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TERNARY RINGS ASSOCIATED TO TRANSLATION PLANE

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Abstract: It is well known that an affine plane is a translation plane if and only if there exists a quasifield coordinatizing it. Simple condition for planary ternary ring with zero coordinatizing a translation plane is deduced by Klucký and Marková in [4]. We shall define a J-ternary ring or JTR to be a PTR that $\exists 0 \in S$ such that
$T(a,0,c) = T(a,b,c)$ implies $T(a,0,y) = T(a,b,y) \quad \forall y \in S$
$T(0,a,c) = T(b,a,c)$ implies $T(0,a,y) = T(b,a,y) \quad \forall y \in S$.
In [5] Martin defines an intermediate ternary ring (ITR). Strucurally, the JTR lie between the PTR and ITR. The purpose of this note is to deduce a necessary and sufficient condition that a given JTR coordinatizes a translation plane. This generalizes the main results of [4] and [5].

Key words: Planar ternary ring, translation plane, intermediate ternary ring, generalized Cartesian group.

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A coordinatization of a projective plane: We shall give a coordinatization to a projective plane of order n. Let S be any set of cardinality n. Let ∞ be any element which is not in S and let $0 \in S$. We pick one point L and one line $\mathcal{L}$ joining through L in the plane. For any $M \in \mathcal{L}$ denote by $\widetilde{M}$ the set of all lines containing M. Let $m \mapsto (m)$ be a bijection of $S \cup \{\infty\}$ onto $\mathcal{L}$ such that $[\infty] = \mathcal{L}$. Let $x \mapsto [x]$ be a bijection of $S \cup \{\infty\}$ onto $\widetilde{L}$ such that $[\infty] = \mathcal{L}$. Let $y \mapsto (0,y)$ be a bijection of S onto $[0] \setminus \{L\}$. We denote by $A \cup B$ ($a \cap b$) the line joining two distinct points A,B (the common point of two distinct lines). Let α_1, α_2: $S \to S$ be two mappings. Then to every point P off $\mathcal{L}$ we assign coordinates (x,y) if and only if $P = [x] \sqcap ((\alpha_1(x)) \sqcup (0,y))$. We shall now dualize the above construction in the following sense. Let $c \mapsto [0,c]$ be a bijection of S onto $(\tilde{O}) \setminus \{\mathcal{L}\}$. Then to every line p off $\tilde{L}$ we assign coordinates [m,c] if and only if $p = (m) \sqcup ([\alpha_2(m)] \sqcap \sqcap [0,c])$.

Planar ternary rings:

Definition 1: Let S be a set containing two different elements at least and let ternary operation T be given on it. An ordered pair (S,T) will be called a planar ternary ring or PTR if it holds:

(1) $\forall a,b,c \in S \ \exists! x \in S \qquad T(a,b,x) = c$

(2) $\forall a,b,c,d \in S; x \in S \qquad T(x,a,b) = T(x,c,d)$

(3) $\forall a,b,c,d \in S;\ a \neq c \ \exists (x,y) \in S^2 \quad T(a,x,y) = b,\ T(c,x,y) = d$

An intermediate ternary ring on ITR (see [5],p.1187) is a PTR (S,T) such that (I_1) and (I_2) holds.

(I_1) $T(m,a,y) = T(m,b,y) = c,\ a \neq b$ implies $T(m,x,y) = c$ $\forall x \in S$

(I_2) $T(a,x,y) = T(b,x,y) = c,\ a \neq b$ implies $T(m,x,y) = c$ $\forall m \in S$

A J-ternary ring or JTR is a PTR (S,T) such that there exists $O \in S$ where

(J_1) $T(m,O,a) = T(m,x,a)$ implies $T(m,O,y) = T(m,x,y)$ $\forall y \in S$

(J_2) $T(O,x,a) = T(m,x,a)$ implies $T(O,x,y) = T(m,x,y)$ $\forall y \in S$

Let (S,T) be a PTR. Then (S,T) defines a projective plane $\pi(S,T)$ as follows.

Points: $(x,y),(m),(\infty)$; $m,x,y \in S$, ∞ not in S

Lines: $[m,c] := \{(x,y) \mid x,y \in S,\ T(m,x,y) = c\}$

$[x] := \{(x,y) \mid y \in S\}$

$[\infty] := \{(\infty)\} \cup \{(m) \mid m \in S\}$

In [2],[3](p. 114-115),[5](p. 1186) there was shown that $\pi(S,T)$ is a projective plane. Thus a solution in (3) is unique.

Proposition 1: *Let π be a projective plane. Then there exists a JTR (S,T) such that $\pi(S,T)$ is isomorphic to π.*

Proof: Let the projective plane π be coordinatized as above by elements from a set S. Define a ternary operation by $T(m,x,y) = c$ if and only if (x,y) is on $[m,c]$. Then it is obvious that the (S,T) is a JTR. One has only to check (1),(2), (3),(J_1),(J_2) in turn.

Remark: Let (S,T) be a JTR. Then there are mappings α_1, α_2: $S \longrightarrow S$ such that $\forall x,y \in S\quad T(\alpha_1(x),0,y) = T(\alpha_1(x),x,y)$

$\forall m,y \in S\quad T(0,\alpha_2(m),y) = T(m,\alpha_2(m),y)$

and such that for every point (x,y) and every line $[m,c]$ in $\pi(S,T)$ is $(x,y) = [x] \cap ((\alpha_1(x)) \sqcup (0,y))$

$[m,c] = (m) \sqcup ((\alpha_2(m)) \sqcap [0,c])$

Proposition 2: *Let (S,T) be an ITR. Then (S,T) is a JTR.*

Proof: The proposition is a direct consequence of Theorem 6 in [5], p. 1188.

Vertically transitive planes: (S,t) is said to be the dual ternary system of PTR (S,T) if $c = T(m,x,t(x,m,c))$

$\forall$ m,c,x$\in$S or equivalently $y = t(x,m,T(m,x,y))$ $\forall$ m,x,y$\in$S.

Proposition 3: *The dual of a JTR is a JTR.*

Proof: The proof is straightforward.

In the following we shall denote by j_a^1 the solution of the equation $t(x,0,0) = t(x,a,a)$ for each $a \in S \setminus \{0\}$ and by j_a^2 the solution of the equation $T(x,0,0) = T(x,a,a)$ for each $a \in S \setminus \{0\}$; additionally we define $j_0^1 = j_0^2 = 0$. Thus for each $a \in S$ is $t(j_a^1,0,0) = t(j_a^1,a,a)$ and $T(j_a^2,0,0) = T(j_a^2,a,a)$. Now let us introduce in S two binary operations $+_1, +_2$ by virtue of

$$a +_1 b := T(a,j_a^1,t(j_a^1,0,b))$$
$$a +_2 b := t(a,j_a^2,T(j_a^2,0,b)) \qquad \forall a,b \in S$$

Remark: It can be easily verified that

(4) $0 +_1 a = a +_1 0 = 0 +_2 a = a +_2 0 = a \qquad \forall a \in S$

(5) $\forall a,b \in S \ \exists! x \in S \qquad a +_1 x = b$

$\forall a,b \in S \ \exists! y \in S \qquad a +_2 y = b$

Definition 2: Let (S,T) be a PTR. The projective plane $\pi(S,T)$ is said to be a vertically transitive plane (by [4], p. 620) if for each $x,y,z \in S$ there exists a translation τ of the affine plane $(S^2, \{[m,c] \mid m,c \in S\} \cup \{[x] \mid x \in S\})$ such that $(x,y)^\tau = (x,z)$.

Let (S,T) be a JTR and (S,t) its dual. By (1) $\wp_1: y \mapsto T(0,0,y)$, $\wp_2: c \mapsto t(0,0,c)$ are bijective mappings and $\wp_1\wp_2 = \wp_2\wp_1 = \text{id}$.

Proposition 4: *Let (S,T) be a JTR. Then the projective plane $\pi(S,T)$ is a vertically transitive plane if and only if*

(6) $\forall m,c,x,y \in S$ $(T\ m,x,y +_2 c) = T(m,x,y) +_1 (0^{\rho_2} +_2 c)^{\rho_1}$

Proof. I. Suppose first that (S,T) (6) holds. We shall see that $(S,+_2)$ is a loop. By (4),(5) it is sufficient to show that $\forall u,c \in S\ \exists!\, v \in S\ v +_2 c = u$.

Let $a +_2 c = b +_2 c$ and let $m,x \in S$ such that $x \neq 0$, $T(m,0,a) = T(m,x,b)$. Then
$T(m,0,a +_2 c) = T(m,0,a) +_1 (0^{\rho_2} +_2 c)^{\rho_1} =$
$= T(m,x,b) +_1 (0^{\rho_2} +_2 c)^{\rho_1} = T(m,x,b +_2 c)$ and by (J_1)
$T(m,0,a) = T(m,x,a) = T(m,x,b)$ hence $a = b$. Now let $u \in S$. Choose $m,x,y \in S$ such that $x \neq 0$
$T(m,0,u) = T(m,x,y +_2 c)$ and denote
$(0,v) := [m,T(m,x,y)] \sqcap [0]$. Then there is $T(m,0,v) =$
$= T(m,x,y)$, $T(m,0,u) = T(m,x,y +_2 c) = T(m,x,y) +_1 (0^{\rho_2} +_2 c)^{\rho_1} =$
$= T(m,0,v) +_1 (0^{\rho_2} +_2 c)^{\rho_1} = T(m,0,v +_2 c)$ from here $v +_2 c =$
$= u$.

Thus, the map $\tau_c : S^2 \longrightarrow S^2$ defined by $(x,y)^{\tau_c} := (x,y +_2 c)$ is a translation. Since $(0,0)^{\tau_c} =$
$= (0,c)$, the $\pi(S,T)$ is a vertically transitive plane.

II. Let $\pi(S,T)$ be a vertically transitive plane. Then for each $a \in S$ there is a translation τ_a mapping (0,0) into (0,a). Then $(y,y)^{\tau_a} = (y,y +_2 a)$ for each $y \in S$ hence $(0,y)^{\tau_a} =$
$= (0,y +_2 a)$ for each $y \in S$ and $(x,y)^{\tau_a} = (x,y +_2 a)$ for each $x,y \in S$. It is obvious that $[0,0]^{\tau_a} = [0,(0^{\rho_2} +_2 a)^{\rho_1}]$
this implies $[m,c]^{\tau_a} = [m,c +_1 (0^{\rho_2} +_2 a)^{\rho_1}]$. Hence,
$(x,y) \in [m,T(m,x,y)]$ for each $m,x,y \in S$ from here $(x,y)^{\tau_a} \in$
$\in [m,T(m,x,y)]^{\tau_a}$ then $(x,y +_2 a) \in [m,T(m,x,y) +_1 (0^{\rho_2} +_2 a)^{\rho_1}]$
consequently $T(m,x,y +_2 a) = T(m,x,y) +_1 (0^{\rho_2} +_2 a)^{\rho_1}$
for each $m,x,y,a \in S$.

Corollary 4.1: *Let* (S,T) *be a JTR and let* $\pi(S,T)$ *be a vertically transitive plane. Then* $(S,+_1)$, $(S,+_2)$ *are groups and* $(S,+_1)$ *is isomorphic to* $(S,+_2)$.

Proof: Consider translations $\varphi: (0,0)\mapsto(0,a)$, $\sigma: (0,0)\mapsto(0,b)$, $\tau: (0,0)\mapsto(0,c)$. Then $(0,(a+_2 b)+_2 c) = (0,0)^{(\varphi\sigma)\tau} = (0,0)^{\varphi(\sigma\tau)} = (0,a+_2(b+_2 c))$.

The second result follows from (6). In particular, for every $a,b\in S$ $(a+_2 b)^{\varphi_1} = T(0,0,a+_2 b) = T(0,0,a)+_1(0^{\varphi_2}+_2 b)^{\varphi_1} = a^{\varphi_1}+_1(0^{\varphi_2}+_2 b)^{\varphi_1}$. Since for each $y,a,b\in S$ $y+_2(a+_2 b)=(y+_2 a)+_2 b$, we have

$$y^{\varphi_1}+_1(0^{\varphi_2}+_2(a+_1 b))^{\varphi_1} = (y+_2(a+_2 b))^{\varphi_1} = (y+_2 a)^{\varphi_1}+_1(0^{\varphi_2}+_2 b)^{\varphi_1} = (y^{\varphi_1}+_1(0^{\varphi_2}+_2 a)^{\varphi_1})+_1 +_1(0^{\varphi_2}+_2 b)^{\varphi_1}.$$

Setting $y = 0^{\varphi_2}$, we have

$$(0^{\varphi_2}+_2(a+_1 b))^{\varphi_1} = (0^{\varphi_2}+_2 a)^{\varphi_1}+_1(0^{\varphi_2}+_2 b)^{\varphi_1}.$$

Remark: The group of all translations of a vertically transitive plane $\pi(S,T)$ is Abelian if and only if $(S,+_1)$ is commutative.

Now let us introduce two binary operations $\cdot_1$, $\cdot_2$ by virtue of

$$T(m,x,0) = m\cdot_1 x \qquad \forall\, m,x\in S$$
$$t(x,m,0) = x\cdot_2 m \qquad \forall\, m,x\in S$$

Corollary 4.2: *Let* (S,T) *be a JTR and let* $\pi(S,T)$ *be a vertically transitive plane. Then*

$$(7)\quad \forall\, m,x,y\in S \quad T(m,x,y) = m\cdot_1 x+_1(0^{\varphi_2}+_2 y)^{\varphi_1}$$
$$t(x,m,y) = x\cdot_2 m+_2(0^{\varphi_1}+_1 y)^{\varphi_2}$$

Proof: Let as set $y = 0$ in (6). Then
$T(m,x,c) = m \cdot_1 x +_1 (0^{\varphi_2} +_2 c)^{\varphi_1}$ for each $m,x,c \in S$.

Proposition 5: *Let* (S,T) *be a JTR. The projective plane* $\pi(S,T)$ *is a vertically transitive plane if and only if*

(8) $(S,+_1)$, $(S,+_2)$ *are groups*

(9) *there exists an isomorphism* $\varphi : (S,+_2) \longrightarrow (S,+_1)$ *such that* $\forall\, m,x,y \in S \quad T(m,x,y) = m \cdot_1 x +_1 y^{\varphi}$.

Proof: I. Let (8),(9) hold for (S,T). Then for each $m,x,y,c \in S$ $T(m,x,y +_2 c) = m \cdot_1 x +_1 (y +_2 c)^{\varphi} = m \cdot_1 x +_1 (y^{\varphi} +_1 c^{\varphi}) = (m \cdot_1 x +_1 y^{\varphi}) +_1 c^{\varphi} = T(m,x,y) +_1 c^{\varphi}$.
Setting $m = x = 0$, $y = 0^{\varphi_2}$, we have $(0^{\varphi_2} +_2 c)^{\varphi_1} = 0 +_1 c^{\varphi}$ thus $c^{\varphi} = (0^{\varphi_2} +_2 c)^{\varphi_1}$ for each $c \in S$ therefore $T(m,x,y +_2 c) = T(m,x,y) +_1 (0^{\varphi_2} +_2 c)^{\varphi_1}$ for each $m,x,y,c \in S$.

II. The second part follows immediately from Corollary 4.1 and Corollary 4.2.

Corollary 5.1: *Let* (S,T) *be a JTR such that* $T(0,0,y) = y$ *for each* $y \in S$. *. Then the projective plane* $\pi(S,T)$ *is a vertically transitive plane if and only if*

(i) $(S,+_1)$ is a group

(ii) $\forall\, m,x,y \in S \qquad T(m,x,y) = m \cdot_1 x +_1 y$

Proof: I. $\forall\, m,x,y,c \in S \qquad T(m,x,y +_1 c) = m \cdot_1 x +_1 (y +_1 c) = (m \cdot_1 x +_1 y) +_1 c = T(m,x,y) +_1 c$.
Hence $\pi(S,T)$ is a vertically transitive plane.

II. If $\pi(S,T)$ is a vertically transitive plane, then by Proposition 5 $(S,+_1)$ is a group and there exists an isomorphism $\varphi : (S,+_2) \rightarrow (S,+_1)$ such that $T(m,x,y) = m \cdot_1 x +_1 y^{\varphi}$ for each $m,x,y \in S$. This yields then $y = T(0,0,y) = 0 +_1 y^{\varphi} = y^{\varphi}$ for each $y \in S$ hence $T(m,x,y) = m \cdot_1 x +_1 y$ for each $m,x,y \in S$.

Corollary 5.2: *Let* (S,T) *be a JTR and* (S,t) *its dual. Let* $\pi(S,T)$ *be a vertically transitive plane, then there exists an isomorphism* $\varphi:(S,+_2)\longrightarrow(S,+_1)$ *such that*

$\forall m,x,y\in S \qquad T(m,x,y) = m\cdot_1 x +_1 y^{\varphi},$

$$t(x,m,y) = x\cdot_2 m +_2 y^{\varphi^{-1}}, \quad m\cdot_1 x +_1 (x\cdot_2 m)^{\varphi} = 0$$

Proof: Since it holds $T(m,x,t(x,m,0)) = 0$ for each $m,x\in S$, we have $m\cdot_1 x +_1 (x\cdot_2 m)^{\varphi} = 0$. Since it holds $T(m,x,t(x,m,y)) = y$ for each $m,x,y\in S$, we obtain $m\cdot_1 x +_1 (t(x,m,y))^{\varphi} = y$ thus $(t(x,m,y))^{\varphi} = -_1\, m\cdot_1 x +_1 y = (x\cdot_2 m)^{\varphi} +_1 y$ from what you say $t(x,m,y) = x\cdot_2 m +_2 y^{\varphi^{-1}}$.

Definition 3: Let S be a set $+,\cdot$ two binary operations on S. $(S,+,\cdot)$ will be called a generalized Cartesian group (see [4],p. 620) if S has two distinct elements at least and if it holds:

(10) $(S,+)$ is a group

(11) $\forall a,b,c\in S;\ a\neq b\ \exists!\, x\in S \qquad -xa + xb = c$

(12) $\forall a,b,c\in S;\ a\neq b\ \exists\, x\in S \qquad ax - bx = c$

Propoposition 6: *Let* $\mathbb{C}:=(S,+,\cdot)$ *be a generalized Cartesian group and let* $\varphi: S\longrightarrow S$ *be a bijection such that* $0^{\varphi}=0$. *If we define* $T(\mathbb{C},\varphi)$, $(m,x,y) = m\cdot x + y^{\varphi}$ *for each* $m,x,y\in S$ *then* $(S,T(\mathbb{C},\varphi))$ *is a JTR and* $\pi(S,T(\mathbb{C},\varphi))$ *is a vertically transitive plane.*

Proof: The proof is straightforward. One has only to check (1),(2),(3),(J_1),(J_2),(8),(9) in turn.

Proposition 5 and Proposition 6 now imply the next

Theorem 1: *Let* (S,T) *be a JTR. Then the projective plane* $\pi(S,T)$ *is a vertically transitive plane if and only if*

(i) $\mathbb{C}:=(S,+_1,\cdot_1)$ *is a generalized Cartesian group*

(ii) there exists a bijection $\varphi: S \longrightarrow S$ *such that* $0^{\varphi} = 0$, $T = T(\mathbb{C}, \varphi)$.

Translation planes: First we give some general remarks. Let us investigate a projective plane $\pi = (P,L)$. Let us distinguish a line ℓ. Then by an affine plane $\pi(\ell)$ we shall as usual mean the restriction of π to the incidence structure $(P \setminus \ell, \{m \setminus (m \cap \ell) \mid m \in L \setminus \{\ell\}\})$. The points from $P \setminus \ell$ will be called proper, the points of ℓ improper or directions. A projective plane $\pi = (P,L)$ is said to be an ℓ-transitive plane if the group of all translations of $\pi(\ell)$ transitively operates on the set of all points of $\pi(\ell)$. Let u, v be affine lines of $\pi(\ell)$ with different directions, then the projective plane π is a ℓ-transitive plane if and only if the group of all translations of $\pi(\ell)$ transitively operates on the lines u, v.

Proposition 7: *Let* $\mathbb{C} = (S,+,\cdot)$ *be a generalized Cartesian group and* $\varphi: S \longrightarrow S$ *a bijection such that* $0^{\varphi} = 0$. *Then the projective plane* $\pi(S, T(\mathbb{C}, \varphi))$ *is a* $[\infty]$-*transitive plane if and only if*

(13) $\forall x,a \in S \ \exists x' \in S \ \forall m \in S$

$mx' - 0x' = ma - 0a + 00 - m0 + mx - 0x$

Proof: I. It suffices to prove that the group of all translations transitively operates on proper points of the line $[0,0]$ In this case it suffices to show that for each line $[a]$ there exists a translation τ such that $[0]^{\tau} = [a]$. Define a mapping $\tau_a: (x,y) \longmapsto (x', (-0x' + 0x + y^{\varphi})^{\varphi^{-1}})$ with $x' \in S$ uniquely determined by (13) (see (12)). Clearly τ_a is bijective. Further it is obvious that the image of the line $[x]$ is the line $[x']$. Let us consider a line $[m,c]$. If $(x,y) \in [m,c]$, then $T(\mathbb{C},\varphi)\quad(m,x,y) =$ $mx + y^{\varphi} = c$. Hence it is

$T(\mathbb{C},\varphi)\quad(m,x',(-0x'+0x+y^{\varphi})^{\varphi^{-1}}) =$

$= (mx'-0x') + 0x + y^{\varphi} = (ma - 0a + 00 - m0 + mx - 0x) +$

$+ 0x + y^{\varphi} = (ma - 0a + 00 - m0) + c$

or equivalently $(x',(-0x'+0x+y^{\varphi})^{\varphi-1}) \in$ $\in [m, ma - 0a + 00 - m0 + c]$. If $x = x'$ for some $x \in S$, then necessarily $a = 0$ therefore $\tau_a = id$. This implies τ_a is a translation. Setting $x = 0$ in (13), we obtain $m0' - 00' = ma - 0a$ for each $m \in S$ then $0' = a$ hence $[0]^{\tau_a} = [a]$ and consequently $\pi(S,T(\mathbb{C},\varphi))$ is a $[\infty]$-transitive plane.

II, Conversely, suppose that $\pi(S,T(\mathbb{C},\varphi))$ is a $[\infty]$-transitive plane. First of all, evidently for $a = 0$ $mx - 0x =$ $= ma - 0a + 00 - m0 + mx - 0x$ for each $m,x \in S$. Thus suppose $a \neq 0$. For $x = 0$ we have $ma - 0a = ma - 0a + 00 - m0 + mx - 0x$ for each $m \in S$. Thus suppose $x \neq 0$. Now choose any element $k \in S \setminus \{0\}$. By (12) there is x' such that

$kx' - 0x' = ka - 0a + 00 - k0 + kx - 0x$

Further let τ_a be a translation for which $(0,0)^{\tau_a} =$ $= (a, (-0a + 00 + 0^{\varphi})^{\varphi-1})$. Then

$(0,0),(x,(-kx + k0 + 0^{\varphi})^{\varphi-1}) \in [k, k0 + 0^{\varphi}]$,

$(0,0),(a,(-0a + 00 + 0^{\varphi})^{\varphi-1}) \in [0, 00 + 0^{\varphi}]$,

$(a(-0a + 00 + 0^{\varphi})^{\varphi-1}),(x',(-0x' + 0x - kx + k0 + 0^{\varphi})^{\varphi-1}) \in$

$\in [k, ka - 0a + 00 + 0^{\varphi}]$

$(x,(-kx + k0 + 0^{\varphi})^{\varphi-1}),(x',(-0x' + 0x - kx + k0 + 0^{\varphi})^{\varphi-1}) \in$

$\in [0, 0x - kx + k0 + 0^{\varphi}]$.

Thus, $(x,(-kx + k0 + 0^{\varphi})^{\varphi-1})^{\tau_a} = (x',(-0x' + 0x - kx +$ $+ k0 + 0^{\varphi})^{\varphi-1})$ hence $[x]^{\tau_a} = [x']$. For $m = 0$ is

$0 = mx' - Ox' = ma - Oa + OO - mO + mx - Ox.$

Thus let be $m \in S \setminus \{0\}$. Then

$(x, (-mx + mO + O^{\varphi})^{\varphi-1}) \in [O, Ox - mx + mO + O^{\varphi}]$,

$(x', (-Ox' + Ox - mx + mO + O^{\varphi})^{\varphi-1}) \in [O, Ox - mx + mO + O^{\varphi}] \cap$

$\cap [x']$. Thus, $(x, (-mO + mO + O^{\varphi})^{\varphi-1})^{\tau_a} =$

$= (x', (-Ox' + Ox - mx + mO + O^{\varphi})^{\varphi-1})$.

But $T(\mathbb{C}, \varphi)\ (m, O, O) = mO + O^{\varphi} =$

$= T(\mathbb{C}, \varphi)\ (m, x, (-mx + mO + O^{\varphi})^{\varphi-1})$ and then it follows

necessarily $T(\mathbb{C}, \varphi)(m, a, (-Oa + OO + O^{\varphi})^{\varphi-1}) =$

$= T(\mathbb{C}, \varphi)\ (m, x', (-Ox' + Ox - mx + mO + O^{\varphi})^{\varphi-1})$, hence

$ma - Oa + OO + O^{\varphi} = mx' - Ox' + Ox - mx + mO + O^{\varphi}$ consequently $mx' - Ox' = ma - Oa + OO - mO + mx - Ox$

Thus Proposition 7 is proved.

Corollary 7.1.: *Let* $(S, +, \cdot)$ *be a generalized Cartesian group such that the condition (13) holds. Then the group* $(S, +)$ *is Abelian.*

Proof: The proof of the preceding corollary depends on the obvious fact that the group of all translations of a $[\infty]$-transitive plane is Abelian.

Proposition 8: *Let* $\mathbb{C} = (S, +, \cdot)$ *be a generalized Cartesian group such that there exists* $e \in S$ *where for each* $x \in S$ $\quad e \cdot x = e \cdot O$. *Further let* $\varphi: S \longrightarrow S$ *be a bijection such that* $O^{\varphi} = O$. *Then the projective plane* $\pi(S, T(\mathbb{C}, \varphi))$ *is a* $[\infty]$*-transitive plane if and only if*

(14) $\forall x, a \in S\ \exists x' \in S\ \forall m \in S\ mx' - mx = ma - mO$

Proof: I. First we note that by (12) for every $a \in S \setminus \{e\}$ and for every $b \in S$ there exists exactly one $x \in S$ such that $ax - ex = b - eO$ it holds if and only if $ax - eO = b - eO$,

$ax = b$. This implies that for each $a \in S \setminus \{e\}$ and for each $b \in S$ there exists exactly one $x \in S$ such that $ax = b$. Define a mapping $\tau_a : (x,y) \longmapsto (x',y)$ with $x' \in S$ uniquely determined by (14). Clearly τ_a is a bijective. Further it is obvious that the image of the line $[x]$ is the line $[x']$. Let us consider a line $[m,c]$. If $(x,y) \in [m,c]$, then $T(\mathbb{C},\varphi)(m,x,y) = mx + y^{\varphi} = c$. Hence it is $T(\mathbb{C},\varphi)(m,x',y) = mx' + y^{\varphi} = ma - m0 + mx + y^{\varphi} = ma - m0 + c$ or equivalently $(x',y) \in [m, ma - m0 + c]$. If $x' = x$ for some $x \in S$ then by (14) $a = 0$, $\tau_a = id$. This implies τ_a is a translation. Setting $x = 0$ in (14), we have $m0' = ma$ for each $m \in S$ hence $[0]^{\tau_a} = [a]$ and consequently $\pi(S,T(\mathbb{C},\varphi))$ is a $[\infty]$-transitive plane.

II. Let $\pi(S,T(\mathbb{C},\varphi))$ be a $[\infty]$-transitive plane. Setting $m = e$ in (13), we obtain $ex' - 0x' = ea - 0a + 00 - e0 + ex - 0x$ then $-0x' = -0a + 00 - 0x$ hence $mx' - 0x' = mx' - 0a + 00 - 0x = ma - 0a + 00 - m0 + mx - 0x$ for each $m \in S$ and by Corollary 7.1 $mx' = ma - m0 + mx$ therefore $mx' - mx = ma - m0$.

Theorem 1 and Proposition 7 now imply

<u>Theorem 2</u>: *Let (S,T) be a JTR. Then the projective plane $\pi(S,T)$ is a $[\infty]$-transitive plane if and only if*

(i) $\mathbb{C} := (S, +_1, \cdot_1)$ *is a generalized Cartesian group*

(ii) there exists a bijection $\varphi : S \longrightarrow S$ such that

$0^{\varphi} = 0$, $T = T(\mathbb{C},\varphi)$

(iii) $\forall\, x,a \in S\ \exists\, x' \in S\ \forall\, m \in S$

$$m \cdot_1 x' -_1 0 \cdot_1 x' = m \cdot_1 a -_1 0 \cdot_1 a +_1 0 \cdot_1 0 -_1 m \cdot_1 0 +_1 m \cdot_1 x -_1 0 \cdot_1 x$$

R e f e r e n c e s

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