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PRERADICALS AND CHANGE OF RINGS

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Abstract: This paper investigates the problem of lifting preradicals under change of rings. The first part determines the relationship between the preradicals for R-modules and S-modules provided that there is given a ring homomorphism $f: R \longrightarrow S$. In particular, there is given a full description of this relationship in case that f is either onto or ker f is a ring direct summand of R . The final part of the paper establishes a one-to-one correspondence between the preradicals of Morita equivalent rings and this correspondence preserves all the elementary properties of preradicals.

Key words: Preradical, ring homomorphism, Morita equivalence.

AMS: 18E40 Ref. Ž.: 2.726.4

The origin of this paper reaches back to [5], where the lifting of idempotent radicals under suitable change of rings was investigated. We decided to study the problem in a greater scope with respect to the general theory of preradicals. Throughout this paper, a ring stands for an associative ring with identity. Let R be a ring. A preradical r for the category of left R-modules, R-mod , is a subfunctor of identity. Suppose that $M \in$ R-mod is chosen arbitrarily and $N \subseteq M$ is a submodule. The preradical r is said to be either idempotent or a radical or hereditary or cohereditary if

$r(r(M)) = r(M)$ or $r(M/r(M)) = 0$ or $r(N) = N \cap r(M)$ or $r(M/N) = (r(M) + N)/N$, respectively, for all $M \in R$-mod and every submodule N of M . We shall say that $T_r = \{ M \mid M \in$ $\in R\text{-mod}, r(M) = M \}$ is r-torsion class and $F_r = \{ M \mid M \in$ $\in R\text{-mod}, r(M) = 0 \}$ is r-torsion free class. If every R-module (R-injective, R-projective) M splits in r (i.e. $r(M)$ is a direct summand of M) then r is said to be splitting (stable, costable). The zero and the identity preradicals will be denoted by zer and id , respectively. Let r and t be two preradicals for R-mod. Then we can define the preradicals $r \circ t$ and $r \vartriangle t$ by $r \circ t(M) = r(t(m))$ and $r \vartriangle t(M) / r(M) = t(M / r(M))$.

Further, $r \subseteq t$ if $r(M) \subseteq t(M)$, for every $M \in R$-mod. Consequently, if $\{r_i\}$, $i \in I$, is a family of preradicals for R-mod we can define the preradicals $\bigcap r_i$ and $\sum r_i$ by $(\bigcap r_i)(M) = \bigcap r_i(M)$ and $(\sum r_i)(M) = \quad r_i(M)$. The idempotent core (radical closure) $\bar{r}(\tilde{r})$ of the preradical r is the largest idempotent preradical (the least radical) contained in r (containing r) defined as follows.

For all $M \in R$-mod , put $\bar{r}(M) = \sum N (\tilde{r}(M) = \bigcap N)$, where N runs through all the submodules N of M with $N \in T_r$ $((M/N) \in F_r)$. In a due course we can define the hereditary closure (cohereditary core) of r by $h(r)(M) = M \cap r(E(M))$ $(ch(r)(M) = r(R). M)$, where $E(M)$ is the injective hull of M . If the ring R is a ring direct sum of its subrings R_i , $i = 1,2$, we shall denote it by $R = E_1 \dot{+} R_2$ to distinguish it from the direct sum as submodules. We shall frequently use the following assertion. If $I \subseteq R$ is a two sided ideal then

R/I is a flat right R-module iff $x \in I.x$, for each $x \in I$. The technicalities about preradicals which are prerequisite for this paper can be found scattered elsewhere. In particular, a systematical treatment of preradicals is presented in [1],[2],[3],[4]. As for the notation and the definitions we refer to [6]. In what follows we shall frequently deal with different rings and therefore we are going to distinguish the modules over them. Namely, the symbols for the modules will be supplied with an index denoting the generating ring. Let $f: R \longrightarrow S$ be a ring homomorphism. Then every ${}_SM \in$ S-mod is naturally an R-module in the structure defined by $r.m = f(r).m$. We shall denote it by ${}_R({}_SM)$. If $A \subseteq$ R-mod then $A \cap$ S-mod is the class of all ${}_RM \in A$ which can be viewed as S-modules ${}_SM$ such that ${}_R({}_SM)$ has the same R-module structure as ${}_RM$. Finally, the R-homomorphism ${}_RM \longrightarrow S \otimes {}_RM$ given by $m \longmapsto 1 \otimes m$ will be denoted by j_M . The last section of the paper enables us to establish p-equivalence of rings which seems to be the desired tool for structural classification of rings.

2. <u>Lifting of preradicals</u>. Let $f: R \longrightarrow S$ be a ring homomorphism and r be a preradical for R-mod. For all $M \in$ S-mod we define $f[r](M) = Sr({}_RM)$ and $f\{r\}(M) = \{m \mid m \in r({}_RM)$ and $am \in r({}_RM)$ for all $a \in S\}$.

2.1. <u>Proposition</u>. (i) $f[r]$ and $f\{r\}$ are preradicals for S-mod and $f\{r\} \subseteq f[r]$.

(ii) $T_r \cap$ S-mod $= T_{f\{r\}}$ and $F_r \cap$ S-mod $= F_{f[r]}$.

(iii) If r is either idempotent or cohereditary then $f[r]$

is so, respectively.

(iv) If r is either a radical or hereditary then $f\{r\}$ is so, resp.

(v) $\overline{f\{r\}} \subseteq f\{\bar{r}\}$ and $f[\bar{r}] \subseteq \overline{f[r]}$.

(vi) $\widetilde{f\{r\}} \subseteq f\{\tilde{r}\}$ and $f[\tilde{r}] \subseteq \widetilde{f[r]}$.

(vii) $f[ch(r)] \subseteq ch(f[r])$.

(viii) If the left R-module S is projective, then $f[ch(r)] = ch(f[r])$.

(ix) $h(f\{r\}) \subseteq f\{h(r)\}$.

Proof. (i) and (ii) are obvious.

(iii) Let r be idempotent. Then $r(f[r](M)) = r(Sr({}_RM)) = r({}_RM)$, and consequently $f[r]$ is idempotent. Now, if r is cohereditary and $N \subseteq M$ is an S-submodule, then $Sr(({}_R(M/n)) = (Sr({}_RM) + N)/N = (f[r](M) + N)/N$.

(iv) Similarly as for (iii).

(v) and (vi) follow by (ii), (iii) and (iv).

(vii) and (viii) by (iii) and (ix) by (iv).

The following two propositions are purely of technical character and hence the proofs are omitted.

2.2. Proposition. The operator $r \longmapsto f\{r\}$ preserves the intersection of preradicals and the operator $r \longmapsto f[r]$ preserves the sum.

2.3. Proposition. Let r and s be two preradicals for R-mod. Then

(i) $f\{r\} \circ f\{s\} \subseteq f\{r \circ s\} \subseteq f[r \circ s] \subseteq f[r] \circ f[s]$.

(ii) $f\{r\} \triangle f\{s\} \subseteq f\{r \triangle s\} \subseteq f[r \triangle s] \subseteq f[r] \triangle f[s]$.

(iii) $\overline{f\{r\} \cap f\{s\}} = \overline{f\{r \circ s\}} = \overline{f\{r\} \circ f\{s\}} = \overline{f\{r \cap s\}}$.

(iv) $\widetilde{f[r] + f[s]} = \widetilde{f[r \triangle s]} = \widetilde{f[r] \triangle f[s]} = \widetilde{f[r + s]}$.

(v) If r is hereditary then $f\{r \circ s\} = f\{r\} \circ f\{s\}$.

(vi) If r is cohereditary then $f[s \triangle r] = f[s] \triangle f[r]$.

2.4. <u>Definition</u>. The ring homomorphism f is called delightful, if $f[r] = f\{r\}$ for every preradical r on R-mod.

2.5. <u>Proposition</u>. If the ring homomorphism f satisfies at least one of the following conditions, then it is delightful.

(i) f is an onto homomorphism.

(ii) The ring S is commutative.

(iii) $\operatorname{Im} f$ is contained in the center of S .

(iv) The R-bimodule S is isomorphic to a free R-module.

(v) There is a set X is generators of the R-module S such that $f(a)x = xf(a)$ for all $x \in X$ and $a \in R$.

<u>Proof</u>. As it is easy to see, we may consider only the fifth condition. With respect to the hypothesis, the map $m \longmapsto xm$ is an R-endomorphism of ${}_RM$ for all $x \in X$ and $M \in$ S-mod . Hence $xr({}_RM) \subseteq r({}_RM)$, and consequently $yr({}_RM) \subseteq r({}_RM)$ for every $y \in S$ since X is a set of generators of ${}_RS$. Thus $r({}_RM)$ is an S-submodule of M .

The following two propositions follow immediately from 2.1 and 2.2.

2.6. Proposition. Let s be a preradical for S-mod and $A(B)$ be the class of all the preradicals r for R-mod with $f\{r\} = s$ $(f[r] = s)$. Suppose that $A(B)$ is non-empty. Then

(i) $r_0 = \bigcap_{r \in A} r$ $(r_1 = \sum_{r \in B} r)$ is the least (largest) element in $A(B)$.

(ii) If s is idempotent (a radical) and $r \in A$ $(r \in B)$ then $\bar{r} \in A$ $(\tilde{r} \in B)$.

(iii) If s is idempotent (a radical) then r_0 is idempotent (r_1 is a radical).

(iv) If s is cohereditary, ${}_RS$ is projective and $r \in B$ then $ch(r) \in B$.

2.7. Proposition. Let s be a radical (an idempotent preradical) for S-mod and $C(D)$ be the class of all the radicals (idempotent preradicals) r for R-mod with $f\{r\} = s$ $(f[r] = s)$. Suppose that $C(D)$ is non-empty. Then

(i) $r^0 = \bigcap_{r \in C} r$ $(r^1 = \sum_{r \in D} r)$ is the least (largest) element in $C(D)$.

(ii) If s is idempotent (a radical) then r^0 (r^1) is an idempotent radical.

(iii) If s is idempotent (a radical) and $r \in C$ $(r \in D)$ then $\bar{r} \in C$ $(\tilde{r} \in D)$.

2.8. Proposition. Let r be a preradical for R-mod and s be a preradical for S-mod. For all $M \in$ R-mod define $[s]f(M) = \{m \mid m \in M$ and $1 \otimes m \in s({}_SS_R \otimes_R M)\}$. Then

(i) $[s]f$ is a preradical for R-mod and $f[[s]f] \subseteq s$.

(ii) $r \subseteq [f[r]]f$ and $f[r] = f[[f[r]]f]$.

(iii) $[s]f = [f[[s]f]]f$.

<u>Proof</u>. (i) is obvious.

(ii) $j_M(r({}_R M)) \subseteq j_M({}_R M) \cap f[r](S \otimes {}_R M)$, and so $r({}_R M) \subseteq [f[r]]f({}_R M)$.

Since $r \subseteq [f[r]]f$, $f[r] \subseteq f[[f[r]]f] \subseteq f[r]$.

(iii) is clear by (i) .

2.9. <u>Proposition</u>. Let r be a preradical for R-mod and s be a preradical for S-mod. For all $M \in$ R-mod define $\{s\}f(M) = \{m \mid m \in M$ and $m = p(1)$ for some $p \in s(\mathrm{Hom}_R({}_R S_S , {}_R M))\}$. Then

(i) $\{s\}f$ is a preradical for R-mod and $s \subseteq f\{\{s\}f\}$.

(ii) $\{f\{r\}\}f \subseteq r$ and $f\{r\} = f\{\{f\{r\}\}f\}$.

(iii) $\{s\}f = \{f\{\{s\}f\}\}f$.

<u>Proof</u>. The proof runs without principal difficulties.

2.10. <u>Proposition</u>. Let s and t be two preradicals for S-mod. Then

(i) $[s]f \circ [t]f \subseteq [s \circ t]f$.

(ii) If the right R-module S_R is flat and s is hereditary, then $[s]f \circ [t]f = [s \circ t]f$.

(iii) $[s]f \triangle [t]f \subseteq [s \triangle t]f$.

(iv) $\{s \circ t\}f \subseteq \{s\}f \circ \{t\}f$.

(v) $\{s \triangle t\}f \subseteq \{s\}f \triangle \{t\}f$.

(vi) If the left R-module ${}_R S$ is projective and t is cohereditary, then $\{s \triangle t\}f = \{s\}f \triangle \{t\}f$.

<u>Proof</u>. (i) Let $M \in$ R-mod and $N = [t]f(M)$. If

$n \in [s]\,f(N)$ then $1 \otimes n \in s(t(S \otimes_R M))$.

(ii) follows from (i).

(iii) Let $N = [s]\,f(M)$. The assertion is an easy consequence of the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "d"] \arrow[d, "j_N"] & M \arrow[r, "q"] \arrow[d, "j_M"] & M/N \arrow[r] \arrow[d, "j_{M/N}"] & 0 \\
 & S \otimes N \arrow[r, "1 \otimes d"] & S \otimes M \arrow[r, "1 \otimes q"] \arrow[dr, "w"] & S \otimes M/N \arrow[r] \arrow[d, "z"] & 0 \\
 & & & (S \otimes M)/s(S \otimes M) &
\end{tikzcd}$$

where q, w and z are natural.

The proofs of (iv),(v) and (vi) are almost dual to those of (i),(ii) and (iii), respectively.

2.11. <u>Corollary</u>. (i) If s is a radical then $[s]\,f$ is so.

(ii) If S_R is flat and s is hereditary then $[s]\,f$ is hereditary.

(iii) If s is idempotent then $\{s\}\,f$ is so.

(iv) If ${}_RS$ is projective and s is cohereditary then $\{s\}\,f$ is cohereditary.

<u>Proof</u>. (i) By 2.10(iii), $[s]\,f \vartriangle [s]\,f \subseteq [s \vartriangle s]\,f = [s]\,f$.
(ii) By 2.10 (ii), $[s]\,f$ is idempotent. Further, $[s]\,f$ -torsion modules are closed under submodules, and so $[s]\,f$ is hereditary.
(iii) and (iv) can be proved similarly.
The following proposition is clear.

2.12. Proposition. The operator $s \longmapsto [s]f$ preserves the intersection of preradicals and the operator $s \longrightarrow \{s\}f$ preserves the sum.

2.13. Definition. The ring homomorphism f is called codelightful, if $[s]f = \{s\}f$ for every preradical s on S-mod.

2.14. Example. Let R be a subring in S and f be the canonical inclusion. Suppose that there is a set $X = \{x_1, \dots, x_n\} \subseteq S$ such that X is a set of free generators of S over R, the elements from X are orthogonal idempotents and $1 = \sum x_i$. Then f is delightful and codelightful.

2.15. Example. The canonical embedding of integers into rational numbers is delightful but not codelightful.

2.16. Problem. To say more about delightful and codelightful homomorphisms, in particular, whether the imbedding of a ring into the maximal quotient ring is delightful.

2.17. Proposition. If S_R is flat and r is a preradical for R-mod, then $f\{h(r)\} = h(f\{r\})$.

Proof. Taking into account the flatness of S_R we see that every S-injective is R-injective, and hence the assertion easily follows by 2.1 (ix).

3. Onto ring homomorphisms. In the following $f: R \twoheadrightarrow S$ denotes an onto ring homomorphism. Further, for all $M \in$ R-mod we define $u(M) = \mathrm{Ker}\, f.M$ and $v(M) = \{m \mid m \in M,\ \mathrm{Ker}\, f.m = 0\}$.

3.1. Proposition. Let r and t be two preradicals for R-mod . Then the following are equivalent:

(i) $r \circ v = t \circ v$.

(ii) $f[r] = f[t]$.

(iii) $u \vartriangle r = u \vartriangle t$.

Proof. The proof needs only a tedious checking.

3.2. Proposition. Let r be a preradical for R-mod such that $\mathrm{Ker}\, f \subseteq r(R)$. Then

(i) $u \vartriangle \tilde{r} = \tilde{r} = \tilde{r} \vartriangle u$, $u \subseteq r$ and $r \vartriangle u = r$.

(ii) $f[\mathrm{ch}(\tilde{r})] = \mathrm{ch}(f[\tilde{r}])$.

(iii) $f[r]$ is a radical iff $u \vartriangle r$ is so. In such a case, $\tilde{r} = u \vartriangle r = r \vartriangle r$.

(iv) If $\mathrm{Ker}\, f$ is idempotent then $f[r]$ is idempotent iff $u \vartriangle r$ is so. In such a case, $\tilde{r}$ is idempotent.

(v) $f[r]$ is cohereditary iff $u \vartriangle r$ is so.

(vi) If S_R is flat then $f[r]$ is hereditary iff $u \vartriangle r$ is so. In such a case, $\tilde{r}$ is hereditary.

Proof. (i) and (ii) are immediate.

(iii) If $u \vartriangle r$ is a radical then $f[u \vartriangle r] = f[u] \vartriangle f[r] = \mathrm{zer} \vartriangle f[r] = f[r]$ by 2.5 and 2.3(ii), and we are ready to use 2.1(iv). Conversely, if $f[r]$ is a radical, then there is a radical t for R-mod such that $f[t] = f[r]$ (see 2.7 (ii)). It is easy to check that 3.1 yields $u \vartriangle r = u \vartriangle t$ is a radical. Further, since $r \subseteq u \vartriangle r \subseteq u \vartriangle \tilde{r} = \tilde{r}$, $\tilde{r} = u \vartriangle r$ and $\tilde{r} = r \vartriangle \tilde{r} = r \vartriangle (u \vartriangle r) = (r \vartriangle u) \vartriangle r = r \vartriangle r$.

(iv) We can proceed similarly as in (iii), using 2.1(iii).

(v) Use (iii) and 2.1(iii).

(vi) Observing 2.5 and 2.17 the result easily follows.

The following proposition is dual to the preceding one and the proof is left to the reader.

3.3. __Proposition.__ Let r be preradical for R-mod such that $r \subseteq v$. Then

(i) $\bar{r} \circ v = \bar{r} = v \circ \bar{r}$ and $v \circ r = r$.

(ii) $f[h(\bar{r})] = h(f[\bar{r}])$.

(iii) $f[r]$ is idempotent iff $r \circ v$ is so. In such a case, $\bar{r} = r \circ v = r \circ r$.

(iv) If $\operatorname{Ker} f$ is idempotent, then $f[r]$ is a radical iff $r \circ v$ is so. In such a case, $\bar{r}$ is a radical.

(v) $f[r]$ is hereditary iff $r \circ v$ is so.

(vi) If ${}_RS$ is projective then $f[r]$ is cohereditary iff $r \circ v$ is so. In this case, $\bar{r}$ is cohereditary, provided R is left perfect.

3.4. __Proposition.__ Let r be a preradical for R-mod. Then

(i) If r is costable cohereditary then $f[r]$ is so.

(ii) If r is a costable radical and $\operatorname{Ker} f \subseteq r(R)$ then $f[r]$ is costable.

(iii) If ${}_RS$ is projective and r is costable then $f[r]$ is so.

(iv) If r is stable hereditary then $f[r]$ is so.

(v) If r is stable idempotent and $\operatorname{Ker} f . r(M) = 0$ for all $M \in$ R-mod then $f[r]$ is stable.

(vi) If S_R is flat and r is stable then $f[r]$ is so.

(vii) If r is splitting then $f[r]$ is so.

__Proof.__ (i),(iii) and (vi) are easy.

(ii) follows by (i) and 3.2(ii).

(iv) As one may verify, a hereditary preradical t is stable iff $I \cap K = L$ for some left ideal $I \neq L$ whenever $L \subseteq K \subsetneqq R$ are left ideals with $K/L = t(R/L)$. Now we can apply 2.5.
(v) follows by (iv) and 3.3(ii) and (vii) is obvious.

3.5. <u>Proposition</u>. Let r be a preradical for R-mod and s be a preradical for S-mod. Then

(i) $\{s\}f \subseteq v$, $\{s\}f \circ v = \{s\}f = v \circ \{s\}f$.

(ii) $f\{\{s\}f\} = f[\{s\}f] = s$ and $\{s\}f(M) = s({}_{S}v(M)))$ for all $M \in$ R-mod.

(iii) $\{s\}f$ is idempotent (hereditary) iff s is so.

(iv) If $\operatorname{Ker} f$ is a left direct summand then $\{s\}f$ is cohereditary iff s is so. In such a case, $\{ch(s)\}f =$
$= ch(\{s\}f)$.

(v) $\{\bar{s}\}f = \overline{\{s\}f}$ and $h(\{s\}f) = \{h(s)\}f$.

(vi) If $\operatorname{Ker} f$ is idempotent then $\{s\}f$ is a radical iff s is so. In such a case, $\{\bar{s}\}f = \widetilde{\{s\}f}$.

(vii) $\{f\{r\}\}f = \{f[r]\}f = r \circ v$.

(viii) If S_R is flat then $\{s\}f$ is stable iff s is so.

(ix) $u \subseteq [s]f$, $[s]f \vartriangle u = [s]f = u \vartriangle [s]f$.

(x) $f[[s]f] = s$ and $[s]f(M/u(M)) = s(M/u(M))$ for all $M \in$ R-mod.

(xi) $[s]f$ is a radical (cohereditary) iff s is so.

(xii) $[\tilde{s}]f = \widetilde{[s]f}$ and $ch([s]f) = [ch(s)]f$.

(xiii) If $\operatorname{Ker} f$ is idempotent then $[s]f$ is idempotent iff s is so. In such a case, $[\bar{s}]f = \overline{[s]f}$.

(xiv) If S_R is flat then $[s]f$ is hereditary iff s is so. In such a case, $h([s]f) = [h(s)]f$.

(xv) If $\operatorname{Ker} f$ is a left direct summand then $[s]f$ is costable iff s is so.

(xvi) $[f[r]]f = u \circ r$.

Proof. (i) and (ii) are evident.

(iii) Use (i),(ii) and 3.3(iii) and (v).

(iv) By (i),(ii) and 3.3(vi).

(v) According to (iii), $\{\bar{s}\}f \subseteq w = \overline{\{s\}f}$ and $h(\{s\}f) = t \subseteq \{h(s)\}f$.

On the other hand, $f[w] = \bar{s}$ and $f[t] = h(s)$. Hence $\{\bar{s}\}f = \{\bar{s}\}f \circ v = w \circ v = \overline{\{s\}f} \circ v = \overline{\{s\}f} = w$ and $\{h(s)\}f = \{h(s)\}f \circ v = t \circ v \subseteq t$.

(vi) By 3.3(iv).

(vii) With respect to (ii), $f\{\{f\{r\}\}f\} = f\{r\}$ and therefore $r \circ v = \{f\{r\}\}f \circ v = \{f\{r\}\}f$.

(viii) If $\{s\}f$ is stable then s is stable by (ii) and 3.4(vi).

Conversely, if s is stable then $\{s\}f$ is stable by (ii) since v is stable.

The remaining points of the proof are dual to some of those preceding, respectively.

The following corollary is an easy consequence of 3.3 and 3.5.

3.6. Corollary. Let A be the class of all the preradicals r for R-mod such that $f[r] = s$. Then

(i) $[s]f$ is the largest element in A .

(ii) $[s]f = u \vartriangle r$ for all $r \in A$.

(iii) If s is a (cohereditary) radical then $[s]f$ is so.

(iv) If s is idempotent then $\overline{[s]f} \in A$.

(v) If s is an idempotent radical then $\overline{[s]f}$ is so and $\overline{[s]f} \in A$.

(vi) If R is left perfect and s is idempotent cohereditary then $\overline{[s]f}$ is so and $\overline{[s]f} \in A$.

(vii) $\{s\}f$ is the least element in A .

(viii) $\{s\}f = r \circ v$ for all $r \in A$.

(ix) If s is idempotent (hereditary) then $\{s\}f$ is so.

(x) If s is a radical then $\widetilde{\{s\}f} \in A$.

(xi) If s is an idempotent radical then $\widetilde{\{s\}f}$ is so and $\widetilde{\{s\}f} \in A$.

(xii) If s is a hereditary radical then $\widetilde{\{s\}f}$ is so and $\widetilde{\{s\}f} \in A$.

4. Ring direct sums

4.1. Definition. Let r be a preradical for R-mod. An elementary property of r is any of the following:

(i) r is a radical.

(ii) r is idempotent.

(iii) r is hereditary.

(iv) r is cohereditary.

(v) r is stable.

(vi) r is costable.

(vii) r is splitting.

(viii) $r = id$.

(ix) $r = zer$.

(x) $r(R) = 0$.

Further, the first seven elementary properties will be called superelementary.

4.2. <u>Proposition</u>. Let $f: R \longrightarrow S$ be an onto ring homomorphism, $\operatorname{Ker} f$ be a ring direct summand in R, r be a preradical for R-mod and s be a preradical for S-mod. Then

(i) If r possesses an elementary property then $f[r]$ has the same one,

(ii) $[s]f$ possesses a superelementary property iff s does so.

(iii) $\{s\}f$ possesses a superelementary property iff s does so.

<u>Proof</u>. By 2.1(iii), (iv), 3.4, and 3.5.

Let $R = R_1 \dotplus R_2 \dotplus \dots \dotplus R_n$ be a ring direct sum, $p_i: R \longrightarrow R_i$ be the canonical projection and $A = \{(r_1, \dots, r_n) \mid r_i$ is a preradical for R_i-mod $\}$. An element from A is said to have an elementary property if each of its components possesses the same one. Further, the intersection, sum, inclusion and the operators $\circ$, Δ can be defined on A in obvious manner.

4.3. <u>Proposition</u>. There is a one-to-one correspondence between preradicals for R-mod and the class A given by

$$r \longmapsto (p_1[r], \dots, p_n[r]) \quad \text{and}$$

$$(r_1, \dots, r_n) \longmapsto \sum \{r_i\} p_i = \bigcap [r_i] p_i .$$

Moreover, this correspondence preserves all the elementary properties, intersections, sums, inclusions and the operators $\circ$, Δ.

<u>Proof</u>. The proposition can be verified directly but one can also use the preceding theory for the convenience.

5. Lifting of preradicals and Morita equivalent rings

5.1. Definition. We shall say that two rings R and S are p-equivalent if there is a one-to-one correspondence between preradicals for R-mod and S-mod, preserving the operators $\circ$ and Δ , inclusions, sums and intersections of preradicals and all the elementary properties, in both directions.

5.2 Proposition. Morita equivalent rings are p-equivalent.

Proof. Let R and S be Morita equivalent rings, F: R-mod $\longrightarrow$ S-mod and G: S-mod $\longrightarrow$ R-mod be the functors which represent this categorical equivalence and $f: GF \longrightarrow 1_{R\text{-mod}}$, $g: FG \longrightarrow 1_{S\text{-mod}}$ be the corresponding functorial isomorphism. If r and s are preradicals for R-mod and S-mod , respectively, for all $M \in$ R-mod and $N \in$ S-mod we define ${}_S r(N) = g(F(r(G(N))))$ and ${}_R s(M) = f(G(s(F(M))))$. Then ${}_S r\left({}_R s\right)$ is a preradical for S-mod (R-mod) and $r = {}_R\left({}_S r\right)$, $s = {}_S\left({}_R s\right)$. The rest follows easily, using the fact that $Gg = fG$ and $Ff = gF$.

5.3. Example. Any two skew-fields are p-equivalent. Thus p-equivalent rings need not be Morita equivalent.

5.4. Remark. The preceding proposition enables us to show trivially that many properties of rings which can be characterized by means of preradicals are Morita invariant. For example, the properties of a ring being QF-3 , semiartinian or with trivial torsion parts for a given class of preradicals (CTF-rings, ATF-rings) are Morita invariant.

R e f e r e n c e s

[1] L. BICAN, P. JAMBOR, T. KEPKA and P. NĚMEC: Preradicals, Comment.Math.Univ.Carolinae 15(1974),75-83.

[2] L. BICAN, P. JAMBOR, T. KEPKA and P. NĚMEC: Hereditary and cohereditary preradicals (to appear).

[3] L. BICAN, P. JAMBOR, T. KEPKA and P. NĚMEC: Stable and costable preradicals (to appear).

[4] L. BICAN, P. JAMBOR, T. KEPKA and P. NĚMEC: Composition of preradicals (to appear).

[5] R.S. CUNNINGHAM, E.A. RUTTER, Jr., and D.R. TURNIDGE: Rings of quotients of endomorphism rings of projective modules, Pac.Jour.Math.(3)41(1972),647-667.

[6] J. LAMBEK: Lectures on rings and modules, Blaisdell,1966.

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