radical classes of abelian groups is continued. A subgroup A of an abelian group B is a p^{α}-pure, for a prime p and an ordinal α, if the equivalence class of the exact sequence

$$0 \to A \to B \to B/A \to 0$$

belongs to $p^{\alpha}\mathrm{Ext}(B/A, A)$; if $p_1, p_2, \dots$ is the natural enumeration of the primes and $(\alpha_n) = (\alpha_1, \alpha_2, \dots)$ is a sequence of ordinals, a subgroup is (α_n)-pure if it is $p_n^{\alpha_n}$-pure for each n. For an arbitrary sequence (α_n), we characterize the radical classes which are closed under formation of (α_n)-pure subgroups.

Key Words: Radical class, abelian group, p^{α}-pure.

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Introduction. In [2] and [3] we examined radical classes of abelian groups with various subgroup closure properties and in the present paper the investigation of such properties is continued. We consider in this context a generalization of purity essentially due to Irwin, Walker and Walker [5]. Let p be a prime, α an ordinal number. A subgroup A of an abelian group B is p^{α} -pure if the equivalence class of the exact sequence

$$0 \to A \to B \to B/A \to 0$$

belongs to $p^{\alpha}\,\mathrm{Ext}\,(B/A, A)$. Let $p_1, p_2, \dots$ be the natural enumeration of the primes, $(\alpha_n) = (\alpha_1, \alpha_2, \dots)$ a sequence of ordinals. We call a subgroup (α_n) -pure if it is $p_n^{\alpha_n}$ -pure for each n. A radical class which is closed under formation of (α_n)-pure subgroups is called (α_n) -pure-hereditary, and similar expressions are used to describe radical classes with other subgroup closure properties. We obtain a complete characterization of the (α_n)-pure-hereditary radical classes for an arbitrary sequence (α_n). The case of ordinary purity, corresponding to the sequence $(\omega, \omega, \dots)$, was treated in [2].

We maintain the notation and conventions of [4] and also use the following terminology: Let P be a set of primes, P^* the semigroup of integers generated by P and 1. A group G is P -divisible if $kG = G$ for each $k \in P^*$ and a P -group if each of its elements has order belonging to P^*; a subgroup A of a group B is P -pure if $A \cap kB = kA$ for each $k \in P^*$; $Q(P)$ is the group of rational numbers with denominators in P^*; T (resp. T_p resp. T_P) is the class of torsion (resp. p -primary, resp. P-) groups and G_P is the maximum P -subgroup of a group G.

The results.

Theorem 1. Let (α_n) be a sequence of ordinals such that $\alpha_n \geq \omega$ whenever $\alpha_n \neq 0$, and let

$$S = \{p_n \mid \alpha_n \geq \omega\} .$$

The following conditions are equivalent for a radical class $R \nsubseteq T$.

(i) R is (α_n)-pure-hereditary.

(ii) R is S-pure-hereditary.

(iii) $R = L(\{Q(P)\} \cup \{Z(p) \mid p \in R\})$, where P, R are sets of primes with $R \subseteq P \subseteq S$.

Proof. (i) $\Longrightarrow$ (ii) : For any $p \in S$, the class $R \cap T_p$ either is hereditary or consists of all divisible p-groups ([1], Theorem 2.6). In either case it is S-pure-hereditary, so $R \cap T_S$ is also. In general, if G is an S-pure subgroup of $H \in R$, then G/G_S is isomorphic to an S-pure subgroup of $H/H_S \in R$ (cf. the proof of Theorem 3.2 in [2]). But in H/H_S, p^ω-purity coincides with p^α-purity for any $p \in S$ and $\alpha \geq \omega$, so G/G_S is isomorphic to an (α_n)-pure subgroup of H/H_S and hence belongs to R. Since $H_S \in R$ ([1], Theorem 5.2), the first part of the proof implies that $G_S \in R$; hence $G \in R$ and R is S-pure-hereditary.

(ii) $\Longrightarrow$ (i): This is clear, since (α_n)-pure subgroups are S-pure.

(ii) $\Longleftrightarrow$ (iii): The proof is closely related to that of Theorem 4.2 of [2] (which is the special case where S contains all primes) and we therefore omit it.

Note that the proof of the implication (ii) $\Longrightarrow$ (iii) given by Jambor ([6], Corollary 3.6) is incomplete. In the proof of Corollary 3.5 the rational groups of nil type are not considered, while in the proof of Corollary 3.6 itself, insufficient evidence is adduced for the conclusion that the type set of M is closed under arbitrary infima.

We can now introduce our principal result.

Theorem 2. Let (α_n) be a sequence of ordinals, $S = \{p_n \mid \alpha_n \neq 0\}$, $T = \{p_n \mid \alpha_n \geq \omega\}$. A radical class $\mathcal{R}$ is (α_n)-pure-hereditary if and only if either

(i) $\mathcal{R} \subseteq T$ and $\mathcal{R} \cap T_p$ is hereditary for all $p \notin S$,

or

(ii) $\mathcal{R} = L(\{Q(P)\} \cup \{Z(p) \mid p \in R\})$, where $P \subseteq \subseteq S$ and $R \subseteq P \cap T$.

Proof. I. Let $\mathcal{R}$ be of type (i). If A is (α_n)-pure in $B \in \mathcal{R}$, then for every prime p, A_p is (α_n)-pure in B_p. Obviously A_p belongs to $\mathcal{R}$ if $\mathcal{R} \cap T_p$ is hereditary. We are therefore interested only in the case where $p \in S$ and $\mathcal{R} \cap T_p$ is the class of divisible p-groups. But the latter is p^1-pure-hereditary and hence p^α-pure-hereditary for any $\alpha > 0$; in particular, it is (α_n)-pure-hereditary.

II. Let $\mathcal{R}$ be of type (ii) with $\mathcal{R} \neq \emptyset$, (α_n) a sequence of ordinals for which $\alpha_n \geq \omega$ whenever $p_n \in R$, and define

$$\beta_n = \begin{cases} \omega & \text{if} \quad \alpha_n \geq \omega & \text{and} \quad p_n \in P \\ 1 & \text{if} \quad 0 < \alpha_n < \omega & \text{and} \quad p_n \in P \\ 0 & \text{if} \quad \alpha_n = 0 & \text{or} \quad p_n \in S \setminus P \end{cases} .$$

We first prove that R is (β_n)-pure-hereditary. Let A be p^1-pure in $B \in R$, where $p \in P \setminus R$. Then B_p belongs to R, so is divisible, whence A_p is also. Direct decompositions $A = A_p \oplus \bar{A}$, $B = B_p \oplus \bar{B}$ can be chosen so that $\bar{A} \subseteq \bar{B}$ and it is straightforward to show that $\bar{A}$ is p^1-pure in $\bar{B}$. But $\bar{B}$ has no elements of order p, so $\bar{A}$ is p^ω-pure in $\bar{B}$. Being divisible, A_p is p^ω-pure in B_p. Thus in R, for $p \in P \setminus R$, p^ω-purity coincides with p^1-purity. Now Theorem 1 says that R is P-pure-hereditary, i.e. (γ_n)-pure-hereditary, where

$$\gamma_n = \begin{cases} \omega & \text{if} \quad \beta_n = \omega \quad \text{or } 1 \\ 0 & \text{if} \quad \beta_n = 0 \end{cases} .$$

By our remarks above, (β_n)-purity and (γ_n)-purity coincide in R, so that R is (β_n)-pure-hereditary, and <u>a fortiori</u> (α_n)-pure-hereditary.

III. Let R be of type (ii) with $R = \emptyset$. Then R is of the class of all P-divisible groups for some set $P \subseteq S$. For any sequence (α_n) of ordinals, let

$$\beta_n = \begin{cases} 1 & \text{if} \quad \alpha_n \neq 0 \\ 0 & \text{if} \quad \alpha_n = 0 \end{cases} .$$

If A is (β_n)-pure in $B \in \mathcal{R}$, then for each $p \in P$ we have $pA = A \cap pB = A \cap B = A$, so $\mathcal{R}$ is (β_n)-pure-hereditary, and hence (α_n)-pure-hereditary.

Thus each of the classes described in the theorem has the specified relative heredity property. We turn now to the converse.

IV. Let $\mathcal{R} \subseteq T$ be an (α_n)-pure-hereditary radical class. If $p \notin S$, then since $\mathrm{Ext}(Z(p^\infty), Z(p))$ is a p-group, we have

$$p_n^{\alpha_n} \mathrm{Ext}(Z(p^\infty), Z(p)) = \mathrm{Ext}(Z(p^\infty), Z(p))$$

for each n, so the natural embedding $Z(p) \to Z(p^\infty)$ is (α_n)-pure. Thus $Z(p) \in \mathcal{R}$ if $\mathcal{R} \cap T_p \neq \{0\}$, i.e. $\mathcal{R} \cap T_p$ is hereditary.

V. Let $\mathcal{R} \nsubseteq T$ be (α_n)-pure-hereditary. Then $\mathcal{R}$ is (β_n)-pure-hereditary, where

$$\beta_n = \begin{cases} \max\{\alpha_n, \omega\} & \text{if } \alpha_n \neq 0 \\ 0 & \text{if } \alpha_n = 0 \end{cases},$$

and so, by Theorem 1, S-pure-hereditary, Thus

$$\mathcal{R} = L(\{Q(P)\} \cup \{Z(p) \mid p \in R\})$$

where $R \subseteq P \subseteq S$.

Suppose there exists such a class for which $R \neq \emptyset$ and $0 < \alpha_n < \omega$ for some n with $p_n \in R$. We shall obtain a contradiction to complete the proof. To simplify notation in what follows, let $p_n = p$ and $\alpha_n = k$.

Let $G = G' \oplus \langle c \rangle$, where $G' \cong Q(P)$ and c has order p^k. Then $G \in \mathfrak{R}$. Let $F = \langle f_0 \rangle$, where $f_0 = p^k a + c$ and a is a fixed non-zero element of G' ,

$$F^* = \{x \in G \mid \exists m \in Z, (p, m) = 1, mx \in F\} .$$

We first prove that F^* is p^k-pure in G .

If $f^* = p^r g$, where $f^* \in F^*$, $g \in G$ and $0 < r \leq \leq k$, then $mp^r g = nf_0$ for some $m, n \in Z$ with $(p, m) = = 1$. Let $g = x + sc$. Then

$$mp^r x + mp^r sc = np^k a + nc ,$$

so $mp^r sc = nc$ and $p^k \mid (mp^r s - n)$. Hence $p^r \mid n$, say $n = n'p^r$. If $up^r + vm = 1$, then

$$vmf_0 = vmf^* = (1 - up^r)f^* ,$$

so $f^* = up^r f^* + vmf_0 = p^r(uf^* + vm'f_0)$ and since $uf^* + + vm'f_0 \in F^*$, F^* is p^k-pure in G , as asserted.

Let q be any prime $\neq p$. If $g \in G$ and $qg \in F^*$, let $mqg \in F$, where $(p, m) = 1$. Then $(p, mq) = 1$, so $g \in F^*$; hence $(G/F^*)_q = 0$. From this it readily follows (see for example [5], Theorem 6) that F^* is q^λ-pure in G for any ordinal λ . In particular, F^* is (α_n)-pure in G .

We now need only to show that $F^* \notin \mathfrak{R}$. Suppose $lx = = 0$ for some $l \in Z$, $x \in F^*$, $x \neq 0$. Let

$$mx = tf_0 = tp^k a + tc ,$$

where $(p, m) = 1$. Then

$$ltp^k a + ltc = mlx = 0 \ ,$$

so $ltp^k a = 0$. If $t = 0$, then $mx = 0$, so $x \in \in \langle c \rangle$ and $p \mid m$, which is impossible. Hence $l = 0$, so F^* is torsion-free. To belong to $\mathbf{R}$, F^* must therefore be $\mathbf{P}$ -divisible; in particular, $pF^* = F^*$. Assuming this, we have $p^k y = p^k a + c = f_0$ for some $y \in F^*$. Thus for some $i, j \in \mathbf{Z}$ with $(p, i) = 1$, we have $iy = = j(p^k a + c)$, so that

$$ip^k a + ic = p^k j(p^k a + c)$$

which implies that $p^k \mid i$, contrary to what is known about i. The group F^* therefore does not belong to $\mathbf{R}$, which accordingly is not (α_m) -pure-hereditary.

This completes the proof.

The example used in part V of the proof just given owes something to the proof of Lemma 4 of [7].

One other special case is worthy of separate mention.

<u>Corollary 3.</u> A radical class is neat-hereditary if and only if it is either the class of all $\mathbf{P}$ -divisible groups for some set $\mathbf{P}$ of primes or a subclass of $\mathbf{T}$.

R e f e r e n c e s

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