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MEAN VALUE THEOREM FOR CONVEX FUNCTIONALS

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Abstract: A mean value theorem for convex functionals, defined on general real linear spaces, resp. on real linear topological spaces is established. As an application the Lipschitz continuity of convex functionals is studied.

Key words: Mean value theorem, convex functionals, subdifferential, Lipschitz continuity.

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In this note we present a mean value theorem for convex functionals on general real linear spaces, resp. on real linear topological spaces. This result extends recent work of Wegge [7] to infinite dimensions.

To give a simple application of this analogue of the classical mean value theorem in convex analysis we deal with Lipschitz continuity of convex functionals. We modify the arguments of Kolomý [3] and obtain a Lipschitz bound without assuming Gâteaux differentiability. Let us note that a more complicated calculation of this estimate has already been given by Orlicz and Ciesielski [5]. Moreover we directly derive another Lipschitz bound, established by Ekeland and Temam in [1], from a simple estimate of the subgradient.

Let us refer to Rockafellar [6] for definitions and notations of convex analysis.

Theorem A. Let E be a real linear space. Let $f: E \longrightarrow$ $\longrightarrow (-\infty, +\infty]$ be a convex functional, and let the restriction $f \mid [a,b]$ be finite and lower semicontinuous in the closed line segment $[a,b] \subset E$ $(a \neq b)$. Suppose, the open line segment (a,b) is contained in the relative interior of dom f. Then there exist a point $\tilde{x} \in (a,b)$, and a linear functional $\tilde{\xi} \in \partial f(\tilde{x})$ which satisfies

$$f(b) - f(a) = \langle \tilde{\xi}, b - a \rangle \ . \tag{1}$$

Proof. With $d = f(b) - f(a)$ we introduce a convex function φ by

$$\varphi(t) = f[a + t(b - a)] - d \cdot t, \quad t \in \mathbb{R} \ .$$

Clearly we have $\varphi(0) = f(a) = \varphi(1)$, and in virtue of lower semicontinuity some $\tilde{t}$ which can be chosen in $(0,1)$ exists such that

$$\varphi(\tilde{t}) \leq \varphi(t), \quad \forall\, t \in [0,1] \ . \tag{2}$$

This inequality remains valid for all $t \in \mathbb{R}$ because of convexity of the function φ. Now by extension we obtain a linear functional δ such that $d = \langle \delta, b - a \rangle$ holds. With $\tilde{x} = a + \tilde{t}(b - a)$, $M = a + \mathbb{R}(b - a)$ and the convex functional g, defined by

$$g(x) = f(x) - \langle \delta, x - a \rangle \ , \quad x \in E$$

inequality (2) reads

$$g(\tilde{x}) \leq g(x), \quad \forall\, x \in M. \tag{3}$$

Let D denote the domain of g, set $E_1 = \operatorname{span}(D - D)$. 0 belongs to the algebraic interior of $D - \tilde{x}$ relative to E_1, and $M - \tilde{x}$ is contained in E_1. Thus, if a subgradient with certain properties is constructed in the algebraic dual E_1', then we obtain the desired subgradient in E' by extension. Therefore, without any loss of generality we can assume that $E_1 = E$ and $\tilde{x}$ is an algebraic interior point of D.

The following sets in $\mathbb{R} \times E$

$$A = \{(r,x) \mid x \in D,\ r > g(x)\}$$
$$B = \{(s,y) \mid y \in M,\ s \leq g(\tilde{x})\}$$

are convex and disjoint. One can show - see the proof of the subdifferentiability theorem in convex analysis [2, p. 23] - that the algebraic interior of A is not empty. By the basic separation principle (cf. [2, p. 15]) a pair $(\varphi, \xi) \in$ $\in (\mathbb{R} \times E') \setminus \{(0,0)\}$ exists such that for any $(r,x) \in A$ and any $(s,y) \in B$

$$\varphi \cdot r + \langle \xi ,x \rangle \geq \varphi \cdot S + \langle \xi ,y \rangle \tag{4}$$

holds. Obviously $\varphi \geq 0$, indeed $\varphi > 0$ is valid, since $(g(\tilde{x}),\tilde{x}) \in B$, and $(g(\tilde{x}) + \varepsilon,\tilde{x})$ belongs to the algebraic interior of A for any $\epsilon > 0$. From (4) we conclude that $\langle \xi ,b - a \rangle = 0$, and $\varphi^{-1} \xi \in \partial g(\tilde{x})$. Hence

$$\tilde{\xi} = \varphi^{-1} \xi + \sigma \tag{5}$$

is a subgradient of f at $\tilde{x}$, and satisfies (1).

If E is endowed with a linear topology we can guarantee the continuity of the functional constructed above under the usual interior point and continuity assumptions. More precisely we have the following

Theorem B. Let E be a real linear topological space. Let the convex functional $f: E \longrightarrow (-\infty,+\infty]$ be finite and continuous in the closed line segment $[a,b] \subset E$ $(a \neq b)$. Suppose, the open line segment (a,b) is contained in the topological interior of dom f. Then there exist a point $\tilde{x} \in$ $\in (a,b)$ and a linear continuous functional $\tilde{\xi} \in \partial f(\tilde{x})$ which satisfies

$$f(b) - f(a) = \langle \tilde{\xi}, b - a \rangle .$$

Proof. We show that the linear functional $\tilde{\xi}$ defined in (5) is continuous. Indeed, it is bounded above on some neighborhood of $\tilde{x}$, for $\tilde{x}$ is a topological interior point of dom f, and $\tilde{\xi}$ is a subgradient of the continuous functional f at $\tilde{x}$.

Now let us turn to Lipschitz continuity of convex real-valued functionals in normed linear spaces.

Let $B(x_0,R) = \{ x \in E \mid \| x - x_0 \| < R \}$ denote an open ball in the normed linear space E. Let the real-valued convex functional f be defined and bounded above by some constant M in the ball $B(x_0,R)$. From the boundedness above it follows that f is continuous in $B(x_0,R)$ (cf. [2, p.82]). Fix an arbitrary $r \in (0,R)$, choose $x \in B(x_0,r)$, $h \in E$ with $\| h \| = 1$. Then for any $t \in (0,R - r)$ the point $x + th$ belongs to $B(x_0,R)$, and for any $\xi \in \partial f(x)$ - continuous subgradients exist in virtue of continuity - we calculate

$$(6) \qquad \| \xi \| = \sup_{\|h\| \leq 1} \langle \xi , h \rangle \leq \sup_{\|h\| \leq 1} \{ \frac{1}{t} [f(x + th) - f(x)] \}$$

Since by convexity

$$f(x + th) - f(x) \leq f(x + th) + f(2x_0 - x) - 2f(x_0)$$
$$\leq 2 [M - f(x_0)]$$

holds, we obtain by letting $t \longrightarrow R - r$

$$\sup \left\{ \|\xi\| \mid \xi \in \partial f(x),\ x \in B(x_0,r) \right\} \leq \frac{2}{R-r}[M - f(x_0)] = C.$$

Theorem B immediately implies the Lipschitz continuity of f in $B(x_0,r)$ with constant C (cf. [3, p.42], [5, p. 336/337]).

The same reasoning shows that the hypothesis of Gâteaux differentiability of the convex functional f in [4, Th.1, p. 78] can be dropped, and this result can be stated in the following more general way.

<u>Proposition</u>. Let X, Y be real normed linear spaces. Let $F: B(x_0,R) \longrightarrow Y$ be a mapping which is Gâteaux differentiable in $B(x_0,r)$ $(0 < r < R)$. Furthermore let a convex functional $f: B(x_0,R) \longrightarrow \mathbb{R}$ be given. Assume f is bounded above in the boundary of $B(x_0,R)$ by some constant M. If F satisfies

$$\|F'(x)\| \leq \sup \left\{ \|\xi\| \mid \xi \in \partial f(x) \right\}$$

for every $x \in B(x_0,r)$, then

$$\sup \left\{ \|F'(x)\| \mid x \in B(x_0,r) \right\} \leq \frac{2}{R-r}[M - f(x_0)] = C$$

holds. Moreover, F and f are Lipschitzian in $B(x_0,r)$ with constant C.

If moreover a lower bound m for the convex functional f in $B(x_0,R)$ is known, then (6) yields at once

$$\sup \left\{ \|\xi\| \mid \xi \in \partial f(x),\ x \in B(x_0,r) \right\} \leq \frac{M-m}{R-r} = C' .$$

And Theorem B implies Lipschitz continuity of f in $B(x_0,r)$ with this constant (cf. [1, p. 12/13] for a more complicated argument).

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