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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

15,4 (1974)

A WEAKLY PSEUDOCOMPACT SUBSPACE OF BANACH SPACE IS WEAKLY COMPACT

David PREISS, Petr SIMON, Praha

Abstract: The aim of the present paper is to prove the theorem mentioned in the title. Beside this, a short and direct proof of an equivalence between the Lindenstrauss' characterization of an Eberlein compact and the Rosenthal's one is given.

Key words and phrases: Eberlein compact, Čech-Stone compactification, pseudocompact space, cozero set, Banach space.

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1. Definition [L] . A compact Hausdorff space X is called an Eberlein compact, if X can be embedded into some cube $[0,1]^{\Gamma}$ in such a way that for each $x \in X$ and for each real $\varepsilon > 0$ the set $\{\gamma \in \Gamma \mid x(\gamma) > \varepsilon\}$ is finite.

2. Theorem [R] . A compact Hausdorff space X is an Eberlein compact if and only if X admits a σ-point-finite family of cozero sets weakly separating points of X .

Proof. Necessity: Suppose $X \subset [0,1]^{\Gamma}$ be an Eberlein compact. Let us define $C_{j,\gamma,n} = \pi_{\gamma}^{-1}\,]\,j-2/n, j/n\,[\,\cap X$, where π_{γ} is the γ-th projection, $\mathcal{C}_n = \{C_{j,\gamma,n} \mid j = 3,4,\dots,n+1, \gamma \in \Gamma\}$, $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n = 2,3,4,\dots\}$.

Clearly each $C_{j,\gamma,n}$ is a cozero set.

The family $\mathcal{C}_n$ is point-finite: Suppose contrary. If $x \in X$ belongs to infinitely many $C_{j,\gamma,n} \in \mathcal{C}_n$, then there must be infinitely many indices γ such that $x(\gamma) > 1/n$, which is a contradiction.

The family $\mathcal{C}$ weakly separates points of X: Let x, $y \in X$, $x \neq y$. Then for some $\gamma \in \Gamma$, $x(\gamma) \neq y(\gamma)$. Assume $x(\gamma) < y(\gamma)$. There exists a natural n such that the following two inequalities take place: $|y(\gamma) - x(\gamma)| > 2/n$, $y(\gamma) > 1/n$. Now it is obvious that for some natural j the point y belongs to $C_{j,\gamma,n}$ and $x \notin C_{j,\gamma,n}$.

Sufficiency: Let $\mathcal{C} = \bigcup \{\mathcal{C}_n \mid n \in \omega\}$ be the system of cozero sets weakly separating the points of X with each $\mathcal{C}_n$ point-finite. For every $C \in \mathcal{C}_n$ there is a continuous real-valued function $f_C : X \longrightarrow [0,1]$ such that $f_C[X] \subset [0, 1/n]$, $C = f_C^{-1}[\,]0,1]\,]$, $X - C = f_C^{-1}[0]$.

Define $\psi : X \longrightarrow [0,1]^{\mathcal{C}}$ by the rule $\psi(x) = \{f_C(x) \mid C \in \mathcal{C}\}$. Then the mapping ψ is an embedding, since ψ is continuous (all f_C are continuous), one-to-one ($\mathcal{C}$ weakly separates points), and both domain and range of ψ are compact Hausdorff spaces.

Let $y = \psi(x)$, $n \geq 1$ be a natural number. The system $\mathcal{B} = \{C \mid C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_n,\ x \in \mathcal{C}\}$ is finite, because all $\mathcal{C}_i$ are point-finite; let $C \in \mathcal{C} - \mathcal{B}$. If $C \in \mathcal{C}_i$ for $i \leq n$, then $y(C) = f_C(x) = 0$, if $C \in \mathcal{C}_i$ for $i > n$, then $y(C) = f_C(x) \leq 1/i < 1/n$.

3. Proposition. Let $X \subset [0,1]^{\Gamma}$ be an Eberlein compact, $x \in X$. Then there exists an embedding ψ of X in-

to some cube $[0,1]^{\Delta}$ such that $\psi[X]$ has the property needed in Definition 1 and $\psi(x)(\sigma) = 0$ for all $\sigma \in \Delta$.

Proof. For $\Delta = \Gamma \times \{0,1\}$ let us define the embedding ψ by the following: $\psi(y) = z$, where $z(\gamma, 0) = \mathrm{Max}(y(\gamma) - x(\gamma), 0)$, $z(\gamma, 1) = \mathrm{Max}(x(\gamma) - y(\gamma), 0)$. Using the obvious inequality $\psi(y)(\gamma, i) \leq x(\gamma) + y(\gamma)$, $i = 0, 1$ one can easily check that $\psi[X]$ has all the desired properties.

4. Lemma. Let $X \subset [0,1]^{\Gamma}$ be an Eberlein compact, $\emptyset \neq A \subset X$, $\varepsilon > 0$. Then there exists a finite set $F(A, \varepsilon)$ with the following properties:

(i) The set $\{x \in A \mid \gamma \in F(A,\varepsilon) \Longrightarrow x(\gamma) > \varepsilon\}$ is non-void,

(ii) If for $x \in A$, $x(\gamma) > \varepsilon$ for all $\gamma \in F(A, \varepsilon)$, then $x(\gamma) \leq \varepsilon$ whenever $\gamma \notin F(A, \varepsilon)$.

Proof. By the method of contradiction, suppose that each finite $F \subset \Gamma$ satisfying (i) does not satisfy (ii). Thus we can inductively construct a strictly increasing sequence $F_1 \subsetneqq F_2 \subsetneqq F_3 \subsetneqq \dots$ of finite subsets of Γ, such that for each n the set $\{x \in A \mid \gamma \in F_n \Longrightarrow x(\gamma) > \varepsilon\}$ is non-void.

Setting $K_n = \{x \in [0,1]^{\Gamma} \mid \gamma \in F_n \Longrightarrow x(\gamma) \geq \varepsilon\}$, we obtain that $K_n \cap X \neq \emptyset$ for all $n \in \omega$, and since K_n is a decreasing sequence of compact subsets of $[0,1]^{\Gamma}$, X is compact, there is a point $y \in X \cap \bigcap \{K_n \mid n \in \omega\}$. But then $y(\gamma) \geq \varepsilon$ for infinitely many indices γ of Γ, which is a contradiction.

5. <u>Theorem</u>. Let X be an Eberlein compact, x non-isolated point of X. Then there exists a sequence $\{U_m \mid m \in \omega\}$ of open sets in X, which converges to x.

<u>Proof</u>. According to Proposition 3 we may assume that $X \subset [0,1]^\Gamma$ and that $x(\gamma) = 0$ for all $\gamma \in \Gamma$.

By induction we shall define for all natural n finite sets of indices F_n, open neighbourhoods V_n of x and open subsets U_n of X.

Define $F_1 = \emptyset$, $U_1 = V_1 = X$.

Let $n \in \omega$ and suppose that F_k, U_k and V_k has been defined for all $k = 1, 2, \dots, n-1$. Define

$V_n = \{y \in X \mid \gamma \in \bigcup_{i=1}^{n-1} F_i \Longrightarrow y(\gamma) < 1/n\}$. By the lemma, there exists an $F_n = F(V_n, 1/n) \subset \Gamma$ with properties (i), (ii).

Clearly $F_n \cap \bigcup_{i=1}^{n-1} F_i = \emptyset$. Define $U_n = \{y \in V_n \mid \gamma \in F_n \Longrightarrow$ $\Longrightarrow y(\gamma) > 1/n\}$. Obviously U_n are open. It remains to prove that U_n converge to x.

Let W be a neighbourhood of x. Then there exist a natural number m and a finite subset D of indices such that

$W \supset \{y \in X \mid \gamma \in D \Longrightarrow y(\gamma) < 1/m\} = W_0$

Since D is finite and since F_n are disjoint, there exists an $n \in N$, $n > m$ such that $F_k \cap D = \emptyset$ whenever $k \geq n$. Let $k \geq n$, $y \in U_k$, $\gamma \in D$. Since $y \in V_k$ and since $\gamma \notin F_k$, we may apply (ii) from Lemma 4 to obtain that $y(\gamma) \leq 1/k \leq 1/n < 1/m$. Thus $y \in W_0 \subset W$.

6. Corollary. A pseudocompact subspace of an Eberlein compact is closed and hence it is an Eberlein compact, too.

Proof. Supose, on the contrary, that $Y \subset X$ is pseudocompact, $Y \neq X$ and $\overline{Y} = X$. By Theorem 5, let $\{\mathcal{U}_m\}$ be a sequence of open sets converging to a point $x \in X - Y$; choose a point $x_m \in \mathcal{U}_m \cap Y$ and let $\mathcal{U} = X - \{x_m \mid m \in N\}$. Then $\{\mathcal{U}_m \cap Y \mid m \in N\} \cup \{\mathcal{U} \cap Y\}$ is an open infinite locally finite cover of Y - a contradiction with pseudocompactness of Y.

7. Corollary. Let X be an Eberlein compact, $Y \subset X$, $Y \neq X$. Then $\beta Y \neq X$.

Proof. If $X = \beta Y$, $Y \subsetneqq X$, then $X = \beta(X - \{x\})$ for any $x \in X - Y$. This implies that $X - \{x\}$ is pseudocompact, which contradicts to Corollary 6.

8. Remark. Now, the theorem stated in the title is an easy consequence of the theorem of Pták:

[P, p.281]. A weak closure of a weakly pseudocompact subspace of Banach space is weakly compact,

of the theorem of Amir and Lindenstrauss:

[AL, p.36], [L, p.236]. Eberlein compacts are exactly the topological spaces which are homeomorphic to weakly compact sets in Banach spaces,

and of Corollary 6.

9. Example. It is not true in an Eberlein compact that to every sequence $\{x_m\}$ of points converging to a point x, one can find a sequence $\{\mathcal{U}_m\}$ of open sets, each $\mathcal{U}_m$

being a neighbourhood of x_m, converging to a point x too, though it may seem to be a natural strengthening of Theorem 5. A counterexample is easy:

Let P be a one-point compactification of an uncountable discrete set, p the non-isolated point of P, and let q be some other point of P. The space $X = P^{\omega_0}$, as a countable product of Eberlein compacts, is an Eberlein compact, too. Let us denote x_m the point of X, whose first m coordinates equal to p, and all other to q. The sequence $\{x_m\}$ converges to a point x, whose all coordinates equal to p.

Now, let U_m be a neighbourhood of x_m; because of uncountable cardinality of P there must be a point $r \in \bigcap\{\pi_1[U_m] \mid m \in N\}$. Thus, whenever U is a neighbourhood of x such that $U \subset \pi_1^{-1}[P-(r)]$, then $U_m - U$ is non-empty for every natural m.

10. <u>Remark</u>. A topological space X is called to be Fréchet, if for each $A \subset X$, $x \in \overline{A} - A$, there is a sequence $\{x_m\}$ of points of A converging to x.

Let us define a topological space X to be strongly Fréchet, if for each $A \subset X$, $x \in \overline{A} - A$ there is a sequence $\{U_m\}$ of sets relatively open in A converging to x.

An Eberlein compact is Fréchet and, according to Theorem 5, strongly Fréchet. There exists a Fréchet, strongly Fréchet compact Hausdorff space which is not an Eberlein compact. But we do not know if each Fréchet, compact Hausdorff space is strongly Fréchet, nor we know any counterexample to this statement.

R e f e r e n c e s

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