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WHEN FINELY CONTINUOUS FUNCTIONS ARE OF THE FIRST CLASS OF BAIRE

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Abstract: Given a metric space (X,ρ) with another "fine topology", a simple method enables to decide whether any finely continuous function on X is of Baire class 1. As application we give new simple proofs that any finely continuous function in any abstract potential theory, any approximately continuous function as well as any approximate derivative is of Baire class 1. Finally, using the same method we prove a generalization of the theorem of Snyder on partial approximate derivatives.

Key words: Functions of Baire class one, fine topology in potential theory, density topology, approximate derivatives, Jarník-Snyder method, partial approximate derivatives.

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1. Introduction. It is known that any approximately continuous function on the real line, or more generally on an euclidean space R^k is of the Baire class 1 (cf. [3],[16], [9]). The class of all approximately continuous functions is exactly equal to the class of all continuous functions in the ordinary density topology on R^k. Since the fine topology in R^k derived from the classical case of harmonic functions is coarser than the density topology, it follows that any finely continuous function is of the first Baire

class. This was observed and proved by B. Fuglede 1971 [4], [5]. This result was generalized to the case of finely continuous functions in the heat equation by I. Netuka and L. Zajíček 1974 [14], and finally again B. Fuglede 1974 [6] proved that finely continuous functions in any axiomatic potential theory are of the Baire class 1. We give here a simplified proof of this theorem, and using certain more general ideas we show how to prove that some types of functions are of Baire class 1. In particular, the result of D. Preiss 1971 [15] (asserting that any approximate derivative, finite or infinite, is of Baire class 1) follows easily by Jarník's method from a theorem on boundary behaviour of functions of two variables (cf. [19],[13]) and we give a simplified proof of this last theorem. Finally, we prove that if a function $f(x,y)$ ($x \in R^k$, $y \in R$) is continuous in x and has a partial approximate derivative in y, then this approximate derivative is of Baire class 1. This theorem in the case of finite partial approximate derivatives and $k = 1$ is due to L.E. Snyder 1966 [18]. Our simplified proof uses again the Jarník's method.

2. General theorem. Let (X, ρ) be a metric space. Considering on X a new topology τ, topological notions referring to the τ-topology will be qualified by the prefix τ to distinguish them from those pertaining to the ρ-topology. In particular, A^o_τ and $\overline{A}_\tau$ denote the τ-interior and the τ-closure of A, while ∂G will be the ρ-boundary of G.

Theorem A. The following assertions are equivalent:

(i) any τ -continuous function on X is of Baire class 1;

(ii) any τ -zero set in X is of type G_δ;

(iii) given any τ -zero set Z and any τ -cozero set C, $C \subset Z$, there is a set G of type G_δ such that $C \subset G \subset Z$.

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are obvious. Assume (iii), let f be τ -continuous function on X. Given $a \in R$ and an integer n, we have

$$F_a := \{x \in X;\ f(x) \geq a\} \subset C_n := \{x \in X;\ f(x) > a - n^{-1}\} \subset$$

$$\subset Z_n := \{x \in X;\ f(x) \geq a - n^{-1}\}.$$

By (iii) there is a G_n of type G_δ such that $C_n \subset G_n \subset Z_n$. Since $F_a = \bigcap_{n=1}^{\infty} Z_n$, we obtain that F_a is of type G_δ. Since $F^a := \{x \in X;\ f(x) \leq a\} = \{x \in X;\ -f(x) \geq -a\}$, F^a is of type G_δ as well.

Corollary. If for any set $A \subset X$ there is a set A^* of type G_δ such that $A^o_\tau \subset A^* \subset \overline{A}_\tau$, then any τ -continuous function on X is of Baire class 1.

Remark. We do not suppose that τ is finer than ρ .

3. Finely continuous functions in potential theory. Let X be a $\mathfrak{P}$-harmonic space with countable base in the sense of the axiomatics of C. Constantinescu and A. Cornea [2]. Given any set $A \subset X$, the base b(A) of A is the set of all points where A is not thin. It is known that b(A) is finely closed and of type G_δ. Moreover, since the fine topology has not the isolated points, b(A) contains the fine interior of A

and, of course, is contained in the fine closure of A (which is exactly $A \cup b(A)$). Using our Corollary, we get the following theorem immediately.

Theorem. Every finely continuous function on X is of Baire class 1.

4. Approximately continuous functions. Consider the ordinary density topology d in an euclidean space R^k (see, e.g., [10],[8],[12]). Given any set $A \subset R^k$, we put

$$A^* = \{ x \in R^k;\ \text{for any natural } n \text{ there is } m > n \text{ such that}$$
$$\frac{\mu(A \cap K(x,m^{-1}))}{\mu K(x,m^{-1})} > \frac{1}{2} \},$$

where μ denotes the outer Lebesgue measure and $K(c,r)$ is an open ball with center c and radius r. Since

$$A_m := \{ x \in R^k;\ \frac{\mu(A \cap K(x,m^{-1}))}{\mu K(x,m^{-1})} > \frac{1}{2} \}$$

is open and $A^* = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$, A^* is a G_δ set. Moreover, $A_d^o \subset A^* \subset \overline{A}_d$, so that our Corollary immediately implies the following theorem.

Theorem. Every approximately continuous function on R^k is of the Baire class 1.

Remark. In an analogous way it can be proved that also in more general "density" or "measure" topologies any continuous function is of Baire class 1. In contrast to our method, the usual proofs of these assertions (see, e.g., [9],[21]) use the Baire characterization of functions of

Baire class 1. Likewise, the similar propositions concerning the mappings to metric spaces can be proved by the same method.

5. Approximate derivatives and Jarník's method. Some theorems on derivatives can be considered as consequences of theorems concerning boundary behaviour of functions of two variables. It seems that this idea is due to V. Jarník 1926 [11]. He proved by his method that any derivative (with finite or infinite values) is of Baire class 1. x) This theorem was rediscovered many times.

The following theorem is the special case of Jarník's theorem on boundary behaviour of functions.

Theorem J. Let $F(x,y)$ be an arbitrary function defined on the open halfplane $H := \{(x,y) \in R^2;\ y < x\}$. Assume that for each $z \in R$ there exists the limit $f(z)$ (finite or infinite) of F in the point (z,z) with respect to the angle

$$U(z) := \{(x,y) \in R^2;\ x > z > y\}.$$

Then the "boundary function" f is of Baire class 1.

Suppose now that a function g has derivative on R. Applying Theorem J to $F(x,y) = g(y) - g(x)/y - x$, we immediately obtain that g' is of Baire class 1.

If we replace in Theorem J "limit" by "approximate limit", the new theorem also holds. We shall refer to this

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x) In another context, the same idea was used by H. Blumberg 1930 [1], cf. also [17], p. 262.

generalization of the Theorem J as to Theorem S. From the Theorem S it follows easily that any approximate derivative (finite or infinite) is of Baire class 1. (G. Tolstov 1938 [20] proved that any finite approximate derivative is of Baire class 1. The simplified proof was given by C. Goffman and C.J. Neugebauer 1960 [7]. L.E. Snyder 1966 [19] proved this result using the modified Jarník's method. Finally, D. Preiss 1971 [15] proved that any, even infinite, approximate derivative is of Baire class 1, while L. Mišík 1972 [13] proved this theorem again using the Jarník-Snyder method.)

We shall show how Theorem S easily follows from our Corollary. Let ρ be a metric defined on $\overline{H}$ which induces on H discrete and on the boundary $\partial H = \{(z,z) \in R^2;\ z \in R\}$ the euclidean topology. Further, define on $\overline{H}$ the topology τ as follows:

a set $G \subset \overline{H}$ is τ-open if any point $(z,z) \in G \cap \partial H$ is the point of density of G with respect to U(z).

If F on H fulfils the assumptions of the Theorem S, then the extension $\tilde{F}$ of F defined on ∂H as $\tilde{F}(z,z) = f(z)$ is τ-continuous. For any $z \in R$ and $r > 0$ put

$U(z,r) = \{t \in U(z);\ \|t - z\| < r\}$.

Given any set $A \subset \overline{H}$, we put

$A^* = (A \cap H) \cup \{(z,z);$ for any natural n there is $m > n$ such that

$$\frac{\mu(A \cap U(z,m^{-1}))}{\mu(U(z,m^{-1}))} > \frac{1}{2}\ \},$$

where μ denotes the outer Lebesgue measure. Since

$$A_m := \{ (z,z);\ \frac{\mu(A \cap U(z,m^{-1}))}{\mu(U(z,m^{-1}))} > \frac{1}{2} \}$$

is open in ∂H and $A^* = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \cup (A \cap H)$, A^* is a G_δ set with respect to $\wp$. Since $A^o_\tau \subset A^* \subset \overline{A}_\tau$, our Corollary immediately implies that $\widetilde{F}$ is of Baire class 1 with respect to $\wp$. Therefore, also f is of Baire class 1.

6. Partial approximate derivatives. In this section we shall prove the following generalization of the assertion that any approximate derivative is of Baire class 1.

Theorem B. Suppose that a function $f(t,x)$ ($t \in R^k$, $x \in R$) is continuous in the variable t and has partial approximate derivative (finite or infinite) in x at any point of $R^k \times R$. Then this derivative is of Baire class 1.

L.E. Snyder 1966 [18] proved this theorem for finite partial approximate derivatives and it seems that his method cannot be modified to prove the general case. In contrast to his method which is similar to that of [7], we use the Jarník's method. In the present problem our Corollary does not work, but if we use the main idea of the Theorem A we obtain a relatively simple proof of the Theorem B.

The theorem on boundary behaviour of functions which corresponds to the Theorem B is the following

Theorem C. Let $F(t,x,y)$ ($t \in R^k$, $x,y \in R$) be a function defined on the open halfspace $H := \{(t,x,y);\ x > y\}$. Assume that F is continuous in t and has for any $t_0 \in R^k$ and $x_0 \in R$ an approximate limit $f(t_0,x_0)$ in (t_0,x_0,x_0) with respect to *the set*

$$U(t_0,x_0):=\{(t_0,x,y);\ x>x_0>y\}$$

and the two-dimensional measure in this set. Then f *is of Baire class* 1 *in* R^{k+1}.

The same argument as in [13] easily gives the Theorem B applying the Theorem C to $F(t,x,y) = f(t,x) - f(t,y)/x - y$.

Proof of the Theorem C. In any $H_t := \{(t,x,y);\ x>y\}$ let τ_t be the topology which corresponds to τ from Section 5 in the identification $(t,x,y) = (x,y)$. Let (H,ω) be the topological sum of all spaces (H_t, τ_t). If $\widetilde{F}$ on $\overline{H}$ is the natural extension of F, $\widetilde{F}$ is continuous with respect to ω. For $t_0 \in R^k$, $x_0 \in R$, $r>0$ put

$$U(t_0,x_0,r) = U(t_0,x_0) \cap \{(t,x,y);\ \|(t,x,y) - (t_0,x_0,x_0)\| < r\}.$$

Given $a \in R$ and a natural s, we have

$$\widetilde{F}_a := \{v \in \overline{H};\ \widetilde{F}(v) \geq a\} \subset C_s := \{v \in \overline{H};\ \widetilde{F}(v) > a - s^{-1}\} \subset$$

$$\subset Z_s := \{v \in \overline{H};\ \widetilde{F}(v) \geq a - s^{-1}\}.$$

For $A \subset \overline{H}$, we put

$A^* = (A \cap H) \cup \{(t,x,x);$ for any natural n there is $m>n$ such that

$$\frac{\mu(A \cap U(t,x,m^{-1}))}{\mu U(t,x,m^{-1})} > \frac{1}{2}\},$$

where μ denotes the outer two-dimensional measure. Obviously, $C_s \subset C_s^* \subset Z_s$, and therefore $\widetilde{F}_a = \bigcap_{s=1}^{\infty} C_s^*$.

For natural m,s we put further

$$R(m,s) = \{(t,x,x);\ \frac{\mu(C_s \cap U(t,x,m^{-1}))}{\mu U(t,x,m^{-1})} > \frac{1}{2}\} .$$

Then

$$\tilde{F}_a \cap \partial H = \bigcap_{s=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} R(m,s).$$

Therefore, to complete the proof it is sufficient to prove that $R(m,s)$ is open in ∂H. Fix $(t_0,x_0,x_0) \in R(m,s)$. Using the continuity of F in the variable t, for any $(t_0,x,y) \in U(t_0,x_0,m^{-1}) \cap C_s$ there is a natural p such that $\| t - t_0 \| < p^{-1}$ implies $(t,x,y) \in C_s$. Since the outer measure is a lower continuous set function, there exists p_0 and $\varepsilon > 0$ such that $\| t - t_0 \| < (p_0)^{-1}$ implies

$$\frac{\mu(C_s \cap U(t,x_0,m^{-1}))}{\mu U(t,x_0,m^{-1})} > \frac{1}{2} + \varepsilon .$$

Therefore, there is $\delta > 0$ such that

$$\frac{\mu(C_s \cap U(t,x,m^{-1}))}{\mu U(t,x,m^{-1})} >$$

$$> \frac{\mu(C_s \cap U(t,x_0,m^{-1})) - \mu(\Delta[U(t,x_0,m^{-1}),U(t,x,m^{-1})])}{\mu U(t,x,m^{-1})} > \frac{1}{2}$$

for any $\| t - t_0 \| < (p_0)^{-1}$ and $\| x - x_0 \| < \delta$.

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